3.9.(2) <u>Proposition</u>: Let S be a subgroup of $\pi_1(\Delta, w)$ where Δ is a connected graph. Then

(a) $\exists$ covering $f : \Gamma \longrightarrow \Delta$ and $v \in f^{-1}(w)$ such that Γ is connected and $f(\pi_1(\Gamma, v)) = S$

(b) If $f' : (\Gamma', v') \longrightarrow (\Delta, w)$ is another such covering then there exists a unique isomorphism $h : (\Gamma', v') \longrightarrow (\Gamma, v)$ such that $f \circ h = f'$

(c) The index $[\pi_1(\Delta, w) : S]$ of S in $\pi_1(\Delta, w)$ has the same cardinality as $f^{-1}(w)$

<u>Proof</u>: (a) Construction of Γ.

<u>Vertices</u> are equiv. classes $\langle p \rangle$ of paths in Δ with $\iota p = w$ where $p \sim p'$ if $\tau p = \tau p'$ and $p\overline{p'}$ represents an element of S

Set $v = \langle c_w \rangle$, c_w the constant path in w.

<u>Edges</u> with initial point $\langle p \rangle$ correspond bijectively to the edges in $st(\tau p, \Delta)$. I.e. edges are pairs

$(\langle p \rangle, e)$, $e \in st(\tau, \Delta)$, and $\iota(\langle p \rangle, e) = \langle p \rangle$

and define $\overline{}$ by $\overline{(\langle p \rangle, e)} = (\langle pe \rangle, \bar{e})$. Then obviously, $\overline{(\langle p \rangle, e)} \neq (\langle p \rangle, e)$ since $e \neq \bar{e}$. Further

$$\overline{\overline{(\langle p \rangle, e)}} = \overline{(\langle pe \rangle, \bar{e})} = (\langle pe\bar{e} \rangle, e) = (\langle p \rangle, e)$$

since $pe\bar{e}$ is homotopic to p.

So we have a graph and f is the obvious map:

$$\langle p \rangle \xmapsto{f} \tau p \quad , \quad (\langle p \rangle, e) \xmapsto{f} e .$$

Clearly, f is a covering with $f(v) = \tau c_w = w$.

Since $\tau(\langle p\rangle, e) = \langle pe\rangle$ a path starting at v has the form

$$(\langle c_w\rangle, e_1)(\langle e_1\rangle, e_2) \cdots (\langle e_1 \cdots e_{n-1}\rangle, e_n)$$

and it is closed if and only if $\langle e_1 \cdots e_{n-1} e_n\rangle = \langle c_w\rangle$, i.e. iff $e_1 \cdots e_n$ represents a path in S. It follows that the map $\pi_1(\Gamma, v) \longrightarrow \pi_1(\Delta, w)$ induced by f - which by abuse of language we call again f - maps $\pi_1(\Gamma, v)$ onto S. Since covering maps induce injective maps on π_1 the map $f: \pi_1(\Gamma, v) \longrightarrow S \subset \pi_1(\Delta, w)$ is an isomorphism.

By definition of the vertices of Γ the graph Γ is connected:

$$\langle p\rangle = \tau\big((\langle c_w\rangle, e_1)(\langle e_1\rangle, e_2) \cdots (\langle e_1 \cdots e_{n-1}\rangle, e_n)\big)$$

if $p = e_1 e_2 \cdots e_n$.

(3.8)

(b) Follows immediately from our general lifting theorem, since Γ and Γ' are connected: i.e. $\exists$ unique $h: \Gamma \to \Gamma'$ and $h': \Gamma' \to \Gamma$ s.t. $f \circ h' = f'$, $f' \circ h = f$ and $h(v) = v'$, $h'(v') = v$. Then $h' \circ h: \Gamma \longrightarrow \Gamma$ maps v to v and $f \circ (h' \circ h) = f' \circ h = f$. Since id_Γ has the same property we have $h' \circ h = id_\Gamma$ and similarly $h \circ h' = id_{\Gamma'}$.

(c) Let x_α, $\alpha \in A$, be a system of representatives of the left cosets $S \backslash \pi_1(\Delta, w)$ of S, and let p_α be a path representing x_α. Then $\langle p_\alpha\rangle \neq \langle p_{\alpha'}\rangle$ if $\alpha \neq \alpha'$. For $p_\alpha \sim p_{\alpha'}$ implies $p_\alpha \overline{p_{\alpha'}}$ represents an $s \in S$, i.e. $x_\alpha = s \cdot x_{\alpha'}$ contradicting the choice of the $x_\beta, \beta \in A$.

Furthermore, each p_α is closed so that $w = \tau p_\alpha =: f\langle p_\alpha\rangle$, i.e. each $\langle p_\alpha\rangle \in f^{-1}(w)$. Furthermore, if $\langle p\rangle \in f^{-1}(w)$ then $\tau p = w$, i.e. p is closed. Then $\exists\, \alpha$ and $s \in S$ such that $[p] = s \cdot x_\alpha$ i.e. $[p \cdot \bar{p}_\alpha] = s$. Therefore, $\langle p\rangle = \langle p_\alpha\rangle$. □

Next we need some notions on group actions. Group actions can be defined in various contexts which is determined by the category we are dealing with. So

<u>3.10 Definition</u>. Let G be a group and X an object of a category $\mathcal{C}$. A G-action on X is a homomorphism

$$\varphi : G \longrightarrow Aut_{\mathcal{C}}(X)$$

Recall that in any category $\mathcal{C}$ and any obj. X of $\mathcal{C}$ the composition of morphisms makes $Aut_{\mathcal{C}}(X)$ into a group

We will exclusively deal with categories where the objects are sets together with some structure and the morphisms are maps of the underlying sets respecting the structure. Examples are TOP, Set, Graphs, Groups, Abelian Groups, Vector spaces, rings, modules etc.

In this situation, for $g \in G$ and $x \in X$ an element of the underlying set of the object X we write

$$gx \quad \text{or } g \cdot x \quad \text{or } g(x) \quad \text{instead of } (\varphi(g))(x).$$

The following notions for an action φ of G on X then make sense

3.11 Definition:

(a) φ is called *effective* if $\ker(\varphi) = 1_G$, i.e. if $g \in G$ with $g(x) = x$ $\forall x \in X$ then $g = 1_G$

(b) φ is called *transitive* if for all $x, y \in X$ there is $g \in G$ with $g(x) = y$

(c) φ is called *free* if the only element of G which fixes an $x \in X$ is 1_G, i.e. if $g \in G$ and there is $x \in X$ with $g(x) = x$ then $g = 1_G$.

(d) The orbit of $x \in X$ is the set $G \cdot x = \{g(x) : g \in G\}$

(e) For a subset $S \subset G$ the fix set of S is the set $X^S := \{x \in X : s(x) = x \ \forall s \in S\}$.

3.12 Definition: A *G-graph* Γ is a graph Γ together with a G-action such that for every edge e and $g \in G$ we have $g(e) \neq \bar{e}$ (i.e. the orbits of e and $\bar{e}$ are distinct, and therefore disjoint)

Then the *quotient graph* Γ/G exists with vertices the sets of orbits of vertices of Γ and edges the orbits of edges of Γ and

$$\overline{G \cdot e} = G \cdot \bar{e} \quad , \quad \iota(G \cdot e) = G \cdot \iota(e).$$

3.13 Remark: If the action on the G-graph is free on the vertex set V_Γ then it is also free on the edge set, and $\Gamma \longrightarrow \Gamma/G$ is a covering. For any G-graph Γ the quotient map is locally surjective

The standard example of a free action is the action of $\pi_1(\Gamma, v)$ on the universal cover $(\tilde{\Gamma}, \tilde{v})$ of (Γ, v), where Γ is connected. The universal cover of (Γ, v) is then the covering $f : (\hat{\Gamma}, \hat{v}) \longrightarrow (\Gamma, v)$ associated to the trivial subgroup of $\pi_1(\Gamma, v)$; i.e. up to isomorphism $\tilde{\Gamma} \xrightarrow{f} \Gamma$ is determined by

f is a covering, $\hat{\Gamma}$ is connected, $\pi_1(\hat{\Gamma}, \tilde{v}) = \{1\}$ for some (and thus all) $\tilde{v} \in V_{\hat{\Gamma}}$.

If $(\tilde{\Gamma}, \hat{v}) \xrightarrow{\quad f \quad} (\Gamma, v)$ is (the) universal covering we get a $G = \pi_1(\Gamma, v)$ action on $\hat{\Gamma}$ as follows.

Let $\tilde{w}$ be a vertex of $\tilde{\Gamma}$ and $[p]$ an element of $\pi_1(\Gamma, v)$. Let $\hat{q}$ be a path in $\hat{\Gamma}$ from $\tilde{v}$ to $\hat{w}$ and q its image in Γ. Let $\widehat{pq}$ be the unique lift of pq with initial point $\tilde{v}$. Then define

$$[p]\cdot\hat{w} = \tau(\widetilde{pq}).$$

For edges $\tilde{e}$ with $\iota\tilde{e} = \tilde{w}$ let $[p]\hat{e}$ be the unique edge with $\iota([p]\tilde{e}) = [p]\cdot\tilde{w}$ and

$$f([p]\hat{e}) = f(\hat{e})$$

<u>Notice</u>: $[p]\,\tilde{w}$ does not depend on the choice of $p \in [p]$ and path $\hat{q}$ since any two paths in $\tilde{\Gamma}$ with the same initial and final points are homotopic.

Also $f([p]\hat{w}) = f(\tilde{w})$, so that $f(st([p]\hat{w}, \hat{\Gamma}) = st(f(w), \Gamma)$