In fact, $\pi_1(\Gamma, v)$ is a free group for any graph Γ.

To see this we need a few more concepts

2.13 Def A path p is called reduced if it does not contain any 1-step backtrack.

As in 1.7 one shows

2.14 Prop. Any homotopy class $[p] \in \Pi(\Gamma)$ of paths contains a unique reduced path

2.15 Definitions. A graph Γ is connected if for any $v, w \in V_\Gamma$ there is a path p in Γ with $\iota p = v$, $\tau p = w$.

A graph Γ is called a forest if all reduced loops have length 0

A tree is a connected forest.

2.16 If v, w are vertices in a tree T then there is a unique reduced path p in T from v to w (i.e. $\iota p = v$, $\tau p = w$).

Proof. 1st there is one.

2nd. Let p, q be reduced paths in T from v to w. Then $p \cdot \bar{q}$ is a loop. So it is homotopic to c_v. Since neither p nor $\bar{q}$ contain 1-step back-tracks we must have the following. If the last edge of p is e then the first edge of $\bar{q}$ is $\bar{e}$, i.e. the last edge of q is also e. Now continue via induction on the length of p. □

We denote this path by $[v, w]_T$.

2.17 (a) Every graph Γ contains a maximal forest

(b) Every maximal forest of Γ contains all vertices of Γ.

(c) If Γ is connected, every maximal forest is a tree.

Proofs: (a) follows easily from Zorn's lemma. I think that the following alternative proof is useful (we use, as we always do in this lecture, the axiom of choice which is equivalent to Zorn's lemma)

Pick for any component Γ_i of Γ a vertex $v_i \in \Gamma_i$. We will construct for each i a tree T_i in Γ_i which is a maximal forest of Γ_i. Since for $i \neq j$ there are no edges connecting a vertex of Γ_i to a vertex of Γ_j the union of the T_i is a maximal forest.

In any graph Γ we define a distance d_Γ on V_Γ by setting

$$d_\Gamma(v,w) = \begin{cases} \operatorname{Min}\{|p| : p \text{ is a path in } \Gamma \text{ from } v \text{ to } w\}, & \text{if there is a path from } v \text{ to } w \text{ in } \Gamma \\ \infty, & \text{otherwise} \end{cases}$$

If Γ is connected, d_Γ is a metric on V_Γ.

Fix a component Γ_i of Γ. We define inductively trees T_{in}, $n = 0, 1, \ldots$, in Γ_i, starting with $T_{i0} = (\emptyset, \{v_i\})$ and having the property that $V_{T_{in}} = \{w \in V_{\Gamma_i} : d_{\Gamma_i}(v_i, w) \leq n\}$ and with $T_{i0} \subseteq \cdots \subseteq T_{in}$

Assume $n \geq 1$ and that $T_{i\,n-1}$ has been constructed with the desired properties.

Let $S^n(\Gamma_i, v_i) := \{ v \in V_{\Gamma_i} : d(v, v_i) = n \}$ since $T_{i\,n-1}$ contains all vertices with distance at most $n-1$ from v_i for every $v \in S^n(\Gamma_i)$ there exists (at least) one edge pair $(e, \bar{e})$ in Γ_i whose one endpoint is v while the other is in $V_{T_{i\,n-1}}$. Choose one such edge pair $(e_v, \bar{e}_v)$. Then set

$$V_{T_i} = V_{T_{i\,n-1}} \cup S^n(\Gamma_i, v_i)$$ (obviously, this is a disjoint union)

$$E_{T_i} = E_{T_{i\,n-1}} \cup \bigcup_{v \in S^n(\Gamma_i, v_i)} \{ e_v, \bar{e}_v \}$$

with the maps $\bar{\ }$ and ι induced from Γ.

You should have no problem to prove that

$$T_i = \bigcup_{n \geq 0} T_{in}$$ is a tree, which

contains every vertex of Γ_i, and therefore is maximal: if e is any further edge of Γ_i let p be the unique reduced path in T_i from $\tau(e)$ to $\iota(e)$. Then $e \cdot p$ is a reduced non-trivial path.

Obviously:

$$\bigcup_i T_i \quad \text{is a maximal forest of } \Gamma.$$

(This proves (a))

(b) is obvious, since any vertex is a tree

(c) Let Γ be connected, and let T be a maximal tree in Γ. Let T_1 be a component of T. If $T_1 \neq T$ there is a vertex v of T with $v \notin V_{T_1}$. Since Γ is connected, for any vertex w of T_1 there is a path from v to w. In particular, there must be an edge e in this path with

$\iota(e) \notin V_{T_1}$, $\tau(e) \in V_{T_1}$. Adding $(e, \bar{e})$ to T gives us a new forest: if p is any reduced loop not containing e or $\bar{e}$ it is trivial since T is a forest. If it contains e it must also contain $\bar{e}$ (or vice versa), and there must be a subpath of the form $e p' \bar{e}$ (or $\bar{e} p'' e$) with p' entirely in T. But then p' is trivial since it is a loop, and the original path was not reduced.

□

A bit more interesting is the next statement.

<u>2.18 Proposition</u>: Let v be a vertex of a connected graph Γ and T a maximal tree in Γ

Fix an orientation $\mathcal{O}$ of Γ and for each edge $e \in \mathcal{O} \setminus E_T$ let p_e be the path (actually loop)

$$p_e = [v, \iota(e)]_T \; e \; [\tau(e), v]_T,$$

where as in 2.16 for any two vertices v_1, v_2 in a tree T we denote by $[v_1, v_2]_T$ the unique reduced path in T from v_1 to v_2.

Then $\{ [p_e] \mid e \in \mathcal{O} \setminus E_T \}$ is a basis of $\pi_1(\Gamma, v)$. In particular, $\pi_1(\Gamma, v)$ is a free group for any graph Γ (connected or not).

<u>Proof</u>: To avoid confusion we choose for every $e \in \mathcal{O} \setminus E_T$ a symbol s_e and consider $F(S)$ where $S = \{ s_e : e \in \mathcal{O} \setminus E_T \}$. Let

$$\varphi : F(S) \longrightarrow \pi_1(\Gamma, v)$$

be the (unique) homomorphism defined by $\varphi([s_e]) = [p_e]$.

<u>Claim 1</u>: φ is surjective. To see this choose a reduced loop $p \in P(\Gamma)$ with $\iota(p) = \tau(p) = v$. Then p can be written in the form

$$p = w_1 e_1 w_2 e_2 \cdots w_{n-1} e_{n-1} w_n$$ where each w_i is a (reduced) path in T and each $e_i \in \mathcal{O} \cup \bar{\mathcal{O}} \setminus E_T$.

Consider the word

$$w_p = s_{e_1} \cdot \dots \cdot s_{e_{n-1}} \quad \text{in } W(S), \text{ where}$$

$s_{e_i} = s_{e_i}$ if $e_i \in \mathcal{V}$ and $s_{e_i} = s_{\bar{e}_i}^{-1}$ if $e_i \in \bar{\mathcal{V}}$. Then $\varphi[w_p]$ is the homotopy class of

$$\varphi w_p = [v, \iota(e_1)]_T\, e_1\, [\tau(e_1), v]_T\, [v, \iota(e_2)]_T\, e_2\, [\tau(e_2), v]_T \cdots$$
$$[v, \iota(e_{n-1})]_T\, e_{n-1}\, [\tau(e_{n-1}), v]_T$$

abuse of notation

For each $1 \le i \le n-2$ $\quad [\tau(e_i), v]_T [v, \iota(e_{i+1})]_T$ is a path in T from $\tau(e_i)$ to $\iota(e_{i+1})$ which is homotopic to the unique reduced path in T from $\tau(e_i)$ to $\iota(e_{i+1})$. But w_{i+1} is such a reduced path. We also conclude that $[v, \iota(e_1)]_T = w_1$, and $[\tau(e_{n-1}), v]_T = w_n$ since in each instance we have reduced paths in T joining the same endpoints. Thus

$\varphi[w_p] = [p]$ and φ is surjective.

<u>Claim 2.</u> φ is injective. So let

$w = s_{e_1} \cdots s_{e_n}$ be a reduced word such that φw can be reduced to the trivial word. We have to show that $w = \emptyset$. Let for $i = 2, \dots, n-1$

w_i be $[\tau(e_{i-1}), \iota(e_i)]_T$. Then φw is homotopic to

$$\underbrace{[v, \iota(e_1)]_T}_{=:w_1} e_1 \, w_2 \, e_2 \cdots w_{n-1} \, e_{n-1} \underbrace{[\tau(e_{n-1}), v]_T}_{=:w_n}$$

with the $w_1, \dots, w_n$ reduced. Any further reduction can only occur at a subpath $e_{i-1} w_i e_i$ with $w_i = \emptyset$ and $e_{i-1} = \bar{e}_i$. But then $s_{e_{i-1}} = s_{e_i}^{-1}$ and w is not reduced. $\square$