Graphs I
Basic Notions

We follow practically verbatim:

J. Stallings: Topology of finite graphs, Invent. math. 71, 551–565 (1983)

<u>2.1 Definition.</u> A graph Γ consists of two sets E and V and two maps $\bar{\ }: E \longrightarrow E$, $e \longmapsto \bar{e}$ and $\iota : E \longrightarrow V$

satisfying the following rules

(i) $\bar{\bar{e}} = e$ (ii) $e \neq \bar{e}$

<u>2.2 Notations:</u>
An element $e \in E$ is called a <u>directed edge</u> of Γ and $\bar{e}$ is called the <u>reverse</u> of e; $\iota(e)$ is called the <u>initial vertex</u> of e. We define the <u>terminal vertex</u> $\tau(e)$ to be $\iota(\bar{e})$, i.e. the initial vertex of the reverse of e

An <u>orientation</u> of Γ is the choice of exactly one edge in each pair $\{e, \bar{e}\}$. Elements of V are called vertices of Γ

<u>2.3 Definition.</u> A map $f : \Gamma \to \Delta$ between graphs consists of two maps $f_E : E_\Gamma \longrightarrow E_\Delta$, $f_V : V_\Gamma \longrightarrow V_\Delta$ (where for a graph Ψ E_Ψ and V_Ψ are the edges and vertices of Ψ) which are compatible with the structure maps $\bar{\ }$ and ι, i.e.

$$f_E(\bar{e}) = \overline{f_E(e)}, \quad f_V(\iota(e)) = \iota f_E(e).$$

Usually we drop the indices E, V from our notation since usually it is clear what f is applied to.

Clearly, compositions of graph maps are graph maps

$(g\circ f(\bar{e}) = g(\overline{f(e)}) = \overline{g\circ f(e)}; \; g\circ f(\iota(e)) = g(\iota(f(e))) = \iota(g\circ f(e)) \;)$

and we have a category with graphs as objects and graph maps as morphism.

There are a number of categorical concepts one might (or might not) want to study for graphs. Most are quite obvious. We will introduce these concepts when we come across them later on. But two concepts appear frequently. We discuss them now.

<u>2.4 Definitions</u>: Let $\mathcal{C}$ be a category. Given morphisms

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \\ C & \end{tikzcd}$$

in $\mathcal{C}$. The <u>push-out</u> of this diagram is an object D in $\mathcal{C}$ with morphisms $b: B \longrightarrow D$ and $c: C \longrightarrow D$ s.t.

(i) $$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[d, "b"] \\ C \arrow[r, "c"] & D \end{tikzcd}$$ commutes and

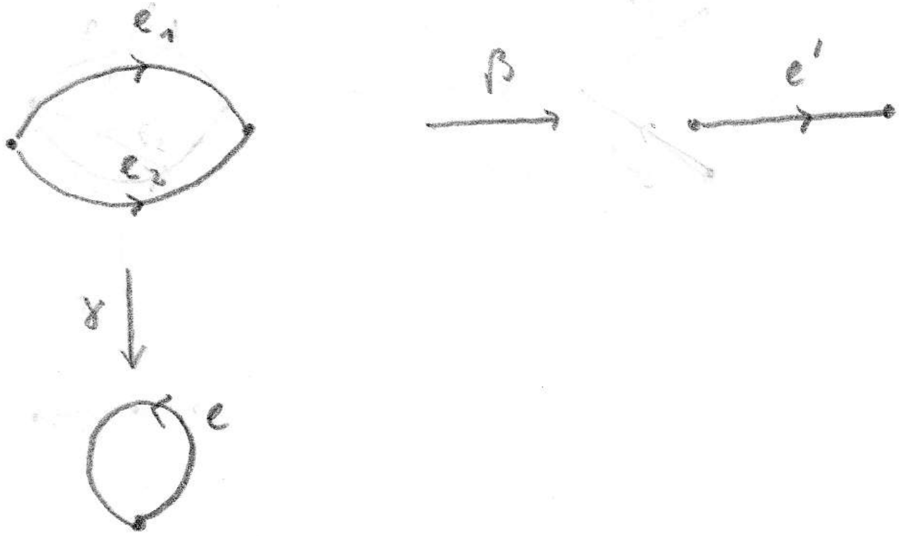
(ii) For any pair of morphisms $c': C \longrightarrow D'$, $b': B \longrightarrow D'$ in $\mathcal{C}$ with $b'\beta = c'\gamma$ there exists a unique morphism $h: D \longrightarrow D'$ such that $h\circ b = b'$, $h\circ c = c'$.

i.e

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[d, "b"] \arrow[ddr, bend left, "b'"] & \\ C \arrow[r, "c"] \arrow[drr, bend right, "c'"'] & D \arrow[dr, dashed, "h", "\exists!"'] & \\ & & D' \end{tikzcd}$$

As before, the drawn out diagram is given and commutes. Then we find a unique dotted arrow h making everything commute.

Push-Outs need not exist in every category. For example consider in the category of graphs:



where β is the obvious map, $\gamma(e_1)=e$, $\gamma(e_2)=\bar{e}$

(We draw for each pair $\{d,\bar{d}\}$ of edges only one, the map ι is indicated by the arrows)

Assume the pushout graph Γ, b, c exists. Then

$c(e) = c\circ\gamma(e_1) = b\circ\beta(e_1) = b(e')$

$c(\bar{e}) = c\circ\gamma(e_2) = b\circ\beta(e_2) = b(e')$. i.e. if c is a graph map we have $\overline{c(e)} = c(\bar{e}) = c(e)$ contradicting 2.1.(ii)

But there is a mild condition on the maps f_1, f_2 in the diagram in the category of graphs

$$\begin{tikzcd} \Gamma \arrow[r, "f_1"] \arrow[d, "f_2"'] & \Delta_1 \\ \Delta_2 & \end{tikzcd}$$

which guarantees the existence of a pushout $g_i : \Delta_i \to \Delta$, $i = 1, 2$.

<u>2.5 Claim</u>: If we can orient $\Gamma, \Delta_1, \Delta_2$ such that f_1 and f_2 preserve orientations then the above diagram admits a pushout.

<u>Proof</u>. Let E_i, V_i be the oriented edges and the vertices of Δ_i

Let V, $g_{V_i} : V_i \to V$ be the push-out of

$$\begin{tikzcd} V_\Gamma \arrow[r, "f_1"] \arrow[d, "f_2"'] & V_1 \arrow[d, "g_{V_1}"] \\ V_2 \arrow[r, "g_{V_2}"'] & V \end{tikzcd}$$

i.e. $V = V_1 \sqcup V_2 / \sim$ where $\sim$ is generated by:

$v_1 \in V_1$, $v_2 \in V_2$ are equivalent if there is $v \in V_\Gamma$ such that $f_i(v) = v_i$, and define $g_{V_i}(w_i) = [w_i]$, $i = 1, 2$, $w_i \in V_i$.

2.6 Exercise: Check that this is in fact a push out in the category of sets.

Similarly we let E be the push-out of the diagram

$$\begin{array}{ccc} E_\Gamma & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ E_2 & \longrightarrow & E \end{array}$$

We define $\bar{\ }: E \longrightarrow E$ via representatives, i.e.

$\overline{[e_i]} = [\bar{e}_i]$.

and $\iota : E \longrightarrow V$ by $\iota[e_i] = [\iota e_i]$.

We have seen in the example, that in general this leads to maps $\bar{\ }$ which have fixed elements. That $(E, V, \bar{\ }, \iota)$ is in fact a graph is a consequence of the orientation conditions in the Claim.

First note that an orientation of (the) target of a graph map induces a unique orientation of the source which makes the map orientation preserving. Thus, our condition says that there are orientations $\mathcal{O}_1$ of Δ_1 and $\mathcal{O}_2$ of Δ_2 such that the induced orientations on Γ are the same. This implies that an $e_i \in E_1 \sqcup E_2$ with $e_i \in \mathcal{O}_1 \cup \mathcal{O}_2$ can be equivalent to $e_j \in E_1 \sqcup E_2$ only if e_j is also in $\mathcal{O}_1 \cup \mathcal{O}_2$. This implies, in particular, that e_i and $\bar{e}_i$ are never equivalent, i.e

$\overline{[e_i]} \neq [e_i]$. $\square$

The other categorical concept which will play a rôle produces no problems. Pullbacks are the duals of pushouts, i.e.

2.6 Definition

Given the diagram $\mathcal{D}$

$$\begin{tikzcd} & B_1 \arrow[d, "\beta_1"] \\ B_2 \arrow[r, "\beta_2"'] & A \end{tikzcd}$$

in the category $\mathcal{C}$. The pullback of $\mathcal{D}$ is an object C of $\mathcal{C}$ with morphisms $\gamma_i : C \to B_i$, $i = 1, 2$, s.t.

(i) $\beta_1 \gamma_1 = \beta_2 \gamma_2$, i.e.

$$\begin{tikzcd} C \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & B_1 \arrow[d, "\beta_1"] \\ B_2 \arrow[r, "\beta_2"'] & A \end{tikzcd}$$

commutes,

and

(ii) For any pair $\delta_i : D \to B_i$, $i = 1, 2$, of morphisms with $\beta_1 \delta_1 = \beta_2 \delta_2$ there exists a unique morphism $D \xrightarrow{\delta} C$ s.t. $\gamma_1 \delta = \delta_1$ and $\gamma_2 \delta = \delta_2$, i.e.

$$\begin{tikzcd} D \arrow[dr, dashed, "\exists! \delta"'] \arrow[drr, bend left, "\delta_1"] \arrow[ddr, bend right, "\delta_2"'] & & \\ & C \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & B_1 \arrow[d, "\beta_1"] \\ & B_2 \arrow[r, "\beta_2"'] & A \end{tikzcd}$$

2.7 Remark: For graphs all pullbacks exist. Here is the construction for given

$$\begin{tikzcd} & \Delta_1 \arrow[d, "\beta_1"] \\ \Delta_2 \arrow[r, "\beta_2"'] & \Gamma \end{tikzcd}$$

Define $\Delta = (E, V, \bar{\ }, \iota)$ by

$$E = \{ (e_1, e_2) \in E_{\Delta_1} \times E_{\Delta_2} : \beta_1(e_1) = \beta_2(e_2) \}$$

$$V = \{ (v_1, v_2) \in V_{\Delta_1} \times V_{\Delta_2} : \beta_1(v_1) = \beta_2(v_2) \}$$

$$\overline{(e_1, e_2)} := (\bar{e}_1, \bar{e}_2) \qquad \iota(e_1, e_2) = (\iota e_1, \iota e_2).$$

The maps $\Delta \xrightarrow{\gamma_i} \Delta_i$ are the obvious projections.

Property 2.6 (ii) is more or less immediate. $\square$

To define the fundamental groupoid and group of a graph we need to notions of path, loop, homotopy of paths.

<u>2.8 Definition</u>. A path p (of length $n = |p|$) of the graph Γ is an n-tuple $p = e_1 \cdot \dots \cdot e_n$ of edges s.t. for $2 \le i \le n$ we have $\tau(e_{i-1}) = \iota(e_i)$; ιe_1 is called the initial vertex ιp of p and $\tau(e_n)$ the terminal vertex τp of p. For every vertex v there is a unique path c_v of length 0 with $\iota c_v = \tau c_v = v$. A loop is a path p with $\iota p = \tau p$.

A <u>1-step backtrack</u> is a path of the form $e\bar{e}$, $e \in E_\Gamma$.

Two paths p_1 and p_2 are called homotopic if there is a finite sequence $q_1 = p_1, q_2, \dots, q_r = p_2$ of paths such that for $i = 2, \dots, r$ the path q_i is obtained from q_{i-1} by inserting or removing somewhere a 1-step backtrack.

Clearly, homotopy is an equivalence relation. If p_1 is homotopic to p_2 ($p_1 \simeq p_2$) then $\iota p_1 = \iota p_2$ and $\tau p_1 = \tau p_2$.

If p_1, p_2 are paths with $\tau p_1 = \iota p_2$ we can form their product by concatenating p_2 with p_1, i.e. we form $p_1 p_2$.

<u>2.9 Remark</u>. As in 1.4 it is easy to show: $p_1 \simeq p_1'$, $p_2 \simeq p_2'$ and $p_1 p_2$ can be formed. Then $p_1' p_2'$ can be formed and $p_1 p_2 \simeq p_1' p_2'$

<u>2.10 Remark and Notation.</u> Let $P(\Gamma)$ be the set of paths of Γ, $\pi(\Gamma)$ the set of homotopy classes of paths. The product in $P(\Gamma)$ is associative and $c_{\iota p}$ is a left unit for p while $c_{\tau p}$ is a right unit for p. The product in $P(\Gamma)$ induces by 2.9 a product in $\pi(\Gamma)$ which makes $\pi(\Gamma)$ into a groupoid (A small category where every morphism is invertible; a category is small, if the objects form a set; here

we view V_Γ as the objects with $[c_v]$ the identity of the vertex v and morphisms from v to w are the homotopy classes of paths p with $\iota p = v$, $\bar{\iota} p = w$. If $p = e_1 \cdots e_r$ then

$[p]^{-1} = [\bar{e}_r \cdots \bar{e}_1] = [c_v] = 1_v$)

If $v \in V_\Gamma$ then any two loops which start in v can be multiplied. Thus, if we denote the set of homotopy classes of loops beginning in v bei $\pi_1(\Gamma, v)$ then the product of paths induces a product on $\pi_1(\Gamma, v)$ which makes it into a group. We call $\pi_1(\Gamma, v)$ the <u>fundamental group</u> of Γ with basepoint v.

<u>2.11 Remark</u>: There is a standard) geometric model $G(\Gamma)$ which describes every graph Γ as a 1-dimensional CW-complex, which captures all features of Γ. It is then a simple (since 1-dimensional) version of simplicial approximation to show that there is a natural isomorphism

See eg. g. page 2-3

$$\pi_1(\Gamma, v) \longrightarrow \pi_1(G(\Gamma), v)$$

where $\pi_1(X, x_0)$ is the usual fundamental group of a topological space X with basepoint $x_0 \in X$.

We will give a short proof of this statement later on.

<u>2.12 Remark</u>: If Γ has exactly one vertex v then by our construction of free groups in 1.3 the group $\pi_1(\Gamma, v)$ is isomorphic to $F(\mathcal{O}_\Gamma)$, where $\mathcal{O}_\Gamma$ is an orientation of Γ.

In fact, $\pi_1(\Gamma, v)$ is a free group for any graph Γ.

To see this we need a few more concepts

2.13 Def A path p is called reduced if it does not contain any 1-step backtrack.

As in 1.7 one shows

2.14 Prop. Any homotopy class $[p] \in \pi(\Gamma)$ of paths contains a unique reduced path

2.15 Definitions. A graph Γ is connected if for any $v, w \in V_\Gamma$ there is a path p in Γ with $\iota p = v$, $\tau p = w$.

A graph Γ is called a forest if all reduced loops have length 0

A tree is a connected forest.

2.16 If v, w are vertices in a tree T then there is a unique reduced path p in T from v to w (i.e. $\iota p = v$, $\tau p = w$).

Proof. 1st there is one.

2nd Let p, q be reduced paths in T from v to w. Then $p \cdot \bar{q}$ is a loop. So it is homotopic to c_v. Since neither p nor $\bar{q}$ contain 1-step back-tracks we must have the following. If the last edge of p is e then the first edge of $\bar{q}$ is $\bar{e}$, i.e. the last edge of q is also e. Now continue via induction on the length of p. $\square$