THIS IS READING MATERIAL Cat 1
(Use it for reference if need be)

Basic Notions from Category Theory
(very basic)

1. Category $\mathcal{C}$ consists of a class of <u>objects</u>

which we denote by $\mathrm{Ob}\,\mathcal{C}$ or-by abuse of notation- simply $\mathcal{C}$

and for each pair X, Y of objects a set

$\mathrm{mor}_{\mathcal{C}}(X,Y)$ (elements of $\mathrm{mor}_{\mathcal{C}}(X,Y)$ are usually depicted as $f : X \to Y$, and f is called a <u>morphism</u> (of $\mathcal{C}$) from X to Y, or with source X and target Y)

and for each triple X, Y, Z of objects with a <u>composition of morphisms</u>

$$\begin{array}{ccc} \mathrm{mor}_{\mathcal{C}}(Y,Z) \times \mathrm{mor}_{\mathcal{C}}(X,Y) & \xrightarrow{-\circ-} & \mathrm{mor}_{\mathcal{C}}(X,Z) \\ (g \quad , \quad f) & \longmapsto & g \circ f \end{array}$$

(for morphisms g, f of $\mathcal{C}$ we say that $g \circ f$ exists or that the composition of g and f exists or that g and f are composable e.t.c, if source g = target f)

Composition is required to be <u>associative</u>

$(h \circ g) \circ f = h \circ (g \circ f)$ if either side exists

Furthermore one requires the existence of and element id_X (or 1_X) in $\mathrm{mor}_{\mathcal{C}}(X,X)$ for each object X which satisfies $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$, whenever the left sides exist.

Examples of categories abound, and you all have been dealing with many of these

(i) $\mathcal{S}et$, the category of <u>sets</u> and maps of sets

(ii) Vect_K , for a field K, is the category of <u>K-vector spaces</u> and K-linear maps

(iii) Top , the category of topological spaces and continuous maps

(iv) Top_0 , the category of topological spaces with basepoint and basepoint preserving continuous maps

(v) $\mathcal{G}p$, the category of groups and group-homomorphisms

(vi) AbG , the category of abelian groups

e.t.c.

2. Also fundamental is the concept of a (covariant) <u>functor</u>

$F : \mathcal{C} \longrightarrow \mathcal{D}$ from the category $\mathcal{C}$ to the category $\mathcal{D}$

F assigns to each object C of $\mathcal{C}$ the object $F(C)$ of $\mathcal{D}$
and to each morphism $f : C \longrightarrow C'$ in $\mathcal{C}$
the morphism $F(f) : F(C) \longrightarrow F(C')$ s.t.

(i) $F(g \circ f) = F(g) \circ F(f)$ whenever $g \circ f$ exists

(ii) $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$ for every object C of $\mathcal{C}$.

A morphism $f: X \to Y$ in $\mathcal{C}$ is called an isomorphism if there exists $g: Y \to X$ in $\mathcal{C}$ s.t. $g \circ f = id_X$ and $f \circ g = id_Y$.

Obviously, any functor $F: \mathcal{C} \to \mathcal{D}$ maps isomorphisms to isomorphisms.

Examples of functors:

(i) $Top \xrightarrow{forget} Set$ simply maps the space to its underlying set and forgets the topology

There are many forgetful functors
$Vect_K \to Set$, $Gp \to Set$, $Top_0 \to Top$

(ii) We have seen the functor

$$Freegp : Set \longrightarrow Gp$$
$$S \longmapsto F(S)$$

(If $f: S \to S'$ is a set map what should $Freegp(f): F(S) \to F(S')$ be ??)

(iii) The fundamental group functor we have seen in Topology 1. It is the functor

$$\pi_1 : Top_0 \longrightarrow Gp$$
$$(X, x_0) \longmapsto \pi_1(X, x_0)$$
$$\big(f: (X,x_0) \to (Y,y_0) \big) \longmapsto \pi_1(f): \pi_1(X,x_0) \to \pi_1(Y,y_0)$$

One more elementary concept will show up often in your future studies

3. Given functors $F, G : \mathcal{C} \to \mathcal{D}$. A <u>natural transformation</u> $\varphi : F \to G$ between F and G assigns to every object C of $\mathcal{C}$ a morphism

$$\varphi_C : F(C) \to G(C) \quad \text{in } \mathcal{D} \text{ s.t.}$$

for every morphism $f : C \to C'$ the diagram

$$\begin{tikzcd} F(C) \arrow[r, "\varphi_C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\ F(C') \arrow[r, "\varphi_{C'}"] & G(C') \end{tikzcd}$$

commutes in $\mathcal{D}$.

One example of a natural transformation shows up in Problem 4 of Homework-Sheet 3.

$$F, G : \mathcal{G}raph_0 \longrightarrow \mathcal{G}p$$

$$F(\Gamma, v_0) = \pi_1(\Gamma, v_0)$$

$$G(\Gamma, v_0) = \pi_1(\Gamma^{(n)}, v_0)$$

You are supposed to show that G is a functor and that there is a natural transformation $\varphi : F \to G$ such that every φ_Γ is an isomorphism. Then φ is called a natural isomorphism.

Functor pairs like $\mathcal{G}p \xrightarrow{\text{forget}} \mathcal{S}et$

and $\text{Freegp}: \mathcal{S}et \longrightarrow \mathcal{G}p$ have analogues in many situations and are useful in studying preservation of certain universal structures. Here we note

(i) There is a (canonical) natural transformation

$\text{Freegp} \circ \text{forget} \longrightarrow id_{\mathcal{G}p}$ which for any group G maps $F(G) \longrightarrow G$ by the obvious homomorphism mapping the basis element $[g] \in F(G)$ to $g \in G$.

(ii) There is a (canonical) natural transformation

$id_{\mathcal{S}et} \longrightarrow \text{forget} \circ \text{Freegp}$ which for a

set S maps $s \in S$ to the element $[s]$ of $F(S)$, where now we consider $F(S)$ as a set.

(iii) If S is a set and G a group then

$$\text{Mor}_{\mathcal{G}p}(F(S), G) \cong \text{Mor}_{\mathcal{S}et}(S, \text{forget}(G))$$

A pair $\mathcal{C} \underset{B}{\overset{A}{\rightleftarrows}} \mathcal{D}$ of functors such that there is a natural isomorphism of sets

$\text{Mor}_{\mathcal{D}}(A(C), D) \cong \text{Mor}_{\mathcal{C}}(C, B(D))$ for any $C \in \mathcal{C}, D \in \mathcal{D}$ is called an <u>adjoint pair</u>, A being the left adjoint of and B the right adjoint of A