

THIS IS READING MATERIAL Cat 1 (use it for reference if needed)

Basic Notions from Category Theory (very basic)

1. Category \mathcal{C} consists of a class of objects

which we denote by $\text{Ob } \mathcal{C}$ or-by abuse of notation- simply \mathcal{C}

and for each pair X, Y of objects a set

$\text{mor}_{\mathcal{C}}(X, Y)$ (elements of $\text{mor}_{\mathcal{C}}(X, Y)$ are usually depicted as $f: X \rightarrow Y$, and f is called a morphism (of \mathcal{C}) from X to Y , or with source X and target Y)

and for each triple X, Y, Z of objects with a composition of morphisms

$$\begin{array}{ccc} \text{mor}_{\mathcal{C}}(Y, Z) \times \text{mor}_{\mathcal{C}}(X, Y) & \xrightarrow{\quad \circ \quad} & \text{mor}_{\mathcal{C}}(X, Z) \\ (g, f) & \longmapsto & g \circ f \end{array}$$

(for morphisms g, f of \mathcal{C} we say that $g \circ f$ exists or that the composition of g and f exists or that g and f are composable etc, if $\text{source } g = \text{target } f$)

Composition is required to be associative

$$(h \circ g) \circ f = h \circ (g \circ f) \quad \text{if either side exists}$$

Furthermore one requires the existence of and element id_X (or 1_X) in $\text{mor}_{\mathcal{C}}(X, X)$ for each object X which satisfies $f \circ \text{id}_X = f$ and $\text{id}_X \circ g = g$, whenever the left sides exist.

Examples of categories abound, and you all have been dealing with many of these

- (i) Set , the category of sets and maps of sets
- (ii) Vect_K , for a field K , is the category of K -vector spaces and K -linear maps
- (iii) Top , the category of topological spaces and continuous maps
- (iv) Top_* , the category of topological spaces with basepoint and basepoint preserving continuous maps
- (v) Grp , the category of groups and group-homomorphisms
- (vi) AbGrp , the category of abelian groups

e.t.c.

2. Also fundamental is the concept of a functor ^(covariant)

$F: \mathcal{C} \longrightarrow \mathcal{D}$ from the category \mathcal{C} to the category \mathcal{D}

F assigns to each object C of \mathcal{C} the object $F(C)$ of \mathcal{D}

and to each morphism $f: C \longrightarrow C'$ in \mathcal{C} the morphism $F(f): F(C) \longrightarrow F(C')$ s.t.

- (i) $F(g \circ f) = F(g) \circ F(f)$ whenever $g \circ f$ exists
- (ii) $F(\text{id}_C) = \text{id}_{F(C)}$ for every object C of \mathcal{C} .

A morphism $f: X \rightarrow Y$ in \mathcal{C} is called an isomorphism if there exists $g: Y \rightarrow X$ in \mathcal{C} s.t. $g \circ f = id_X$ and $f \circ g = id_Y$.

Obviously, any functor $F: \mathcal{C} \rightarrow \mathcal{D}$ maps isomorphisms to isomorphisms.

Examples of functors:

(i) $Top \xrightarrow{\text{forget}} Set$ simply maps the space to its underlying set and forgets the topology

There are many forgetful functors

$Vect_K \rightarrow Set$, $Grp \rightarrow Set$, $Top_0 \rightarrow Top$

(ii) We have seen the functor Free group

$$\begin{aligned} \text{Free}_{gp} : Set &\longrightarrow Grp \\ S &\longmapsto F(S) \end{aligned}$$

(If $f: S \rightarrow S'$ is a set map what should

$\text{Free}_{gp}(f) : F(S) \rightarrow F(S')$ be ??)

(iii) The fundamental group functor we have seen in Topology 1. It is the functor

$$\begin{aligned} \pi_1 : Top_0 &\longrightarrow Grp \\ (X, x_0) &\longmapsto \pi_1(X, x_0) \end{aligned}$$

$$(f: (X, x_0) \rightarrow (Y, y_0)) \longmapsto \pi_1(f) : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

One more elementary concept will show up often in your future studies

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3. Given functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$. A natural transformation $\varphi: F \rightarrow G$ between F and G assigns to every object C of \mathcal{C} a morphism

$$\varphi_C: F(C) \rightarrow G(C) \quad \text{in } \mathcal{D} \quad \text{s.t.}$$

for every morphism $f: C \rightarrow C'$ the diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{\varphi_C} & G(C) \\ F(f) \downarrow & & \downarrow G(f) \\ F(C') & \xrightarrow{\varphi_{C'}} & G(C') \end{array}$$

commutes in \mathcal{D} .

One example of a natural transformation shows up in Problem 4 of Homework-Sheet 3.

$$F, G: \text{Graph}_0 \longrightarrow \mathcal{G}P$$

$$F(\Gamma, v_0) = \pi_0(\Gamma, v_0)$$

$$G(\Gamma, v_0) = \pi_1(\Gamma^{(n)}, v_0)$$

You are supposed to show that G is a functor and that there is a natural transformation $\varphi: F \rightarrow G$ such that every φ_Γ is an isomorphism. Then φ is called a natural isomorphism.

