Case 2:

A as before, B is obtained from C by attaching cells of dim. $\geq n+1$.

Surj. $(I^i, \partial I^i, J^{i-1}) \xrightarrow{f} (X, B)$

hits only finitely many cells. So there is a subcomplex B' of B obtained from C by att. fin. many cells containing image f. Let e^k be one of highest dimension.

and $C' = C \cup$ other cells of B'. Then (B', C') is k-1 connected. Apply Case 1 to avoid e^k and continue inductively.

Injectivity: Use again case 1 to avoid all cells not in A for the homotopy as above.

Case 3: A from C by attaching cells of dim $\geq m+1$

B as in case 2.

We may assume that cells in $A \setminus C$ have dim $\leq m+n+1$ since higher dim cells play no rôle in htpy groups

$\pi_i(A, C)$, $\pi_i(X, B)$, $i \leq m+n$.

Denote $A_k = C \cup$ cells of A of dim $\leq k$

$$X_k = A_k \cup B \quad \text{and prove}$$

$$\pi_i(A_k, C) \longrightarrow \pi_i(X_k, B) \quad \begin{matrix} \text{surj. } i \leq m+n \\ \text{inj. } \quad < m+n \end{matrix}$$

inductively starting with $k = m+1$ by case 2

Now assume $k > m+1$, assume the desired result for $k-1$ and look at the exact htpy seqn. of the triples (A_k, A_{k-1}, C) and (X_k, X_{k-1}, B)

$$\begin{tikzcd}
\pi_{i+1}(A_k, A_{k-1}) \arrow[r, "\partial"] \arrow[d] & \pi_i(A_{k-1}, C) \arrow[r] \arrow[d] & \pi_i(A_k, C) \arrow[r] \arrow[d, dashed] & \pi_i(A_k, A_{k-1}) \arrow[r] \arrow[d] & \pi_{i-1}(A_{k-1}, C) \arrow[d] \\
\pi_{i+1}(X_k, X_{k-1}) \arrow[r, "\partial"] & \pi_i(X_{k-1}, B) \arrow[r] & \pi_i(X_k, B) \arrow[r] & \pi_i(X_k, X_{k-1}) \arrow[r] & \pi_{i-1}(X_{k-1}, B)
\end{tikzcd}$$

(A_k, A_{k-1}) is $(k-1)$-conn. , (X_{k-1}, A_{k-1}) is n-connected

thus if $i+1 < k-1+n$ then vertical map 1 and 4

are iso's, (in particular if $i < m+n \leq k-2+n$)

and by induction vert. maps 2 and 5 are iso's. Thus the middle map is an iso. , and

if $i = m+n$ map 2 is surj., map 5 is an iso.

map 4 is an iso (surj. suffices). Thus, the middle

map is surjective.

The proof works also if $i = 2$, where the 5-th vert. map is not between groups. But the argument extends

If $i = 1$, one proves the excision theorem in case 3 directly

If $m \geq 1$, then both sets are trivial

If $m = 0$, then we have $n \geq 1$, if we have anything to prove. Since C is connected, also A is connected (since (A, C) is 0-connected). Thus, map on π_0 is bijective)

And since $n \geq 1$, B is obtained from C by attaching cells of dim ≥ 2, and similarly X from A. So the result follows using cellular approx.

On page 5.4 there was the following Claim

5.4 Proposition. Let (X, A) be an n-connected CW-pair. Then there exists a CW-complex X' obtained from A by attaching cells of dimension $\geq n+1$ and an extension of $A \hookrightarrow X$ to a homotopy equivalence $f: (X', A) \longrightarrow (X, A)$ rel A; i.e. there is $g: (X, A) \longrightarrow (X', A)$ extending $A \hookrightarrow X'$ such that $g \circ f$ and $f \circ g$ are homotopic rel A to the corresponding identities.

Proof. Use 4.4 to get an n-connected CW model $f: (X', A) \longrightarrow (X, A)$ of (X, A) which was constructed by attaching only cells of dim $\geq n+1$ to A. It is easy to see that then f is a weak htpy equivalence (and thus a htpy equivalence). We may also assume that f is cellular since $A \hookrightarrow X$ is cellular.

Now consider the quotient of the mapping cylinder

$$M_f := X' \times [0,1] \sqcup X \Big/ _{(x',1) \sim f(x')}$$

by collapsing every $a \times [0,1]$, $a \in A$, to a point.

Call this new space N. N deformation retracts to X by factoring the standard def. retr. of M_f to X

through N. Further N is a CW-complex (since f is cellular) and X' is a subcomplex of N.

The map $X' \hookrightarrow N$ is a weak htpy equivalence since $X' \hookrightarrow N \longrightarrow X$ is a weak htpy equiv. and $N \longrightarrow X$ is a weak htpy equiv.

Thus by Whitehead's theorem (Theorem 3.9) $X' \hookrightarrow N$ is a deformation retraction, i.e. a mod X' homotopy equivalence; in particular a mod A htpy equivalence. The same holds for $X \hookrightarrow N$. $\square$

A few easy applications of the homotopy excision theorem.

<u>5.5 Freudenthal suspension Theorem:</u>

The map $\pi_i(X, x_0) \xrightarrow{S} \pi_{i+1}(SX, x_0)$

$$[f : (S^i, *) \to (X, x_0)] \mapsto [Sf : (S^{i+1}, *) \to (SX, x_0)]$$

is a homomorphism. If X is $(n-1)$-connected CW-complex then S is an isomorphism for $i < 2n-1$ and an epim. for $i = 2n-1$. (Here we may use the standard suspension functor or the reduced suspension functor S_r ($S_r X = C_r X / X \times \{1\}$, $C_r X = X \times [0,1] / (X \times \{0\} \cup \{x_0\} \times [0,1])$))

<u>Proof</u> SX is the union of two cones C^+X and C^-X intersecting in X. The suspension map

is the composition of

$$\pi_i(X) \xrightarrow[\cong]{(\partial^+_{i+1})^{-1}} \pi_{i+1}(C^+X, X) \xrightarrow{exc} \pi_{i+1}(SX, C^-X) \xrightarrow[\cong]{\pi_{i+1}(j)^{-1}} \pi_{i+1}(SX)$$

where $\pi_{i+1}(C^+X, X) \xrightarrow{\partial^+_{i+1}} \pi_i(X)$

is the bdy map of the sequence of the pair (C^+X, X)

the middle map is the excision map, and

j is the inclusion $(SX, x_0, x_0) \to (SX, C^-X, x_0)$

Since C^+X and C^-X are contractible the first and last map are inverses of isomorphisms.

Thus: the suspension map is a homomorphism

Looking at the exact sequences of $(C^{\pm}X, X)$ we see that (C^-X, X) and (C^+X, X) are n-connected. $\square$

The theorem follows.

5.6 Corollary. (a) $\pi_n(S^n) \cong \mathbb{Z}$.

(b) $\pi_n\left(\bigvee_{\alpha \in A} S^n\right) \cong \bigoplus_{\alpha \in A} \mathbb{Z}$

Pf (a) follows from the fact that $\pi_i(S^1) \cong \begin{cases} \mathbb{Z} & i=1 \\ 0 & i \neq 1 \end{cases}$

the Hopf fibration $S^1 \hookrightarrow S^3 \xrightarrow{h} S^2$

which shows that $\pi_2(S^2) \xrightarrow{\partial} \pi_1(S^1)$ is an iso

and Freudenthal which shows that

$$\pi_n(S^n) \xrightarrow{S} \pi_{n+1}(S^{n+1}) \quad \text{is surjective if } n=1$$

and an isom. for $n \geq 2$. (Remark: S for $n=1$ is then also an isom.)

(b) Use the inclusion onto last summand and proj onto the summands

$$S^n \hookrightarrow \bigvee^{k+1} S^n \longrightarrow \bigvee^k S^n$$

to show inductively the claim for finite A.

For infinite A use the fact that a compact set meets only finitely many cells. □

Remark: We know that $[\mathrm{id}_{S^1}]$ generates $\pi_1(S^1)$. Since

$S\,\mathrm{id}_{S^n} = \mathrm{id}_{S^{n+1}}$ we see that $[\mathrm{id}_{S^n}]$ generates $\pi_n(S^n)$. Furthermore $\deg(z \overset{r_n}{\longmapsto} z^n) = n$

$r_n : S^1 \to S^1$, so $\deg : \pi_1(S^1) \to \mathbb{Z}$ is an isomorphism. Also $\deg(S^{n-1} r_n) = n$ (via homological considerations) Thus $\deg : \pi_n(S^n) \to \mathbb{Z}$ is an isomorphism

and we have:
$$h_n : \pi_n(S^n) \longrightarrow H_n(S^n)$$
$$[f] \longmapsto H_n(f)(\iota_n)$$

is an isomorphism. $\iota_n \in H_n(S^n)$ the fundamental class (i.e. image of $[\mathrm{id}_{\Delta^{n+1}}] \in H_{n+1}(\Delta^{n+1}, \dot{\Delta}^{n+1})$
$$\cong$$
$$(D^{n+1}, S^n)$$

under $\partial_{n+1} : H_{n+1}(\Delta^{n+1}, \dot{\Delta}^{n+1}) \longrightarrow H_n(\dot{\Delta}^{n+1})$. □

5.7 Construction of Eilenberg-MacLane Spaces

Any abelian group G is isomorphic to a quotient of free abelian group

$$\bigoplus_{\alpha \in A} \mathbb{Z} \twoheadrightarrow G$$

The kernel is again a free abelian gp, say isomorphic to $\bigoplus_{\beta \in B} \mathbb{Z}$. Identify

$$\bigoplus_{\alpha \in A} \mathbb{Z} \quad \text{with} \quad \pi_n\left(\bigvee_{\alpha \in A} S^n\right)$$

For any $\beta \in B$ the generator of the β-th summand is an element $[f_\beta] \in \pi_n(\bigvee S^n)$

Attach for every $\beta \in B$ an $(n+1)$-cell e_β^{n+1} via f_β.

and let X be the resulting CW-complex.

Claim: $\pi_i(X) = 0$ for $0 \leq i < n$ (by cellular approx.)

$$\pi_n(X) \cong G$$

__Pf.__ By cellular approx.

$$\pi_n\left(\bigvee_{\alpha \in A} S^n\right) \twoheadrightarrow \pi_n(X)$$

and obviously all $[f_\beta]$ are in the kernel.

Looking at the exact sequence

$$\pi_{n+1}(X, \bigvee_\alpha S^n) \xrightarrow{\partial} \pi_n(\bigvee_\alpha S^n) \twoheadrightarrow \pi_n(X)$$

we see that the kernel is equal to the image of ∂.

To proceed further, we use the following fact:

<u>5.7.a</u> If (Y, B) is a CW-pair with B contractible then the map $Y \longrightarrow Y/B$ is a homotopy equivalence

(Remark: Here it suffices that $B \hookrightarrow Y$ is a closed cofibration)

Using 5.7.a and excision we obtain

<u>5.7.b</u> <u>Proposition</u> If (Y, B) is m-connected and B is $(n-1)$-connected then the map

$$\pi_i(Y, B, x_0) \longrightarrow \pi_i(Y/B, [x_0])$$

is injective for $i < m+n$ and surj. for $i \leq m+n$.

<u>Proof</u> Attach the cone CB to Y along B to obtain $Y \cup CB$. Look at (Y, B) and (CB, B), which are m resp. n-connected. Thus

$$\pi_i(Y, B) \longrightarrow \pi_i(Y \cup CB, CB)$$ is iso for $i < m+n$, epi for $i \leq m+n$.

□

and then use 5.7.a

Returning to our exact sequence

If $n \geq 2$ then $n+1 < 2n$ and therefore

$$\pi_{n+1}(X, \bigvee_\alpha S^n) \longrightarrow \pi_{n+1}(X/\bigvee_\alpha S^n) = \pi_{n+1}(\bigvee_\beta S^{n+1})$$

is an isomorphism. Thus $\pi_{n+1}(X, \bigvee_\alpha S^n)$ is a free abelian group with basis the homotopy classes of the attaching maps of the $(n+1)$-cells e^{n+1}_β. Thus $\pi_n(X) \cong G$.

Attaching cells of dimensions $\geq n+2$ we can embed X into a (conn.) CW-complex Y s.t. $\pi_i(Y) = \begin{cases} G & i = n \\ 0 & \text{otherwise} \end{cases}$

A CW-complex with these properties is called a $K(G,n)$.

For $n=1$ and any group G using Seifert-van Kampen one can obtain a 2-dim. CW-complex X with $\pi_1(X) \cong G$. Attaching cells of dim ≥ 3 one finds Y as above for $n=1$. Y is then a $K(G,1)$.

Finally, the second statement in the following Theorem is not hard to prove.

<u>5.8 Theorem</u> Given any group G for $n=1$ or an abelian group for $n \geq 2$ there exists a $K(G,n)$ with n-skeleton a wedge of n-spheres. Further, if X is $(n-1)$-connected and $\varphi: \pi_n(X) \longrightarrow \pi_n(K(G,n)) = G$ any homomorphism there is up to homotopy a unique map $f: X \to K(G,n)$ with $\pi_n(f) = \varphi$. $\square$