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Homotopy Theory

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Homework 3 Due: May 7, 2015

Problem 5

Let M_f^u be the unreduced mapping cylinder of the map $f : X \to Y$, i.e.

$$M_f^u := X \times I \sqcup Y/(x,1) \sim f(x), \quad x \in X$$

with $I = [0,1]$ the unit interval, and let $X \sqcup Y \xrightarrow{k} M_f^n$ be given by

$$k(x) = [(x,0)], \quad x \in X, \qquad k(x) = [y], \quad y \in Y$$

where $[z]$ means the equivalence class of $[z] \in X \times I \sqcup Y$ in M_f^u.
Show that k is a cofibration.

Problem 6

Let PC be a polish circle, i.e. a space homeomorphic to

$$X = X_1 \cup X_2 \cup X_3 \cup X_4 \subset \mathbb{R}^2$$

with

$$\begin{aligned} X_1 &= \{(x, \sin\tfrac{1}{x}) \mid 0 < x \leq \tfrac{1}{\pi}\} \\ X_2 &= \{(0,y) \mid |y| \leq 1\} \\ X_3 &= \{(x,0) \mid -2 \leq x \leq 0 \quad \text{or} \quad \tfrac{1}{\pi} \leq x \leq 2\} \\ X_4 &= \{(x,y) \mid y \leq 0 \quad \text{and} \quad x^2 + y^2 = 4\} \end{aligned}$$

Show that

(i) X is path connected

(ii) All homotopy groups of X are trivial

(iii) X is *not* contractible

Remark: In the next lecture we will show as a trivial corollary of a theorem of Whitehead that any connected CW-complex with trivial homotopy groups is contractible.

Hint for iii): Assume that there is a contraction. Follow the paths of points in X_1 during the contraction. Then show what this implies for paths of points in X_2 during the contraction.