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Homotopy Theory

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Homework 2 Due: April 30, 2015

Problem 3

(i) Assume we are given a pointed map

$$\alpha : (S^n, *) \to (S^n \vee S^n, *)$$

with the property that the composition of α with each of the maps which collapse the second and first factor respectively to the base point

$$q_1 : (S^n \vee S^n, *) \to (S^n, *) \quad \text{and} \quad q_2 : (S^n \vee S^n, *) \to (S^n, *)$$

are both homotopic to the identity map on S^n. For $f, g : (S^n, *) \to (X, x_0)$ we can define a new map

$$f \oplus g : S^n \xrightarrow{\alpha} S^n \vee S^n \xrightarrow{(f,g)} X$$

where (f, g) maps the first wedge factor via f and the second via g. Show that this gives rise to a well defined operation

$$\oplus : \pi_n(X, x_0) \times \pi_n(X, x_0) \to \pi_n(X, x_0), \quad ([f], [g]) \mapsto [f \oplus g]$$

(ii) Let $n \geq 2$. The n-th homology group of the n-sphere can be identified with

$$H_n(S^n) \cong \mathbb{Z}$$

and we will denote by ι_n the element in $H_n(S^n)$ corresponding under the above identification to $1 \in \mathbb{Z}$. Let X be a topological space with base point x_0. Define the *Hurewicz map*

$$h : \pi_n(X, x_0) \to H_n(X), \quad [f] \mapsto f_*(\iota_n) = H_n(f)(\iota_n).$$

Show that $h([f] \oplus [g]) = h([f]) + h([g]) \in H_n(X)$.

Hint: It might be helpful to consider the (universal) example $X = S^n \vee S^n$ first, with $f, g : S^n \to S^n \vee S^n$ being the inclusion to the first, respectively second summand.

Remark: It was indicated in the lecture that $\oplus$ is an equivalent definition of addition on $\pi_n(X, x_0)$. Thus this exercise shows that the Hurewicz map is in fact a group homomorphism.

Problem 4

Let $n \geq 2$ and denote by $\widetilde{S^1 \vee S^n}$ the universal covering space of $S^1 \vee S^n$. Show that the map

$$\pi_n(\widetilde{S^1 \vee S^n}) \to H_n(\widetilde{S^1 \vee S^n})$$

is surjective and conclude that $\pi_n(S^1 \vee S^n)$ is not finitely generated as an abelian group.