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Homotopy Theory

Summer 2015

Homework 10 Due: June 25, 2015

We call a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ of maps between sets, where C is a set with distinguished element c_0, *exact* if $\operatorname{im}(\alpha) = \beta^{-1}(c_0)$.

In the set $\big[(X, x_0), (Y, y_0)\big]$ of homotopy classes of maps $(X, x_0) \to (Y, y_0)$, the homotopy class of the (unique) constant map will be the distinguished element.

Given $f : (X, x_0) \to (Y, y_0)$, let M_f be the mapping cylinder as defined in Section 3 of the lecture and call the quotient $M_f/(X \times \{0\} \cup \{x_0\} \times I)$ the *reduced mapping cone* C_f^r, where $M_f^r = M_f/(\{x_0\} \times I\})$ is called the *reduced mapping cylinder*.

If $f = \mathrm{id}_X$ then C_f^r is called the *reduced cone of* X and is denoted by $C^r X$ or $C(X, x_0)$. Similarly, $S(X, x_0) = SX/(\{x_0\} \times I) = C(X, x_0)/(X \times \{1\})$ is called the *reduced suspension*. $S(X, x_0)$ is a quotient of $X \times I$ and we denote the equivalence class of (x, t) in $S(X, x_0)$ by $[x, t]$.

A map $f : (X, x_0) \to (Y, y_0)$ induces a map Sf, the *suspension of* f via

$$Sf : S(X, x_0) \to S(Y, y_0), \quad [x, t] \mapsto [f(x), t].$$

Problem 19

Given $f : (X, x_0) \to (X', x_0')$ let $i : (X', x_0') \to C_f^r$ be the composition

$$i : (X', x_0') \hookrightarrow M_f \to M_f/(X \times \{0\} \cup \{x_0\} \times I) = C_f^r.$$

Prove: For any (Y, y_0) the sequence

$$\big[(C_f^r, *), (Y, y_0)\big] \xrightarrow{i^*} \big[(X', x_0'), (Y, y_0)\big] \xrightarrow{f^*} \big[(X, x_0), (Y, y_0)\big]$$

(Please turn over.)

is exact, where for a map $a : (Z, z_0) \to (Z', z_0')$ the induced map

$$a^* : \big[(Z', z_0'), (Y, y_0)\big] \to \big[(Z, z_0), (Y, y_0)\big]$$

is defined by $a^*[\alpha] = [\alpha \circ a]$ and the basepoint $* \in C_f$ is given by $i(x_0')$.

Problem 20

A sequence of maps $(X, x_0) \xrightarrow{f} (X', x_0') \xrightarrow{g} (X'', x_0'')$ is called *coexact* if for any (Y, y_0) the sequence

$$\big[(X'', x_0''), (Y, y_0)\big] \xrightarrow{g^*} \big[(X', x_0'), (Y, y_0)\big] \xrightarrow{f^*} \big[(X, x_0), (Y, y_0)\big]$$

is exact.

Prove: If $(X, x_0) \xrightarrow{f} (X', x_0') \xrightarrow{g} (X'', x_0'')$ is coexact, then so is

$$S(X, x_0) \xrightarrow{Sf} S(X', x_0') \xrightarrow{Sg} S(X'', x_0'').$$