Surfaces and Automorphisms

Problem Set 5 H. Reich/F. Lenhardt
SS 2013

Exercise 1
Let M be an oriented surface. We define the underlying set of the Teichmüller space $\mathcal{T}(M)$ of M as follows: A point in the Teichmüller space is represented by a complex surface S together with an orientation-preserving diffeomorphism $\phi\colon M \to S$. Two such maps $\phi\colon M \to S$, $\phi'\colon M \to S'$ are identified in the Teichmüller space if and only if there is a biholomorphic map $\tau : S \to S'$ such that $\tau \circ \phi$ and ϕ' are homotopic.
Let $\mathrm{Diff}_0(M) \subset \mathrm{Diff}(M)$ be the group of self-diffeomorphisms of M which are homotopic to the identity. This group acts on the set of complex structures inducing the given orientation $\Gamma^\infty(X^{cx}(TM))^+$. Show that Teichmüller space is in bijection with $\Gamma^\infty(X^{cx}(TM))^+ / \mathrm{Diff}_0(M)$.

Exercise 2
Let M be an oriented surface and J_0 a complex structure on TM inducing the given orientation. Show that $\Gamma^\infty(X^{cx}(TM))^+$ is in bijection with the space of Beltrami forms $M(TM, J_0)$.

Exercise 3
Remember the formulas in Exercise 3, sheet 4.
For $z_0 \in \mathbb{C}$, we consider the complex-valued 1-form

$$\chi = \frac{dz}{z - z_0}$$

on $\mathbb{C} - \{0\}$.

(i) Show that $d\chi = 0$.

For $r > 0$, consider the loop $\gamma(t) = z_0 + re^{it}$ around z_0.

(ii) Show that

$$\int_\gamma \chi = 2\pi i$$

(iii) Conclude that χ is not of the form df for a complex-valued function f on $\mathbb{C} - \{0\}$.
Hint: Stokes' theorem.

Now pick any smooth function $f\colon D \to \mathbb{C}$ defined on a small closed disc D around z_0. Consider the one-form $\omega = f\chi$.

(iv) Show that

$$d(\omega) = d(f\chi) = \frac{\frac{\partial f}{\partial \bar{z}}}{z - z_0} d\bar{z} \wedge dz = 2i \frac{\frac{\partial f}{\partial \bar{z}}}{z - z_0} dx \wedge dy$$

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Using Stokes' theorem again, we conclude

$$\int_D 2i \frac{\frac{\partial f}{\partial \bar{z}}}{z - z_0} dx \wedge dy = \int_{\partial D} f\chi = \int_{\partial D} \frac{f(z)}{z - z_0} dz$$

In particular, if f is holomorphic, we get

$$\int_{\partial D} \frac{f(z)}{z - z_0} dz = 0$$

since $\frac{\partial f}{\partial \bar{z}} = 0$.

(v) Check the formula for $f(z) = z + 1, z_0 = 0$ and D the unit disc. What is wrong with our argument?

To fix the problem, pick $r > 0$ small enough such that $D(z_0, r) \subset D$. Set $D_r = D - D(z_0, r)$.

(vi) Apply Stokes' theorem to ω over D_r to conclude

$$\int_{D_r} \frac{\frac{\partial f}{\partial \bar{z}}}{z - z_0} d\bar{z} \wedge dz = \int_{\partial D} \frac{f(z)}{z - z_0} dz - \int_{\partial D(z_0, r)} \omega$$

(vii) Show that for $r \to 0$, the term $\int_{\partial D(z_0,r)} \omega$ converges to $2\pi i f(z_0)$.

Altogether, we get the Cauchy-Pompeiu formula:

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i} \int_D \frac{\frac{\partial f}{\partial \bar{z}}}{z - z_0}$$

which, if f is holomorphic, specializes to the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz$$