Surfaces and Automorphisms

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Exercise 1

Let V, W be complex vector spaces, with complex structures J_V, J_W. The complex structure J_W determines a conformal class $[s(-,-)]$ and hence a quadratic form $q(v) = s(v,v)$ well-defined up to a positive scalar. The level sets $q^{-1}(c), c \in \mathbb{R}$ of q of this quadratic form yield a foliation $\mathcal{F}_{J_W}$ of $W - \{0\}$ into ellipses.

(i) Verify all claims above and show that the foliation $\mathcal{F}_{J_W}$ only depends on the conformal class of $s(-,-)$ and hence only on J_W.

(ii) Let $f\colon V \to W$ be a real linear map. We can pull back the foliation $\mathcal{F}_{J_W}$ to a foliation $f^*(\mathcal{F}_{J_W})$. Show that this pulled-back foliation only depends on $\mu(f)$.

Exercise 2

A complex structure J on a real vector space V determines the complex scalar multiplication map

$$\begin{aligned} m_J\colon \mathbb{C} \otimes_{\mathbb{R}} V &\to V \\ (a+ib) \otimes v &\mapsto av + bJ(v) \end{aligned}$$

Discuss in which sense this map is complex linear.
There is also the involution

$$\begin{aligned} \overline{}\colon \mathbb{C} \otimes_{\mathbb{R}} V &\to \mathbb{C} \otimes_{\mathbb{R}} V \\ \lambda \otimes v &\mapsto \overline{\lambda} \otimes v \end{aligned}$$

(i) Show that $K = \ker(m_J)$ is an n-dimensional complex subspace of $\mathbb{C} \otimes_{\mathbb{R}} V$ and that $\mathbb{C} \otimes_{\mathbb{R}} V = K \oplus \overline{K}$.

(ii) Show that an n-dimensional complex subspace $K \subset \mathbb{C} \otimes_{\mathbb{R}} V$ with

$$K \cap \overline{K} = 0$$

determines a (real linear) isomorphism from a real vector space to a complex vector space

$$V \cong \mathbb{R} \otimes_{\mathbb{R}} V \to \mathbb{C} \otimes_{\mathbb{R}} V \to (\mathbb{C} \otimes_{\mathbb{R}} V)/K$$

and therefore a complex structure J on V.

Exercise 3
Interpret and prove:

(i) $\frac{\partial}{\partial x}(u+iv) = \frac{\partial}{\partial x}u + i\frac{\partial}{\partial x}v$ and $d(u+iv) = du + idv$

(ii) $\frac{\partial}{\partial \overline{z}} f = \frac{1}{2}(\frac{\partial}{\partial x} f + i\frac{\partial}{\partial y} f)$

(iii) $df = \frac{\partial}{\partial z} f dz + \frac{\partial}{\partial \overline{z}} f d\overline{z} = \frac{\partial}{\partial x} f dx + \frac{\partial}{\partial y} f dy$

(iv) $\frac{\partial}{\partial \overline{z}}(f \circ g) = \frac{\partial}{\partial \overline{z}} g \circ \frac{\partial}{\partial z} f + \frac{\partial}{\partial z} g \circ \frac{\partial}{\partial \overline{z}} f = \frac{\partial}{\partial \overline{z}} g \cdot \overline{\frac{\partial}{\partial z} f} + \frac{\partial}{\partial z} g \cdot \frac{\partial}{\partial \overline{z}} f$

Exercise 4
Let V, W, X be one-dimensional complex vector spaces, with complex structures given by J_V, J_W, J_X. For orientation-preserving isomorphism $f\colon V \to W, g\colon W \to X$. Show that $\mu(g \circ f) = \mu(f)$ if and only if $g^*(J_X) = J_W$.