Surfaces and Automorphisms

Problem Set 1 — H. Reich/F. Lenhardt
WS 2013/14 — Due: 30.10.2013

Exercise 1
Let $\mathcal{G}$ be a pseudogroup on $\mathbb{R}^n$. For two topological n-manifolds M, N and a homeomorphism $f\colon N \to M$, we obtain a pullback map

$$f^*\colon \{\mathcal{G}\text{-structures on M}\} \to \{\mathcal{G}\text{-structures on N}\}, \mathcal{A} \mapsto f^*\mathcal{A}$$

(i) Show that f^* is a bijection.

(ii) The homeomorphism f also induces a group homomorphism

$$\kappa(f)\colon \operatorname{Top}(M) \to \operatorname{Top}(N)$$
$$g \mapsto f^{-1} \circ g \circ f$$

Furthermore, $\operatorname{Top}(M)$ acts on the $\mathcal{G}$-structures on M and $\operatorname{Top}(N)$ on the $\mathcal{G}$-structures on N. Show that f^* is equivariant with respect to these actions and the map κ.

(iii) Show that f^* descends to a well-defined map

$$\overline{f^*}\colon \{\mathcal{G}\text{-structures on } M\}/\operatorname{Top}(M) \to \{\mathcal{G}\text{-structures on } N\}/\operatorname{Top}(N)$$

(iv) Show that $\overline{f^*}$ does not depend on the choice of f.

Exercise 2
Let X be the space of complex structures on $\mathbb{R}^{2n}$; in other words,

$$X = \{J \in Gl_{2n}(\mathbb{R}) \mid J^2 = -Id\} \subset Gl_{2n}(\mathbb{R})$$

(i) Show that $Gl_{2n}(\mathbb{R})$ acts on X via $A \cdot (\mathbb{R}^{2n}, J) = (\mathbb{R}^{2n}, AJA^{-1})$.

(ii) Show this action is transitive.

(iii) Determine the stabilizer of the complex structure J_{stand} coming from the canonical identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

(iv) Show that X is diffeomorphic to $Gl_{2n}(\mathbb{R})/Gl_n(\mathbb{C})$.

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Exercise 3

Let $\omega = e_1^* \wedge e_2^* \wedge \cdots \wedge e_n^*$ be the canonical volume form on $\mathbb{R}^n$. We say that a diffeomorphism $g\colon U \to V$ between open subsets of $\mathbb{R}^n$ is oriented volume-preserving if $g^*(\omega|_V) = \omega|_U$.

(i) Show that the collection of all such g forms a pseudogroup $\mathcal{G}$.

(ii) Show that a manifold M with a $\mathcal{G}$-structure can be equipped with a canonical n-form τ which under a $\mathcal{G}$-chart corresponds to a restriction of ω.

(iii) How would you define the volume of an open subset of M?

(iv) Would all of this work with any nowhere vanishing smooth n-form on $\mathbb{R}^n$?

Exercise 4

(i) Define a structure on n-dimensional $\mathbb{R}$-vector spaces such that the linear maps preserving the standard such structure is $G = \mathbb{R}_{>0} \times O(n)$.

(ii) Show that for the G just defined we have

$$\mathcal{C}_{\mathbb{R}^2}^{\infty,G,*} = \mathcal{C}_{\mathbb{R}^2}^{HOL}$$

Hint: Look at inner products up to scale.