# 7. The homotopy Lie algebra of $F_k(\mathbb{R}^{n+1})$, $n \geq 2$
part IV

Recall: Fibration sequence

$$\begin{array}{ccccccc}
F_k(\mathbb{R}^{n+1}) & \hookleftarrow F_{k-1}(\mathbb{R}^{n+1}_1) \hookleftarrow & \cdots & \hookleftarrow F_2(\mathbb{R}^{n+1}_{k-2}) & \hookleftarrow & F_1(\mathbb{R}^{n+1}_{k-1}) \\
\downarrow p_0 & \downarrow p_1 & & \downarrow p_{k-2} & & \downarrow p_{k-1} = id \\
\mathbb{R}^{n+1} & \mathbb{R}^{n+1}_1 & & \mathbb{R}^{n+1}_{k-2} & & \mathbb{R}^{n+1}_{k-1}
\end{array}$$

$$F_{k-r}(\mathbb{R}^{n+1}_r) = \{(x_1,\dots,x_k) \in F_k(\mathbb{R}^{n+1}) : x_i = q_i,\ i=1,\dots,r\}$$

$$p_i(x_1,\dots,x_k) = x_{k+1}, \quad i=0,\dots,k-1.$$

We had sections $s_i$ of $p_i$ given by

$$s_i(y) = (q_1,\dots,q_i, y, q_{i+2},\dots,q_k), \quad y=(y_1,\dots,y_{n+1}) \in \mathbb{R}^{n+1}_i = \mathbb{R}^{n+1}_i \setminus \{q_1,\dots,q_i\}$$
$$\text{if } y_1 \leq 4(i-1)+2$$

and we can extend $s_i$ to all of $\mathbb{R}^{n+1}_i$ by

$$s_i(y) = (q_1,\dots q_i, y, q_{i+2} + (y_1 - 4(i-1)+2)e_1, \dots,$$
$$q_k + (y_1 - 4(i-1)+2)e_1),$$

$$\text{if } y_1 \geq 4(i-1)+2$$

recall that $q_j = 4(j-1)e_1$ so that

$$q_j + (y_1 - 4(i-1)+2)e_1 = (y_1 + (j-i)+2)e_1.$$

Thus $s_i : \mathbb{R}^{n+1}_i \longrightarrow F_{k-i}(\mathbb{R}^{n+1}_i)$ is a globally defined section of $p_i$.

We used these sections to define our basis $\alpha_{rs}$, $1 \leq s < r$, of $\pi_n(F_k(\mathbb{R}^{n+1}))$

with

$\alpha_{rs}$ the homotopy class of

$$\alpha'_{rs} : S^n \longrightarrow F_{k-r+1}(\mathbb{R}^{n+1}_{r-1}) \hookrightarrow F_k(\mathbb{R}^{n+1}_{r-1})$$

$$\xi \longmapsto (q_1, \dots, q_{r-1}, q_s + \xi, q_{r+1}, \dots, q_k)$$

i.e. $\alpha'_{rs} = s_{r-1} \circ \tilde{\alpha}'_{rs}$ where

$$\tilde{\alpha}'_{rs}(\xi) = q_s + \xi \in \mathbb{R}^{n+1}_{r-1} = \mathbb{R}^{n+1} \setminus \{q_1, \dots, q_{r-1}\}.$$

The homotopy classes $\tilde{\alpha}_{rs}$ of the $\tilde{\alpha}'_{rs}$ form a basis for $\pi_n(\mathbb{R}^{n+1}_{r-1})$, where now $r$ is fixed and $1 \leq s < r$.

Corresponding to the sequence $F_k(\mathbb{R}^{n+1}) \hookleftarrow F_{k-1}(\mathbb{R}^{n+1}_1) \hookleftarrow$ we have the sequence of homotopy Lie algebras

$$L_k = L_{k-1} \hookleftarrow L_{k-2} \hookleftarrow \cdots \hookleftarrow L_1 \qquad \text{with}$$

$$L_i := \pi^*(F_i(\mathbb{R}^{n+1}_{k-i}))$$

and for each $i$ we have a splitting of $L_i$ as graded abelian groups but not necessarily of Lie algebras

$$L_i = \pi_*(s_{k-i})(\pi_*(\mathbb{R}^{n+1}_{k-i})) \oplus L_{i-1}$$

Since the Whitehead product is functorial we see that the image under $\pi_*(s_{k-i})$ of $\pi_*(\mathbb{R}^{n+1}_{k-i})$ is a sub-Liealgebra of $L_i$ which we denote by $L_{i,i-1}$.

Denoting $\pi_*(\mathbb{R}^{n+1}_{k-i})$ by $\tilde{L}_{i,i-1}$ our fibration $p_{k-i}$ gives a short exact sequence of Lie algebras

$$0 \longrightarrow L_{i-1} \longrightarrow L_i \underset{\pi_*(s_{k-i})}{\overset{\pi_*(p_{k-i})}{\rightleftarrows}} \tilde{L}_{i,i-1} \longrightarrow 0$$

which splits as a graded abelian group.
Consequently $L_{i-1}$ is an ideal in $L_i$.
i.e.

7.1. $X \in L_{i-1}$, $Y \in L_i$ then $[X,Y] \in L_{i-1}$
(since $L_{i-1}$ is the kernel of the Lie algebra homomorphism $\pi_*(p_{k-i})$).

We will see shortly that $L_{i,i-1}$ is not an ideal of $L_i$.

For this we need to know a little about the homotopy Lie algebra $\pi_*(\mathbb{R}^{n+1}_i) \cong \pi_*(\underbrace{S^n \vee \cdots \vee S^n}_{i})$

Our modest aim is to show the following:
let $\iota'_j : S^n \longrightarrow X := \underbrace{S^n \vee \cdots \vee S^n}_{m}$
be the inclusion into the $j$-th summand

$$(\iota'_j(\xi) = (e_1, \ldots, \underset{\substack{\uparrow \\ j\text{-th coordinate}}}{\xi}, e_1, \ldots, e_1) \in X \subset S^n \times \cdots \times S^n$$

7.2 Proposition. Let $\iota_j \in \pi_n(X,*)$ be represented by $\iota_j'$. If $1 \leq i,j \leq m$, $i \neq j$, then

$$[\iota_i, \iota_j] \in \pi_{2n-1}(X,*) \text{ is non-zero.}$$

Actually, with the same effort, we can show a little more

7.2' Let $n_1, \dots, n_m \geq 2$, $X = S^{n_1} \vee \dots \vee S^{n_m}$ and

$\iota_j' : S^{n_j} \longrightarrow X$ the inclusion into the j-th summand and $\iota_j \in \pi_{n_j}(X,*)$ its homotopy class.

Then $[\iota_i, \iota_j] \in \pi_{n_i+n_j-1}(X,*)$ is non-zero if $1 \leq i \neq j \leq m$.

The proof is similar to many proofs in elementary algebraic topology. We show that the hypothesis that $[\iota_i, \iota_j] = 0$ implies the existence of a homomorphism between abelian groups which maps 0 to a non-zero element.

We will use the product structure in

$$H^*(S^{n_1} \times \dots \times S^{n_m}) \text{ and } H^*(S^{n_i} \vee S^{n_j}).$$

The first we know from the Künneth theorem for cohomology. For the second we use:

7.3 Lemma: Let $Y = U_1 \cup U_2$, with $U_i \subset Y$ open and $U_i \hookrightarrow Y$ nullhomotopic (homotopic to a constant map). Let $y_i \in H^i(Y)$, with $i > 0$.

Then $y_1 \cup y_2 = 0$. (In short: any interesting cup product in $H^*(Y)$ vanishes: notice that

$$H^0(Y) = \text{group of 0-dimensional cocycles}$$
$$\cong \mathbb{Z} \quad \text{canonically, where } n \in \mathbb{Z} \text{ is}$$
the cocycle mapping all points to $n$.)

<u>Proof.</u> Consider for $i=1,2$ the following part of the cohomology exact sequence

$$H^{n_i}(Y, U_i) \xrightarrow{\ r\ } H^{n_i}(Y) \longrightarrow H^{n_i}(U_i)$$

where $r^{n_i}$ is induced by $(Y, \emptyset) \hookrightarrow (Y, U_i)$. Since $n_i > 0$ and $U_i \hookrightarrow Y$ is nullhomotopic the second map is $0$. Therefore $r^{n_i}$ is surjective and there is $\tilde{y}_i \in H^{n_i}(Y, U_i)$ with $r^{n_i}(\tilde{y}_i) = y_i$.

Thus (by naturality of the cup-product and the fact that $\tilde{y}_1 \cup \tilde{y}_2$ is defined as an element of $H^{n_1+n_2}(Y, U_1 \cup U_2)$ since $U_1$ and $U_2$ are open) we have

$$r^{n_1+n_2}(\tilde{y}_1 \cup \tilde{y}_2) = y_1 \cup y_2.$$

But $\tilde{y}_1 \cup \tilde{y}_2 \in H^*(Y, U_1 \cup U_2) = H^*(Y, Y) = 0$. $\square$

<u>7.4 Remark</u>: $X = S^{n_1} \vee \dots \vee S^{n_m}$ is the union of two open sets each of which is contractible in $X$ (the sets themselves need not be contractible).

<u>Proof</u>. You should have no problem to prove this

Proof of 7.2':

Assume that $[\iota_i, \iota_j] = 0$. Then in the following diagram the map $g$ can be found such that the diagram commutes

$$\begin{tikzcd}
S^{n_i} \times S^{n_j} \arrow[drr, dashed, "g"] & & \\
S^{n_i} \vee S^{n_j} \arrow[u, hook] \arrow[rr, hook] & & S^{n_1} \vee \cdots \vee S^{n_m}
\end{tikzcd}$$

The two inclusions are the obvious maps

We also have a projection $S^{n_1} \vee \cdots \vee S^{n_m} \xrightarrow{p} S^{n_i} \vee S^{n_j}$ where all summands $S^{n_k}$, $k \neq i, j$, are mapped to the basepoint, and the summands $S^{n_i}$ and $S^{n_j}$ are mapped via the identity; so we have a commutative diagram

$$\begin{tikzcd}
S^{n_i} \times S^{n_j} \arrow[dr, "p \circ g"] & \\
S^{n_i} \vee S^{n_j} \arrow[u, hook] \arrow[r, "id"'] & S^{n_i} \vee S^{n_j}
\end{tikzcd}$$

$$\tilde{H}^*(S^{n_i} \vee S^{n_j}) \cong \tilde{H}^*(S^{n_i}) \oplus \tilde{H}^*(S^{n_j})$$

additively. Let $x_i \in H^{n_i}(S^{n_i})$, $x_j \in H^{n_j}(S^{n_j})$ be generators. The Künneth theorem for cohomology gives us the isomorphism

$$\left(H^*(S^{n_i}) \otimes H^*(S^{n_j})\right)^k \longrightarrow H^k(S^{n_i} \times S^{n_j})$$

$$a \otimes b \longmapsto a \times b$$

since there is no torsion in $H^*(S^l)$, any $l$.

Thus, in positive degrees, we have $H^k(S^{n_i} \times S^{n_j})$ generated by $x_i \times 1$, $1 \times x_j$ and $x_i \times x_j$

and $x_i \times 1 \longrightarrow x_i$, $1 \times x_j \longrightarrow x_j$

under the map induced in cohomology by the inclusion $S^{n_i} \vee S^{n_j} \longrightarrow S^{n_i} \times S^{n_j}$.

Thus $(p \circ g)^*(x_i) = x_i \times 1$, $(p \circ g)^*(x_j) = 1 \times x_j$,

and since $x_i \cup x_j = 0$ in $H^*(S^{n_i} \vee S^{n_j})$ we get

$$0 = (p \circ g)^*(x_i \cup x_j) = (x_i \times 1) \cup (1 \times x_j) = x_i \times x_j \neq 0$$

$\square$