Surfaces and Automorphisms

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WS 2013/14

Exercise 1
Let M be a smooth manifold. Show that $X^{or}(TM)$ is an orientable smooth manifold.

Exercise 2
Let M be a smooth manifold. Consider pairs (N, ω) where N is a smooth manifold diffeomorphic to M and ω a volume form on N. Let $\mathcal{S}$ be the set of such pairs modulo volume form preserving diffeomorphisms, i.e. we identify (N, ω) and (N', ω') if there is a diffeomorphism $\phi \colon N \to N'$ with $\phi^*(\omega') = \omega$. Show that there is a bijection between $\mathcal{S}$ and $\Gamma^\infty(X^{vol}(TM))/\operatorname{Diff(M)}$.

Exercise 3
Let $s \in \Gamma^\infty(X^{\text{inn}}(TM))$ be a Riemannian metric on the smooth manifold M. Let $\phi \colon U \to V$ be a chart. Show that there is a bundle isomorphism

$$\begin{array}{ccc} TU & \xrightarrow{f} & TV \\ \downarrow & & \downarrow \\ U & \xrightarrow{\phi} & V \end{array}$$

such that $f^*(s) = s_0$. Why does this *not* imply that every Riemannian metric comes from a $\mathcal{C}^{\infty, \langle -,- \rangle_{st}}$-structure?

Exercise 4
Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. Then $GL(n+1, \mathbb{F})$ operates naturally on the projective space $\mathbb{F}P^n$. Show that with respect to the canonical identifications $\mathbb{R}P^1 \cong \mathbb{R} \cup \{\infty\}$ and $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$, this action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{F})$ and $z \in \mathbb{F} \cup \infty$. Show that we obtain group homomorphisms

$$GL(2, \mathbb{C}) \to \operatorname{Bihol}(\mathbb{C}\mathrm{P}^1)$$
$$GL(2, \mathbb{R}) \to \operatorname{Diff}(\mathbb{R}P^1)$$

and compute their kernels.