Configuration spaces

Problem Set 5 E. Vogt/F. Lenhardt
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Exercise 1
Let $X = \mathbb{R}^n - \{-2, 2\}, n \geq 3$, and let $\alpha_i' \colon S^{n-1} \to X$ be given by $\xi \mapsto 2i + \xi$ for $i = -1, 1$. Let α_i be the homotopy class of α_i'. Let $\beta' \colon S^{n-1} \to X$ be given by $\xi \mapsto 3\xi$ and let β be the homotopy class of β'. Show that $\beta = \alpha_1 + \alpha_{-1}$ in $\pi_{n-1}(X)$, where addition is defined as in lecture 5.

Exercise 2
Show that there is no injective map $\beta' \colon S^{n-1} \to X$ such that its homotopy class β is $\alpha_1 - \alpha_{-1}$.
You may use the generalized Jordan curve theorem: For any embedding $S^k \to S^n$, we have $\tilde{H}_i(S_n - f(S^k)) \cong \tilde{H}_i(S^{n-k-1})$.

Exercise 3
As a preparation to understanding $\pi_k(S^n \vee S^n)$, we investigate the following problem.

Definition: A free Lie algebra over $\mathbb{Z}$ generated by $x_1, \ldots, x_n$ is a Lie algebra L_n over $\mathbb{Z}$ containing $x_1, \ldots, x_n$ with the following property. If L' is any Lie algebra over $\mathbb{Z}$ and $y_1, \ldots, y_n \in L'$, there is a unique Lie algebra homomorphism $\phi \colon L \to L'$ such that $\phi(x_i) = y_i$ for all i.

(i) Show that L_n is uniquely determined up to isomorphism of Lie algebras.

(ii) Show that L_2 is additively a free abelian group.

(iii) Determine the rank of the subgroup $L_2^{(4)}$ of L_2 generated by four-fold products of the two generators x and y.