The homotopy Lie algebra of $F_k(\mathbb{R}^{n+1})$, $n \geq 2$

Part 2

To reduce the amount of writing we denote $\mathbb{R}^{n+1} \setminus \{q_1, \dots, q_r\}$ by $\mathbb{R}^{n+1}_r$, where $(q_1, \dots, q_k)$ is the basepoint of $F_k(\mathbb{R}^{n+1})$ and $0 \leq r \leq k$.

We also choose a particular base point:

$$q_r = 4(r-1) \cdot e_1 ,$$ where $\{e_1, \dots, e_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$.

Consider again the sequence of fibrations

$$\begin{array}{ccccccc}
F_k(\mathbb{R}^{n+1}) & \hookleftarrow & F_{k-1}(\mathbb{R}^{n+1}_1) \hookleftarrow \cdots & \hookleftarrow & F_{k-r}(\mathbb{R}^{n+1}_r) & \hookleftarrow \cdots \hookleftarrow & F_1(\mathbb{R}^{n+1}_{k-1}) \\
\downarrow p_0 & & \downarrow p_1 & & \downarrow p_r & & \downarrow p_{k-1} \\
\mathbb{R}^{n+1} & & \mathbb{R}^{n+1}_1 & & \mathbb{R}^{n+1}_r & & \mathbb{R}^{n+1}_{k-1}
\end{array}$$

Here we consider $F_{k-r}(\mathbb{R}^{n+1}_r)$ as a subspace of $F_k(\mathbb{R}^n)$:

$$F_{k-r}(\mathbb{R}^{n+1}_r) = \left\{ (x_1, \dots, x_k) \in F_k(\mathbb{R}^{n+1}) \mid x_i = q_i , i = 1, \dots, r \right\}$$

and $p_i(q_1, \dots, q_i, x_{i+1}, \dots, x_k) = x_{i+1}$.

We have seen in Lecture 4 (Corollary 4.7) that for each $0 \leq r < k$ and every $p \geq 0$

$$\pi_p(F_{k-r}(\mathbb{R}^{n+1}_r)) \cong \pi_p(F_{k-r-1}(\mathbb{R}^{n+1}_{r+1})) \oplus \pi_p(\mathbb{R}^{n+1}_r)$$

where the inclusion $\pi_p(\mathbb{R}^{n+1}_r) \longrightarrow \pi_p(F_{k-r}(\mathbb{R}^{n+1}_r))$

was induced by a section $s_r : \mathbb{R}^{n+1}_r \to F_{k-r}(\mathbb{R}^{n+1}_r)$

of p_r (i.e. $p_r \circ s_r = id_{\mathbb{R}^{n+1}_r}$).

Thus

$$\pi_p(F_k(\mathbb{R}^{n+1})) \cong \pi_p(F_{k-1}(\mathbb{R}^{n+1}_1)) \oplus \pi_p(\mathbb{R}^{n+1})$$

$$= \pi_p(F_{k-1}(\mathbb{R}^{n+1}_1))$$

$$= \pi_p(\mathbb{R}^{n+1}_1) \oplus \pi_p(F_{k-2}(\mathbb{R}^{n+1}_2))$$

$$\vdots$$

$$= \bigoplus_{i=1}^{k-1} \pi_p(\mathbb{R}^{n+1}_i) \qquad \text{since } F_1(\mathbb{R}^{n+1}_{k-1}) = \mathbb{R}^{n+1}_{k-1}$$

So additively the homotopy groups of $F_k(\mathbb{R}^{n+1})$

look like the homotopy groups of

$$\prod_{i=1}^{k-1} \mathbb{R}^{n+1}_i$$

We will use Whitehead products of elements in $\pi_n(F_k(\mathbb{R}^{n+1}))$ to see that none of the fibrations $F_{k-r}(\mathbb{R}^{n+1}_r) \longrightarrow \mathbb{R}^{n+1}_r$ is trivial for $1 \leq r < k-1$.

For this we need to describe nice elements in $\pi_n(F_k(\mathbb{R}^{n+1}))$ which constitute a basis of the (as we will see) free abelian group $\pi_n(F_k(\mathbb{R}^{n+1}))$.

To do this we first take a little detour to rid ourselves of having to pay attention to base points.

Let X be path-connected, $x_0, x_1 \in X$ and $\omega : [0,1] \longrightarrow X$ a path from x_0 to x_1.

Consider $h_\omega : \pi_p(X, x_0) \longrightarrow \pi_p(X, x_1)$ defined as follows. Let

$f : (B^p, S^{p-1}) \longrightarrow (X, x_0)$ represent $[f] \in \pi_p(X, x_0)$

Consider

$$h'_\omega(f) : B^p \to X, \quad h'_\omega(f)(r\cdot\xi) = \begin{cases} f(2r\cdot\xi), & 0 \leq r \leq \frac{1}{2} \\ \omega(2r-1), & \frac{1}{2} \leq r \leq 1 \end{cases}$$

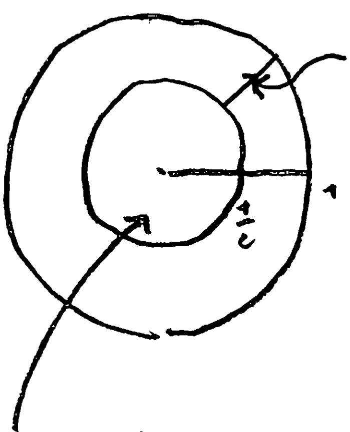
$\xi \in S^{p-1}$, $r \in [0,1]$.

Since $f(\xi) = x_0$ for all $\xi \in S^{p-1}$ $h'_w(f)$ is continuous

Since $w(1) = x_1$ we have $h'_w(f)(\xi) = x_1$ for all $\xi \in S^{p-1}$

Therefore $h_w[f] := [h'_w(f)] \in \pi_p(X, x_1)$

Here is the picture for $p = 2$



have f (after blowing the small disk up by a factor of 2)

You should have no difficulty to establish:

5.1 $h'_w(f) \simeq h'_w(f')$ if $[f] = [f']$ and

5.2 $h_w = h_{w'}$ if w and w' are homotopic as paths.

5.3 If w is a path from x_0 to x_1, and w' a path from x_1 to x_2 then

$$h_{w * w'} = h_{w'} \circ h_w$$

So in particular

5.4 Each h_ω is an isomorphism with

$$h_\omega^{-1} = h_{\omega^-} \quad , \quad \omega^-(t) = \omega(1-t).$$

5.5 If X is path connected and $\pi_1(X, x_0) = 1$ (Then X is called simply connected) then all groups $\pi_p(X,x)$ and $\pi_p(X,y)$ are canonically isomorphic, and we can identify each $\pi_p(X,x)$ with the set of homotopy classes of maps $f : S^p \longrightarrow X$ with addition as follows. Given $[f], [g]$. Take any path ω from $f(1)$ to $g(1)$. Then $[g], h_\omega[f] \in \pi_p(X, g(1))$. Then define $[f]+[g] = [g]+[h_\omega f]$.

We know (exercises) that $\mathbb{R}^{n+1}_r \simeq \underbrace{S^n \vee \cdots \vee S^n}_{r \text{ summands}}$

It is rather easy to show that

$$\pi_p\left(\bigvee_{i=1}^{r} S_i^n\right) = 0 \qquad 0 \le p \le n-1$$

where $S_i^n = S^n$.

and a little harder to show that

$$\pi_n\left(\bigvee_{i=1}^{r} S_i^n\right) \cong \mathbb{Z}^r$$

One way to do this (using may be an oversized tool) is to use the Hurewicz-Theorem in its simple form:

<u>5.6 Hurewicz:</u> Let X be simply connected and $H_i(X) = 0$ for $i < n$. Then $\pi_i(X) = 0$ for $i < n$ and

$$\begin{aligned} \pi_n(X) &\longrightarrow H_n(X) \quad \text{is an isomorphism} \\ [f] &\longmapsto H_n(f)\,[c_n], \end{aligned}$$

where $[c_n] \in H_n(S^n) \cong \mathbb{Z}$ is a generator.
($[c_n]$ is the homology class of $\Delta^n \longrightarrow \Delta^n/\dot{\Delta}^n \xrightarrow{\cong} S^n \quad + \frac{1}{2}((-1)^{n+1}-1)(\Delta^n \to *)$)

To apply this to $\mathbb{R}^{n+1}_r$ we may use the so called <u>Mayer-Vietoris sequence</u> for singular homology, which says:

Let $X = U \cup V$, $U, V \subset X$ open. Then there is for each i a map $\partial_i : H_i(X) \to H_{i-1}(U \cap V)$ such that

$$\cdots \to H_{i+1}(X) \xrightarrow{\partial_{i+1}} H_i(U \cap V) \xrightarrow{\begin{pmatrix} j_{U*} \\ -j_{V*} \end{pmatrix}} H_i(U) \oplus H_i(V) \longrightarrow H_i(X) \xrightarrow{\partial_i} \cdots$$

is exact.

Now use $\mathbb{R}^{n+1}_{r-1} = \mathbb{R}^{n+1}_r \cup \mathring{B}(q_r)$

To obtain the following result.

5.7 After disregarding base points (which we may do if $n+1 \geq 3$) the maps

$$\alpha'_s : S^n \longrightarrow \mathbb{R}^{n+1}_r ,$$
$$\xi \longmapsto q_s + \xi$$

$s = 1, \dots, r$, represent a basis of $\pi_n(\mathbb{R}^{n+1}_r)$.

This allows us to give an additive basis of $\pi_n(F_k(\mathbb{R}^{n+1}))$

For this we needed sections for all our fibrations

$$p_i : F_{k-i}(\mathbb{R}^{n+1}_i) \longrightarrow \mathbb{R}^{n+1}_i$$

Clearly, it suffices to produce a section on a subspace $U_i \subset \mathbb{R}^{n+1}_i$ such that $U_i \hookrightarrow \mathbb{R}^{n+1}_i$ is a strong deformation retract.

Choose $U_i := \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_i \mid x_1 \geq q_i + 2 \}$

This is clearly a deformation retract, and we obtain a section by mapping $x \in \mathbb{R}^{n+1}_i$ to $s_i(x) = (q_1, \dots, q_i, x, q_{i+1}, \dots, q_{k-1})$

5.8 Definition Let $1 \leq s < r \leq k$. Define

$$\alpha'_{rs} : S^n \longrightarrow F_{k-r+1}(\mathbb{R}^{n+1}_{r-1}) \quad \text{by}$$

$$\xi \longmapsto (q_1, \dots, q_{r-1}, q_s + \xi, q_r, \dots, q_{k-1})$$

and let $\alpha_{rs} = [\alpha'_{rs}]$.

Since the maps $\alpha'_s : \xi \longmapsto q_s + \xi$, $1 \leq s < r$ represent generators of the free abelian group $\pi_n(\mathbb{R}^{n+1}_{r-1})$ and

$$S_{r-1} \circ \alpha'_s = \alpha'_{rs} \quad \text{we have}$$

5.9 Proposition: The homotopy classes α_{rs} form a basis for the free abelian group $\pi_n(F_k(\mathbb{R}^{n+1}))$, $1 \leq s < r \leq k$.

To calculate relations among Whitehead products of elements of $\pi_n(F_k(\mathbb{R}^{n+1}))$ it is useful to introduce also elements α_{rs} for $1 \leq r < s \leq k$ which are defined as follows

5.10 Definition:

Let $1 \leq r < s \leq k$. Then

$$\alpha'_{rs} : S^n \longrightarrow F_{k-r+1}(\mathbb{R}^{n+1}_{r-1}) \quad \text{is defined by}$$

$$\xi \longmapsto (q_1, \dots, q_{r-1}, q_{s-1} + \xi, q_r, \dots, q_{k-1})$$

Remark: This definition differs from the one given in the book of Fadell & Husseini, but it is the correct one if one wants their propositions and theorems in chapter II to hold.

5.11 Proposition: Let $\sigma \in S_k$, and consider σ as a map

$$F_k(\mathbb{R}^{n+1}) \longrightarrow F_k(\mathbb{R}^{n+1})$$

$$(x_1, \dots, x_k) \longmapsto (x_{\sigma 1}, \dots, x_{\sigma k})$$

Then $\pi_n(\sigma)(\alpha_{rs}) = \alpha_{\sigma r\, \sigma s}$

Proof. It suffices to prove this for a σ of the form $(t\ t{+}1)$, $1 \leq t < k$, i.e. σ exchanges t and $t+1$ and leaves everything else unchanged.

If $\{s,r\} \cap \{t,t+1\} = \emptyset$ then $\sigma \circ \alpha_{rs}(\xi)$

$$\begin{array}{ccccc} s & & t & t+1 & & r \\ \downarrow & & \downarrow & \downarrow & & \downarrow \end{array}$$

$$= \{ \cdot\, q_s \cdots q_{t+1} \; q_t \qquad q_s + \xi$$