Freie Universität Berlin
Prof. Dr. Nicolas Perkowski

Stochastics IV: SPDEs
Summer Semester 2020

Exercise Sheet 1

To be discussed April 29

Exercise 1. Let W be a Gaussian martingale measure with covariance K. Let $(A_n)_{n\in\mathbb{N}}\subset\mathcal{A}_K$ be disjoint sets with $\bigcup_n A_n\in\mathcal{A}_K$. Show that for all $t\geqslant 0$:

$$W_t\Big(\bigcup_n A_n\Big)=\sum_n W_t(A_n),$$

where the series on the right hand side converges in $L^2(\mathbb{P})$.

Exercise 2.

a) Let $B^1,\ldots,B^n$ be independent one-dimensional Brownian motions, and let $\sigma_1,\ldots,\sigma_n\in C_b(\mathbb{R}^d)$ be such that for each k either $\sigma_k(x)\geqslant 0$ for all $x\in\mathbb{R}^d$, or $\sigma_k\in L^1(\mathbb{R}^d)$. Show that

$$W_t(x):=\sum_{k=1}^n \sigma_k(x)B_t^k, \qquad W_t(A):=\int_A W_t(x)dx$$

defines a Gaussian martingale measure.

b) Let $\alpha\in(0,d)$. Show that

$$K(dx,dy)=|x \quad y|^{\quad\alpha}dx\,dy$$

is a covariance measure.

Hint: Find $C,\beta>0$ *such that* $|x|^{\quad\alpha}=C(|\cdot|^{\quad\beta}*|\cdot|^{\quad\beta})(x)$.

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Exercise Sheet 2

To be discussed May 6

Exercise 1. Show that the fractional kernel $K(dy_1, dy_1) = |y_1 \quad y_2|^{\quad \alpha} dy_1 dy_2$ with $\alpha \in [0, d)$ is in $\mathcal{K}^{\alpha/2}$.

Exercise 2. Show that for $p \in [1, \infty)$ and $x_1, \ldots, x_n \geqslant 0$:

$$(x_1 + \cdots + x_n)^p \leqslant n^{p \quad 1}(x_1^p + \cdots + x_n^p) \leqslant n^{p \quad 1}(x_1 + \cdots + x_n)^p.$$

Deduce that for $q \in (0, 1]$:

$$x_1^q + \cdots + x_n^q \leqslant n^{1 \quad q}(x_1 + \cdots + x_n)^q \leqslant n^{1 \quad q}(x_1^q + \cdots + x_n^q).$$

In other words, for all $p > 0$ and all $n \in \mathbb{N}$ we have

$$(x_1 + \cdots + x_n)^p \simeq x_1^p + \cdots + x_n^p.$$

Exercise 3. Let

$$p(t, x) = (4\pi t)^{\quad \frac{d}{2}} \exp\left(\quad \frac{|x|^2}{4t} \right)$$

be the heat kernel.

a) Show that for all $m \geqslant 0$ and $c > 1$ there exists $C(m, c) > 0$ with

$$\left(\frac{|x|^2}{t} \right)^m p(t, x) \leqslant C(m, c) p(ct, x).$$

b) Deduce that for all $\mu \in \mathbb{N}_0^d$ and $\nu \in \mathbb{N}_0$:

$$|\partial_x^\mu \partial_t^\nu p(t, x)| \lesssim t^{\quad \frac{|\mu|}{2} \quad \nu} p(ct, x).$$

Here we use multi-index notation for the partial derivatives: $\partial_x^\mu = \partial_{x_1}^{\mu_1} \cdots \partial_{x_d}^{\mu_d}$.

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Exercise Sheet 3

To be discussed May 13

Exercise 1. Prove the following generalization of Theorem 1.25:

Let $\alpha \in [0,1)$ and let W be a Gaussian martingale measure with covariance measure $K \in \mathcal{K}^\alpha$. Let u_0 be $\mathcal{F}_0 \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and L^2-bounded. Let $f, g\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ and $h\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be measurable functions which are Lipschitz continuous in u, uniformly in x, and assume also that $f(\cdot,0), g(\cdot,0), h(\cdot,0)$ are bounded: There exists $L > 0$ such that

$$
\begin{aligned}
|f(x,u) - f(x,v)| + |g(x,u) - g(x,v)| + |h(x,u) - h(x,v)| &\leqslant L|u - v|,\\
|f(x,0)| + |g(x,0)| + |h(x,0)| &\leqslant L,
\end{aligned}
$$

for all $x \in \mathbb{R}^d, u, v \in \mathbb{R}$. We call a locally L^2-bounded predictable process u a mild solution to

$$
\partial_t u(t,x) = \Delta u(t,x) + f(x, u(t,x)) + \nabla \cdot h(x, u(t,x)) + g(x, u(t,x)) \partial_t W(t,x), \qquad u(0,x) = u_0(x),
$$

if

$$
\begin{aligned}
u(t,x) = {} & \int_{\mathbb{R}^d} p(t, x - y) u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} p(t - s, x - y) f(y, u(s,y)) dy ds\\
& + \int_0^t \int_{\mathbb{R}^d} \nabla p(t - s, x - y) \cdot h(y, u(s,y)) dy ds\\
& + \int_{[0,t] \times \mathbb{R}^d} p(t - s, x - y) g(y, u(s,y)) W(ds, dx).
\end{aligned}
$$

Mimick the proof of Theorem 1.25 to show that there exists a unique mild solution.

Hint: If some intermediate results follow from the exact same argument as in Theorem 1.25, then maybe there is no need to write them out completely.

Recall Exercise 2.3.

Exercise 2. Deduce that if $b, h\colon \mathbb{R} \to \mathbb{R}$ are bounded and measurable, W is a real-valued Brownain motion, and u_0 is L^2-bounded, then the Zakai equation

$$
\partial_t u(t,x) = \partial_{xx} u(t,x) - \partial_x (b(x)\, u(t,x)) + h(x) u(t,x) \partial_t W, \qquad u(0,x) = u_0(x),
$$

has a unique mild solution.

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Exercise Sheet 4

To be discussed May 13

Exercise 1. Complete the discussion of Example 1.36 from the lecture: For $T>0$ we consider the space

$$\mathcal{X} := C([0,T], L^2(\Omega; L^2(\mathbb{T})) \cap L^2([0,T]; L^2(\Omega; H^1(\mathbb{T}))),$$

equipped with the norm

$$\|u\|_{\mathcal{X}}^2 := \sup_{t\in[0,T]} \mathbb{E}[\|u(t)\|_{L^2}^2] + \int_0^T \mathbb{E}[\|u(t)\|_{H^1}^2] dt.$$

We call a predictable process $u \in \mathcal{X}$ a weak solution to the SPDE

$$\partial_t u = \frac{1}{2}\partial_{xx} u + \sigma \partial_x u \partial_t W, \qquad u(0) = u_0,$$

where W is a one-dimensional Brownian motion, if for all $\varphi \in H^1(\mathbb{T})$:

$$\langle u(t), \varphi\rangle_{L^2} = \langle u(0), \varphi\rangle_{L^2} - \int_0^t \langle \partial_x u(s), \partial_x \varphi\rangle_{L^2} ds + \sigma \int_0^t \langle \partial_x u(s), \varphi\rangle_{L^2} dW_s,$$

where all terms on the right hand side are well defined for $u \in \mathcal{X}$. Show that if $\sigma^2 < 1$ and $\mathbb{E}[\|u_0\|_{L^2}^2] < \infty$, then there exists a unique weak solution.

Hint: You could look at $u_N - u_M$, as defined in the lecture.

Exercise 2. (Voluntary, difficult) Let Z be the stochastic convolution as defined in the lecture, and for $\varphi \in C_c^\infty$ we define

$$Z(t)(\varphi) := \int_{[0,t]\times\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \varphi(x) p(t-s, x-y) dx \right) W(ds, dy).$$

Let

$$\mathcal{B}_r = \left\{ \varphi \in C_c^\infty : \operatorname{supp}(\varphi) \subset B(0,1), \|\varphi\|_{C_b^r} := \sum_{\mu \in \mathbb{N}_0^d : |\mu| \leqslant r} \|\partial^\mu \varphi\|_\infty \leqslant 1 \right\}.$$

Show that for all $T>0$, $\kappa>0$ there exist $\varepsilon>0$ and $C>0$ such that for all $p \geqslant 1$ and $\lambda \in (0,1]$ and $s,t \in [0,T]$

$$\sup_{x\in\mathbb{R}^d} \sup_{\varphi\in\mathcal{B}_0} \|Z(t)(\varphi_x^\lambda) - Z(s)(\varphi_x^\lambda)\|_{L^p} \leqslant C|t-s|^\varepsilon \lambda^{1-\frac{d}{2}-\kappa},$$

where

$$\varphi_x^\lambda(y) = \lambda^{-d} \varphi(\lambda^{-1}(y-x)).$$

Based on this we could show via a sort of Kolmogorov continuity criterion that a.s. $Z(t) \in C_{\mathrm{loc}}^{1 \quad \frac{d}{2} \quad \varepsilon}$ for all $t \geqslant 0$ and $\varepsilon > 0$, where for $\alpha > 0$ the space $C_{\mathrm{loc}}^{\alpha}$ consists of generalized functions of distributional regularity.

See Theorem 2.7 in [Chandra, Weber. Stochastic PDEs, regularity structures, and interacting particle systems. *Annales de la Faculté des sciences de Toulouse: Mathématiques*. Vol. 26. No. 4. 2017] for a version of such a Kolmogorov criterion. Later in the lecture we will prove a similar result.

Hint: Young's convolution inequality and interpolation may both be useful.

An easier version of the exercise would be to show that $\|Z(t)(\varphi_x^{\lambda})\|_{L^p} \lesssim \lambda^{1 \quad \frac{d}{2} \quad \kappa}$. *This would yield a.s.* $Z(t) \in C_{\mathrm{loc}}^{1 \quad \frac{d}{2} \quad \varepsilon}$*, but with a null set which depends on* t*.*

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Exercise Sheet 5

To be discussed May 27

Exercise 1. In the setting of Theorem 2.8 from the lecture, show that the solution x depends locally Lipschitz continuously on (x_0, y), i.e. complete the proof of Theorem 2.8.

Hint: Consider a small time interval first. You may assume that $\|x\|_\infty + \|x\|_\alpha + \|\tilde{x}\|_\infty + \|\tilde{x}\|_\alpha \leqslant C$ *on the entire interval* $[0, T]$*, where* $C > 0$ *is a constant which only depends on the data of our problem:* $b, \sigma, x_0, y, \tilde{x}_0, \tilde{y}, T$*. Why?*

Exercise 2. Consider the functions

$$x_t^n = \frac{1}{n}\cos(n^2 t), \qquad y_t^n = \frac{1}{n}\sin(n^2 t).$$

Show that for all $\alpha < \frac{1}{2}$ we have $x^n, y^n \to 0$ in $C^\alpha([0, T], \mathbb{R})$, but

$$\lim_{n \to \infty} \int_0^t x_s^n d y_s^n = \frac{t}{2}.$$

This shows that unlike for $\alpha > \frac{1}{2}$, for $\alpha < \frac{1}{2}$ there cannot exist a Young integral which is a continuously bilinear operator on $C^\alpha \times C^\alpha$ that agrees with the Lebesgue-Stieltjes integral whenever $y \in C^\alpha$ is of finite variation.

Exercise 3. (Voluntary) The functions $(\sqrt{2}\sin(2\pi k))_{k \in \mathbb{Z} \setminus \{0\}}, (\sqrt{2}\cos(2\pi k))_{k \in \mathbb{Z}}$ form an orthonormal basis of $L^2(\mathbb{T}; \mathbb{R})$. Conside $u: \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$

$$\partial_t u = \frac{1}{2}\partial_{xx} u + \sigma \partial_x u \partial_t W, \qquad u(0, \cdot) = u_0,$$

where W is a one-dimensional Brownian motion, and assume that

$$\langle u(0, \cdot), \sqrt{2}\sin(2\pi k \cdot)\rangle = \begin{cases} 1, & |k| \leqslant 5, \\ 0, & \text{else}, \end{cases} \qquad \text{and} \qquad \langle u(0, \cdot), \sqrt{2}\cos(2\pi k \cdot)\rangle = 0, \qquad k \in \mathbb{Z}.$$

Simulate $u(1, \cdot)$ for $\sigma = 0.9$, $\sigma = 1$ and $\sigma = 1.1$. If you are very motivated you could also make a movie showing how $u(t, \cdot)$ evolves in t.

Hint: Derive SDEs for the processes $X^k = \langle u, \sqrt{2}\cos(2\pi k \cdot)\rangle$ *and* $Y^k = \langle u, \sqrt{2}\sin(2\pi k \cdot)\rangle$*.*

Exercise Sheet 6

To be discussed June 3

Exercise 1.

a) Let $n = d = 1$ and consider the equation

$$\partial_t X_t = \sigma(X_t) \partial_t Y_t, \qquad X_0 = \xi, \tag{1}$$

for a Lipschitz-continuous function σ and $Y \in C^1([0, T], \mathbb{R})$. Let $u: \mathbb{R}^2 \to \mathbb{R}$ be the solution to

$$\partial_y u(x, y) = \sigma(u(x, y)), \qquad u(x, 0) = x.$$

Show that $X_t = u(\xi, Y_t \quad Y_0)$ solves (1). Conclude that X depends continuously on Y in the supremum norm.

b) Can we extend this to $d > 1$ and/or $n > 1$?

Hint: You may assume that if u exists, then $u \in C^2(\mathbb{R}^d \times \mathbb{R}^n)$.

c) Show that if $(Y^m) \subset C^1([0, T], \mathbb{R})$ converges uniformly to $Y \in C([0, T], \mathbb{R})$, then the iterated integrals $\mathbb{Y}^m_{s,t} := \int_s^t Y^m_{s,r} dY^m_r$ also converge to a limit that only depends on Y but not on the approximating sequence (Y^m).

d) If $Y_t(\omega) = B_t(\omega)$ for a Brownian sample path, does the process $X_t(\omega)$ solve the Itô SDE $dX_t = \sigma(X_t) dB_t$? If not, in which sense does it solve the equation?

Exercise 2. Let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ and let $(Y, \mathbb{Y})$ be a d-dimensional α-rough path.

a) Let $Z \in C^{2\alpha}([0, T], (\mathbb{R}^d)^{\otimes 2})$. Show that $(Y, \tilde{\mathbb{Y}})$ is also an α-rough path, where $\tilde{\mathbb{Y}}_{s,t} = \mathbb{Y}_{s,t} + Z_{s,t}$.

b) Let $(Y, \tilde{\mathbb{Y}})$ also be an α-rough path. Show that there exists $Z \in C^{2\alpha}([0, T], (\mathbb{R}^d)^{\otimes 2})$ (i.e. an additive function) such that

$$\tilde{\mathbb{Y}}_{s,t} = \mathbb{Y}_{s,t} + Z_{s,t}.$$

Exercise Sheet 7

To be discussed June 10

Exercise 1. In the setting of Theorem 2.23 let $\tilde{\boldsymbol{Y}} = (\tilde{Y}, \tilde{\mathbb{Y}})$ be another d-dimensional α-rough path, and let $(\tilde{X}, \tilde{X}') \in \mathcal{D}_Y^{2\alpha}([0,T], L(\mathbb{R}^d, \mathbb{R}^m))$. Define

$$\rho_\alpha(\boldsymbol{Y}, \tilde{\boldsymbol{Y}}) = \|Y \quad \tilde{Y}\|_\alpha + \|\mathbb{Y} \quad \tilde{\mathbb{Y}}\|_{2\alpha},$$
$$d_{2\alpha}(X, X', \tilde{X}, \tilde{X}') = \|X' \quad \tilde{X}'\|_\alpha + \|R^X \quad R^{\tilde{X}}\|_{2\alpha},$$

and $M = \max\{\|Y\|_\alpha, \|\mathbb{Y}\|_{2\alpha}, |X_0'|, \|X, X'\|_{\mathcal{D}_Y^{2\alpha}}, \|\tilde{Y}\|_\alpha, \|\tilde{\mathbb{Y}}\|_{2\alpha}, |\tilde{X}_0'|, \|\tilde{X}, \tilde{X}'\|_{\mathcal{D}_Y^{2\alpha}}\}$. Set

$$(Z, Z') = \left(\int_0^\cdot X_s d\boldsymbol{Y}_s, X\right), \qquad (\tilde{Z}, \tilde{Z}') = \left(\int_0^\cdot \tilde{X}_s \tilde{\boldsymbol{Y}}_s, \tilde{X}\right).$$

Show that

$$d_{2\alpha}(Z, Z', \tilde{Z}, \tilde{Z}') \lesssim_T (1 + M)(\rho_\alpha(\boldsymbol{Y}, \tilde{\boldsymbol{Y}}) + |X_0' \quad \tilde{X}_0'| + d_{2\alpha}(X, X', \tilde{X}, \tilde{X}')).$$

Exercise 2. Let $\boldsymbol{Y} = (Y, \mathbb{Y})$ be an α-rough path. Show that $(Y, \mathbb{I}_d) \in \mathcal{D}_Y^{2\alpha}([0,T], \mathbb{R}^d)$, where $\mathbb{I}_d$ denotes the unit matrix on $\mathbb{R}^d$. Let $\int_0^\cdot Y_s \otimes d\boldsymbol{Y}_s$ be the controlled path integral. Show that

$$\int_s^t Y_r \otimes d\boldsymbol{Y}_r = \mathbb{Y}_{s,t} + Y_s \otimes Y_{s,t}$$

for all $(s,t) \in \Delta_T$. In other words, our definition of the integral is consistent with the interpretation of $\mathbb{Y}$ as the iterated integrals of Y.

Exercise 3. Let $Y_t = t$. Find $\mathbb{Y}: \Delta_T \to \mathbb{R}$ such that $\boldsymbol{Y} = (Y, \mathbb{Y})$ is a one-dimensional α-rough path, and find $(X, X') \in \mathcal{D}_Y^{2\alpha}([0,T], \mathbb{R})$ such that $\int_0^\cdot X_s d\boldsymbol{Y}_s \neq \int_0^\cdot Y_s ds$.

Exercise Sheet 8

To be discussed June 17

Exercise 1. Check that in $d=1$ we have $\mathcal{F}^{-1}(1_{[0,1)}(|\cdot|)) \in \bigcap_{p>1} L^p(\mathbb{R})$ but $\mathcal{F}^{-1}(1_{[0,1)}(|\cdot|)) \notin L^1(\mathbb{R})$.

Exercise 2. Let δ denote the Dirac delta, $\delta(\varphi) = \varphi(0)$. Show that $\delta \in B_{p,\infty}^{-d(1-1/p)}$ for all $p \in [1, \infty]$.

Exercise 3. Let $\alpha \in (0,1)$. Then $\mathcal{C}^\alpha = C_b^\alpha$ is the space of bounded α-Hölder continuous functions, and

$$\|u\|_\alpha \simeq \|u\|_{C_b^\alpha}.$$

Hint: For $j \geqslant 0$ *we have* $\int K_j(x) dx = 0$ *(why?).*

To control $\|u\|_{C_b^\alpha}$ *it suffices to control* $\|u\|_\infty$ *and* $|u(x) - u(y)|$ *for* $|x - y| \leqslant 1$*. Given such* x, y*, consider the two cases* $2^{-j} \leqslant |x - y|$ *and* $2^{-j} > |x - y|$ *separately.*

Exercise Sheet 9

To be discussed June 24

Exercise 1. Let ξ be a white noise on $\mathbb{T}^d$, i.e. a centered Gaussian process $(\xi(\varphi))_{\varphi \in L^2(\mathbb{T}^d)}$ with covariance

$$\mathbb{E}[\xi(\varphi)\xi(\psi)] = \int_{\mathbb{T}^d} \varphi(x)\psi(x)dx.$$

a) Show that there exists a process $\tilde{\xi}: \Omega \to \mathcal{S}'$ such that for each $\varphi \in C^\infty(\mathbb{T}^d)$ we have almost surely $\tilde{\xi}(\varphi) = \xi(\varphi)$.

Hint: Have a look at Remark 3.24.

b) Show that almost surely $\tilde{\xi} \in \mathcal{C}^{-\frac{d}{2}-\varepsilon}$ for all $\varepsilon > 0$.

Hint: First compute $\mathbb{E}[\|\tilde{\xi}\|_{B^\alpha_{p,p}}^p]$ for $p < \infty$, and use that $\tilde{\xi}$ is Gaussian.

Exercise 2. Let $(p(t,x): t > 0, x \in \mathbb{T}^d)$ be the heat kernel on $\mathbb{T}^d$,

$$p(t,x) = \sum_{k \in \mathbb{Z}^d} e^{2\pi i k \cdot x} e^{-|2\pi k|^2 t} = \sum_{k \in \mathbb{Z}^d} (4\pi t)^{-d/2} \exp\left(-\frac{|x+k|^2}{4t}\right).$$

a) Show that

$$\|\partial_t^\nu \partial_x^\mu p(t,\cdot)\|_{L^1(\mathbb{T}^d)} \lesssim t^{-\frac{|\mu|}{2}-\nu}.$$

Hint: There is a short solution without many computations. Have a look at Exercise 2.3.

b) Using an “inverse version” of our Bernstein type inequality (Lemma 2.1 of Bahouri et al), one can show that for all $k \in \mathbb{N}_0$:

$$\|u\|_\alpha \simeq \|\Delta_{-1} u\|_{L^\infty(\mathbb{T}^d)} + \|\mathrm{D}^k u\|_{\alpha-k}, \tag{1}$$

where $\mathrm{D}^k u$ is the k-th distributional derivative of u, formally consisting of $(\partial^\mu u)_{|\mu|=k}$. You do not have to prove this, but rather use it to deduce that for all $k \in \mathbb{N}_0$:

$$\|P_t u\|_{\alpha+k} := \|p(t,\cdot) * u\|_{\alpha+k} \lesssim (t^{-\frac{k}{2}} \vee 1)\|u\|_\alpha.$$

Hint: Young's inequality for convolutions also holds on $\mathbb{T}^d$. Note also that $\Delta_j P_t u = P_t \Delta_j u$ (why?).

c) Using an interpolation argument, deduce that for all $\beta \geqslant 0$:

$$\|P_t u\|_{\alpha+\beta} \lesssim (t^{-\frac{\beta}{2}} \vee 1)\|u\|_\alpha.$$

Exercise Sheet 10

To be discussed July 1

Exercise 1. Consider the Φ_2^4 equation $\phi\colon \mathbb{R}_+ \times \mathbb{T}^2 \to \mathbb{R}$,

$$\phi(t) = P_t \phi_0 + \int_0^t P_{t \quad s}(\quad \phi(s)^3) ds + Z(t).$$

a) Which equation does $v = \phi \quad Z$ solve?

b) Recall that $Z \in C_T \mathcal{C}^{\alpha}$ for $\alpha = 1 \quad \frac{d}{2} \quad = 0 \quad$. Therefore, the products Z^2 and Z^3 are not well defined. But if you naively apply the paraproduct estimates to compute the regularity of Z^2 and Z^3, without bothering about the "$\alpha + \beta > 0$ condition" for the resonant product, then what regularity do you obtain?

It may simplify notation in the next step to define $\varepsilon = \quad 3\alpha$.

c) Let $\beta \in (0, 2)$. Assume that $\phi_0 \in \mathcal{C}^{\beta}$ and that Z, Z^2, Z^3 are given with their natural regularities. Show that there exists a unique solution $v \in C_T \mathcal{C}^{\beta}$ to the equation for $v = \phi \quad Z$, up to a possibly finite explosion time T^* (the proof is very similar to the one for the Φ_1^4 equation).

Exercise Sheet 11

To be discussed July 8

Exercise 1. Show that for $t > 0$

$$H_n(x,t) = (-t)^n e^{x^2/(2t)} \partial_x^n e^{-x^2/(2t)}. \tag{1}$$

Conclude that the family $(H_n(\cdot,t))_{n\in\mathbb{N}_0}$ is orthogonal with respect to the centered Gaussian measure with variance t. Show also that

$$\partial_x H_n = n H_{n-1}, \qquad \partial_t H_n = -\frac{n(n-1)}{2} H_{n-2} = -\frac{1}{2}\partial_x^2 H_n, \tag{2}$$

i.e. that each Hermite polynomial solves the backward heat equation $\left(\partial_t + \frac{1}{2}\partial_x^2\right) H_n = 0$.

Exercise 2. Show that for $t > 0$ and $x, y \in \mathbb{R}$ and $n \in \mathbb{N}_0$ we have

$$H_n(x+y,t) = \sum_{k=0}^{n} \binom{n}{k} H_k(x,t) y^{n-k}.$$

Exercise 3. Let ξ be a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^d$ and set $\mathcal{S}_\lambda \phi(t,x) := \phi(\lambda^2 t, \lambda x)$ and $\xi_\lambda := \lambda^{1+d/2} \mathcal{S}_\lambda \xi$, interpreted rigorously as $\xi_\lambda(f) = \lambda^{1+d/2} \xi(\lambda^{-2-d} \mathcal{S}_{\lambda^{-1}} f)$. Show that ξ_λ is a new space-time white noise.

Hint: It suffices to show that ξ_λ is centered Gaussian with the right covariance.

Exercise Sheet 12

To be discussed July 15

Exercise 1.

a) What is the critical dimension for the KPZ equation

$$(\partial_t \quad \Delta)u = |\nabla u|^2 + \xi?$$

b) What is the critical dimension for the Parabolic Anderson model

$$(\partial_t \quad \Delta)u = u\eta,$$

where η is a *space white noise*.

c) What is the critical dimension for the equation

$$(\partial_t \quad \Delta)u = f(u) + \xi,$$

for a general nonlinear function f (not a polynomial)? Here you should think of $f(u) = u^\infty$ in terms of regularity power counting.

d) For which regularity of ξ is the ODE

$$\partial_t u = f(u)\xi$$

subcritical/critical/supercritical? You may use that integration in time gains one derivative.

e) In $d = 4$, for polynomials P of which degree is the equation

$$(\partial_t \quad \Delta)\phi = P(\phi) + \xi$$

subcritical/critical/supercritical?

Exercise 2. Find the paracontrolled ansatz for the KPZ equation $h: \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$,

$$(\partial_t \quad \Delta)h = (\partial_x h)^2 + \xi,$$

where ξ is a space-time white noise.

Hint: You have to subtract more terms than for Φ^4_3 before you bring in paraproducts. It may be a good idea to adapt the tree notation, for example a wiggly line could indicate convolution against $\partial_x p(t)$.

Exercise 3. (Link between resonant product and iterated integrals) Let $\alpha \in \ \frac{1}{3}, \frac{1}{2})$ and u, $v \in \mathcal{C}^\alpha(\mathbb{R})$. Assume that we are a priori given a candidate for the resonant product $u \odot \partial_t v \in \mathcal{C}^{2\alpha \quad 1}$. Then we postulate the iterated integral

$$\mathbb{U}_{s,t} := \int_s^t (u \odot \partial_t v + u \varolessthan \partial_r v + u \varogreaterthan \partial_r v) dr \quad u(s)(v(t) \quad v(s)),$$

see below for the interpretation of the time integral. Show that

$$|\mathbb{U}_{s,t}| \lesssim |t - s|^{2\alpha}(\|u\|_\alpha \|v\|_\alpha + \|u \odot \partial_t v\|_{\alpha+\beta}).$$

An inverse inequality holds as well, but you do not have to show the inverse inequality.

You may use without proof the following two facts:

- Integration in time is a continuous map on $\mathcal{C}^\alpha(\mathbb{R})$ for $\alpha \in (0,1)$:

$$\left| \int_s^t f(r) dr \right| \lesssim |t - s|^\alpha \|f\|_{\alpha - 1}, \qquad \alpha \in (0,1).$$

- Integration in time "commutes" with the paraproduct:

$$\left| \int_s^t (f \varolessthan \partial_t g)(r) dr - f(s)(g(t) - g(s)) \right| \lesssim |t - s|^{\alpha+\beta} \|f\|_\alpha \|g\|_\beta, \qquad \alpha, \beta > 0, \alpha + \beta < 1.$$