

Pathwise stochastic integrals for model free finance *

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Abstract

We present two different approaches to stochastic integration in frictionless model free financial mathematics. The first one is in the spirit of Itô’s integral and based on a certain topology which is induced by the outer measure corresponding to the minimal superhedging price. The second one is based on the controlled rough path integral. We prove that every “typical price path” has a naturally associated Itô rough path, and justify the application of the controlled rough path integral in finance by showing that it is the limit of non-anticipating Riemann sums, a new result in itself. Compared to the first approach, rough paths have the disadvantage of severely restricting the space of integrands, but the advantage of being a Banach space theory.

Both approaches are based entirely on financial arguments and do not require any probabilistic structure.

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1 Introduction

In this paper we use Vovk’s [Vov12] game-theoretic approach to mathematical finance to develop two different approaches to stochastic integration in frictionless model free finance. A priori the integration problem is highly non-trivial in the model free context since we do not want to assume any probabilistic respectively semimartingale structure. Therefore, we do not have access to Itô integration and most known techniques completely break down. There are only two general solutions to the integration problem in a non-probabilistic continuous time setting that we are aware of. One was proposed by [DS14] who simply restrict themselves to trading strategies (integrands) of bounded variation. While this already allows to solve many interesting problems, it is not a very natural assumption to make in a frictionless market model. Indeed, while in [DS14] a general duality approach is developed for pricing path-dependent derivatives that are Lipschitz continuous in the supremum norm, this approach does not allow to treat derivatives depending on the volatility.

Another interesting solution was proposed by [DOR13] (using an idea which goes back to [Lyo95b]). They restrict the set of “possible price paths” to those admitting a quadratic variation. This allows them to apply Föllmer’s pathwise Itô calculus [Föll81] to define pathwise stochastic integrals of the form $\int \nabla F(S) dS$. In [Lyo95b] that approach was used to derive prices for American and European options under volatility uncertainty. In [DOR13] the given data is a finite number of European call and put prices and the derivative to be priced is a weighted variance swap. The restriction to the set of paths with quadratic variation is justified by referring to Vovk [Vov12], who proved that “typical price paths” (to be defined below) admit a quadratic variation.

In our first approach we do not restrict the set of paths and work on the space Ω of d -dimensional continuous paths (which represent the possible asset price trajectories). We follow Vovk who introduces an outer measure on Ω . The crucial point is that this outer measure is defined as a minimal superhedging price (in a suitable sense), and therefore has a purely financial interpretation and does not come from an artificially imposed probabilistic structure. Our first observation is that Vovk’s outer measure allows us to define a topology on processes on Ω , and that the “natural Itô integral” on step functions is in a certain sense continuous in that topology. This allows us to extend the integral to càdlàg adapted integrands, and we call the resulting integral “model free Itô integral”. We stress that the entire construction is based on purely financial arguments.

Let us also stress that it is the *continuity* of our integral which is the most important aspect. Without reference to any topology the construction would certainly not be very useful, since already in the classical probabilistic setting virtually all applications of the Itô integral (SDEs, stochastic optimization, duality theory, . . .) are based on the fact that it is a continuous operator.

This also motivates our second approach, which is more in the spirit of [Lyo95b, DOR13, DS14]. While in the first approach we do have a continuous operator, it is only continuous with respect to a sequence of pseudometrics and it seems impossible to find a Banach space structure that is compatible with it. However, using the model free Itô integral we are able to show that every “typical price path” has a natural Itô rough path associated to it. Since in financial applications we can always restrict ourselves to typical price paths, this observation opens the door for the application of the controlled rough path integral [Lyo98, Gub04] in model free finance. Controlled rough path integration has the advantage of being an entirely linear Banach space theory which simultaneously extends

- the Riemann-Stieltjes integral of S against functions of bounded variation which was used by [DS14];
- the Young integral [You36]: typical price paths have finite p -variation for every $p > 2$, and therefore for every F of finite q -variation for $q < 2$ (so that $1/p + 1/q > 1$), the integral $\int FdS$ is defined as limit of non-anticipating Riemann sums;
- Föllmer’s [Föl81] pathwise Itô integral which was used by [Lyo95b, DOR13]. That this last integral is a special case of the controlled rough path integral is, to the best of our knowledge, proved rigorously for the first time in this paper, although also [FH14] contains some preliminary observations in that direction.

In other words, our second approach covers all previously known techniques of integration in model free financial mathematics, while the first approach is much more general but at the price of leaving the Banach space world.

There is only one pitfall: the rough path integral is usually defined as a limit of compensated Riemann sums which have no obvious financial interpretation. This sabotages our entire philosophy of only using financial arguments. That is why we show that under some weak condition every rough path integral $\int FdS$ is given as limit of non-anticipating Riemann sums that do not need to be compensated – the first time that such a statement is shown for general rough path integrals. While this will not change anything in concrete applications, it is of utmost importance from a philosophical point of view. Indeed, the justification for using the Itô integral in classical financial mathematics is crucially based on the fact that it is the limit of non-anticipating Riemann sums, even if in “every day applications” one never makes reference to that; see for example the discussion in [Lyo95b].

Plan of the paper

Below we present a very incomplete list of solutions to the stochastic integration problem under model uncertainty and in a discrete time model free context (both a priori much simpler problems than the continuous time model free case), and we introduce some notations and conventions that will be used throughout the paper. In Section 2 we briefly recall Vovk’s game-theoretic approach to mathematical finance and introduce our outer measure. We also construct the topology induced by the outer measure. Section 3 is devoted to the construction of the model free Itô integral. Section 4 recalls some basic results from rough path theory, and continues by constructing rough paths associated to typical price paths. Here we also prove that the rough path integral is given as a limit of non-anticipating Riemann sums. We also compare Föllmer’s pathwise Itô integral with the rough path integral and prove that the latter is an extension of the former. Appendix A recalls Vovk’s pathwise Hoeffding inequality.

In Appendix B we show that a result of Davie which also allows to calculate rough path integrals as limit of Riemann sums is a special case of our results in Section 4.

Stochastic integration under model uncertainty

The first works which studied the option pricing problem under model uncertainty were [ALP95] and [Lyo95b], both considering the case of volatility uncertainty. As described above, [Lyo95b] is using Föllmer’s pathwise Itô integral, while in [ALP95] the problem is reduced to the classical setting by deriving a “worst case” model for the volatility.

A powerful tool in financial mathematics under model uncertainty is Karandikar’s pathwise construction of the Itô integral [Kar95, Bic81] which allows to construct the Itô integral of a càdlàg integrand simultaneously under all semimartingale measures. The crucial point that makes the construction useful is that the Itô integral is a continuous operator under every semimartingale measure. While its pathwise definition would allow us to use the same construction also in a model free setting, it is not even clear what the output should signify in that case (for example the construction depends on a certain sequence of partitions and changing the sequence will change the output). Certainly it is not obvious whether the Karandikar integral is continuous in any topology once we dispose of semimartingale measures. A more general pathwise construction of the Itô integral was given in [Nut12], but it suffers from the same drawbacks with respect to applications in model free finance.

A general approach to stochastic analysis under model uncertainty was put forward in [DM06], and it is based on quasi sure analysis. While this approach is extremely helpful when working under model uncertainty, it also does not allow us to define stochastic integrals in a model free context.

In a related but slightly different direction, in [CDGR11] non-semimartingale models are studied (which do not violate arbitrage assumptions if the set of admissible strategies is restricted). While the authors work under one fixed probability measure, the fact that their price process is not a semimartingale prevents them from using Itô integrals, a difficulty which is overcome by working with the Russo-Vallois integral [RV93].

Of course all these technical problems disappear if we restrict ourselves to discrete time, and indeed in that case [BHL13] develop an essentially fully satisfactory duality theory for the pricing of derivatives under model uncertainty.

Notation and conventions

Throughout the paper we fix $T \in (0, \infty)$ and we write $\Omega := C([0, T], \mathbb{R}^d)$ for the space of d -dimensional continuous paths. The coordinate process on Ω is denoted by $S_t(\omega) = \omega(t)$, $t \in [0, T]$. For $i \in \{1, \dots, d\}$, we also write $S_t^i(\omega) = \omega^i(t)$, where $\omega = (\omega^1, \dots, \omega^d)$. The filtration $(\mathcal{F}_t)_{t \in [0, T]}$ is defined as $\mathcal{F}_t := \sigma(S_s : s \leq t)$, and we set $\mathcal{F} := \mathcal{F}_T$. Stopping times τ and the associated σ -algebras \mathcal{F}_τ are defined as usually.

Unless explicitly stated otherwise, inequalities of the type $F_t \geq G_t$, where F and G processes on Ω , are supposed to hold for all $\omega \in \Omega$, and not modulo null sets, as it is usually assumed in stochastic analysis.

The indicator function of a set A is denoted by $\mathbf{1}_A$.

A *partition* π of $[0, T]$ is a finite set of time points, $\pi = \{0 = t_0 < t_1 < \dots < t_m = T\}$. Occasionally, we will identify π with the set of intervals $\{[t_0, t_1], [t_1, t_2], \dots, [t_{m-1}, t_m]\}$, and write expressions like $\sum_{[s, t] \in \pi}$.

For $f: [0, T] \rightarrow \mathbb{R}^n$ and $t_1, t_2 \in [0, T]$, denote $f_{t_1, t_2} := f(t_2) - f(t_1)$ and define the p -variation of f restricted to $[s, t] \subseteq [0, T]$ as

$$\|f\|_{p\text{-var}, [s, t]} := \sup \left\{ \left(\sum_{k=0}^{m-1} |f_{t_k, t_{k+1}}|^p \right)^{1/p} : s = t_0 < \dots < t_m = t, m \in \mathbb{N} \right\}, \quad p > 0, \quad (1)$$

(possibly taking the value $+\infty$). We set $\|f\|_{p\text{-var}} := \|f\|_{p\text{-var}, [0, T]}$. We write $\Delta_T = \{(s, t) : 0 \leq s \leq t \leq T\}$ for the simplex and define the p -variation of a function $g: \Delta_T \rightarrow \mathbb{R}^n$ in the same manner, replacing $f_{t_k, t_{k+1}}$ in (1) by $g(t_k, t_{k+1})$.

For $\alpha > 0$, the space C^α consists of those functions that are $\lfloor \alpha \rfloor$ times continuously differentiable, with $(\alpha - \lfloor \alpha \rfloor)$ -Hölder continuous partial derivatives of order $\lfloor \alpha \rfloor$. The space C_b^α consists of those functions in C^α that are bounded, together with their partial derivatives, and we define the norm $\|\cdot\|_{C_b^\alpha}$ by setting

$$\|f\|_{C_b^\alpha} := \sum_{k=0}^{\lfloor \alpha \rfloor} \|D^k f\|_\infty + \|D^{\lfloor \alpha \rfloor} f\|_{\alpha - \lfloor \alpha \rfloor},$$

where $\|\cdot\|_\beta$ denotes the β -Hölder norm for $\beta \in (0, 1)$, and $\|\cdot\|_\infty$ denotes the supremum norm.

For $x, y \in \mathbb{R}^d$, we write $xy := \sum_{i=1}^d x_i y_i$ for the usual inner product. However, often we will encounter terms of the form $\int S dS$ or $S_s S_{s,t}$ for $s, t \in [0, T]$, where we recall that S denotes the coordinate process on Ω . Those expressions are to be understood as the matrix $(\int S^i dS^j)_{1 \leq i, j \leq d}$, and similarly for $S_s S_{s,t}$. The interpretation will be usually clear from the context, otherwise we will make a remark to clarify things.

We use the notation $a \lesssim b$ if there exists a constant $c > 0$, independent of the variables under consideration, such that $a \leq c \cdot b$, and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. If we want to emphasize the dependence of c on the variable x , then we write $a(x) \lesssim_x b(x)$.

We make the convention that $0/0 := 0 \cdot \infty := 0$ and $\inf \emptyset := \infty$.

2 Superhedging and typical price paths

2.1 The outer measure and its basic properties

In a recent series of papers, Vovk [Vov08, Vov11, Vov12] has introduced a model free, hedging based approach to mathematical finance that uses arbitrage considerations to examine which properties are satisfied by “typical price paths”. This is achieved with the help of an outer measure given by the cheapest superhedging price.

Recall that $T \in (0, \infty)$ and $\Omega = C([0, T], \mathbb{R}^d)$ is the space of continuous paths, with coordinate process S , natural filtration $(\mathcal{F}_t)_{t \in [0, T]}$, and $\mathcal{F} = \mathcal{F}_T$. A process $H: \Omega \times [0, T] \rightarrow \mathbb{R}^d$ is called a *simple strategy* if there exist stopping times $0 = \tau_0 < \tau_1 < \dots$, and \mathcal{F}_{τ_n} -measurable bounded functions $F_n: \Omega \rightarrow \mathbb{R}^d$, such that for every $\omega \in \Omega$ we have $\tau_n(\omega) = \infty$ for all but finitely many n , and such that

$$H_t(\omega) = \sum_{n=0}^{\infty} F_n(\omega) \mathbf{1}_{(\tau_n(\omega), \tau_{n+1}(\omega)]}(t).$$

In that case, the integral

$$(H \cdot S)_t(\omega) := \sum_{n=0}^{\infty} F_n(\omega)(S_{\tau_{n+1} \wedge t}(\omega) - S_{\tau_n \wedge t}(\omega)) = \sum_{n=0}^{\infty} F_n(\omega) S_{\tau_n \wedge t, \tau_{n+1} \wedge t}(\omega)$$

is well defined for all $\omega \in \Omega$, $t \in [0, T]$. Here $F_n(\omega) S_{\tau_n \wedge t, \tau_{n+1} \wedge t}(\omega)$ denotes the usual inner product on \mathbb{R}^d . For $\lambda > 0$, a simple strategy H is called λ -admissible if $(H \cdot S)_t(\omega) \geq -\lambda$ for all $\omega \in \Omega$, $t \in [0, T]$. The set of λ -admissible simple strategies is denoted by \mathcal{H}_λ .

Definition 2.1. The *outer measure* of $A \subseteq \Omega$ is defined as the cheapest superhedging price for $\mathbf{1}_A$, that is

$$\bar{P}(A) := \inf \left\{ \lambda > 0 : \exists (H^n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_\lambda \text{ s.t. } \liminf_{n \rightarrow \infty} (\lambda + (H^n \cdot S)_T(\omega)) \geq \mathbf{1}_A(\omega) \forall \omega \in \Omega \right\}.$$

A set of paths $A \subseteq \Omega$ is called a *null set* if it has outer measure zero.

The term outer measure will be justified by Lemma 2.3 below. Our definition of \bar{P} is very similar to the one used by Vovk [Vov12], but not quite the same. For a discussion see Section 2.4 below.

By definition, every Itô stochastic integral is the limit of stochastic integrals against simple strategies. Therefore, our definition of the cheapest superhedging price is essentially the same as in the classical setting, with one important difference: we require superhedging for *all* $\omega \in \Omega$, and not just almost surely.

Remark 2.2 ([Vov12], p. 564). *An equivalent definition of \bar{P} would be*

$$\tilde{P}(A) := \inf \left\{ \lambda > 0 : \exists (H^n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_\lambda \text{ s.t. } \liminf_{n \rightarrow \infty} \sup_{t \in [0, T]} (\lambda + (H^n \cdot S)_t(\omega)) \geq \mathbf{1}_A(\omega) \forall \omega \in \Omega \right\}.$$

Clearly $\tilde{P} \leq \bar{P}$. To see the opposite inequality, let $\tilde{P}(A) < \lambda$. Let $(H^n)_{n \in \mathbb{N}} \subset \mathcal{H}_\lambda$ be a sequence of simple strategies such that $\liminf_{n \rightarrow \infty} \sup_{t \in [0, T]} (\lambda + (H^n \cdot S)_t) \geq \mathbf{1}_A$, and let $\varepsilon > 0$. Define $\tau_n := \inf \{ t \in [0, T] : \lambda + \varepsilon + (H^n \cdot S)_t \geq 1 \}$. Then the stopped strategy $G_t^n(\omega) := H_t^n(\omega) \mathbf{1}_{t < \tau_n(\omega)}$ is in $\mathcal{H}_\lambda \subseteq \mathcal{H}_{\lambda + \varepsilon}$ and

$$\liminf_{n \rightarrow \infty} (\lambda + \varepsilon + (G^n \cdot S)_T(\omega)) \geq \liminf_{n \rightarrow \infty} \mathbf{1}_{\{\lambda + \varepsilon + \sup_{t \in [0, T]} (H^n \cdot S)_t \geq 1\}}(\omega) \geq \mathbf{1}_A(\omega).$$

Therefore $\bar{P}(A) \leq \lambda + \varepsilon$, and since $\varepsilon > 0$ was arbitrary $\bar{P} \leq \tilde{P}$, and thus $\bar{P} = \tilde{P}$.

Lemma 2.3 ([Vov12], Lemma 4.1). \bar{P} is in fact an outer measure, i.e. a nonnegative function defined on the subsets of Ω such that

- $\bar{P}(\emptyset) = 0$;
- $\bar{P}(A) \leq \bar{P}(B)$ if $A \subseteq B$;
- if $(A_n)_{n \in \mathbb{N}}$ is a sequence of subsets of Ω , then $\bar{P}(\bigcup_n A_n) \leq \sum_n \bar{P}(A_n)$.

Proof. Monotonicity and $\bar{P}(\emptyset) = 0$ are obvious. So let (A_n) be a sequence of subsets of Ω . Let $\varepsilon > 0$, $n \in \mathbb{N}$, and let $(H^{n,m})_{m \in \mathbb{N}}$ be a sequence of $(\bar{P}(A_n) + \varepsilon 2^{-n-1})$ -admissible simple

strategies such that $\liminf_{m \rightarrow \infty} (\bar{P}(A_n) + \varepsilon 2^{-n-1} + (H^{n,m} \cdot S)_T) \geq \mathbf{1}_{A_n}$. Define for $m \in \mathbb{N}$ the $(\sum_n \bar{P}(A_n) + \varepsilon)$ -admissible simple strategy $G^m := \sum_{n=0}^m H^{n,m}$. Then by Fatou's lemma

$$\begin{aligned} \liminf_{m \rightarrow \infty} \left(\sum_{n=0}^{\infty} \bar{P}(A_n) + \varepsilon + (G^m \cdot S)_T \right) &\geq \sum_{n=0}^k (\bar{P}(A_n) + \varepsilon 2^{-n-1}) + \liminf_{m \rightarrow \infty} (H^{n,m} \cdot S)_T \\ &\geq \mathbf{1}_{\bigcup_{n=0}^k A_n} \end{aligned}$$

for all $k \in \mathbb{N}$. Since the left hand side does not depend on k , we can replace $\mathbf{1}_{\bigcup_{n=0}^k A_n}$ by $\mathbf{1}_{\bigcup_n A_n}$ and the proof is complete. \square

Maybe the most important property of \bar{P} is that there exists an arbitrage interpretation for sets with outer measure zero:

Lemma 2.4. *A set $A \subseteq \Omega$ is a null set if and only if there exists a sequence of 1-admissible simple strategies $(H^n)_n \subset \mathcal{H}_1$ such that*

$$\liminf_{n \rightarrow \infty} (1 + (H^n \cdot S)_T) \geq \infty \cdot \mathbf{1}_A(\omega), \quad (2)$$

where we recall that by convention $0 \cdot \infty = 0$.

Proof. If such a sequence exists, then we can scale it down by an arbitrary factor $\varepsilon > 0$ to obtain a sequence of strategies in \mathcal{H}_ε that superhedge $\mathbf{1}_A$, and therefore $\bar{P}(A) = 0$.

If conversely $\bar{P}(A) = 0$, then for every $n \in \mathbb{N}$ there exists a sequence of simple strategies $(H^{n,m})_{m \in \mathbb{N}} \subset \mathcal{H}_{2^{-n-1}}$ such that $2^{-n-1} + \liminf_{m \rightarrow \infty} (H^{n,m} \cdot \omega)_T \geq \mathbf{1}_A(\omega)$ for all $\omega \in \Omega$. Define $G^m := \sum_{n=0}^m H^{n,m}$, so that $G^m \in \mathcal{H}_1$. For every $k \in \mathbb{N}$ we obtain

$$\liminf_{m \rightarrow \infty} (1 + (G^m \cdot S)_T) \geq \sum_{n=0}^k (2^{-n-1} + \liminf_{m \rightarrow \infty} (H^{n,m} \cdot S)_T) \geq (k+1) \mathbf{1}_A.$$

Since the left hand side does not depend on k , the sequence (G^m) satisfies (2). \square

In other words, if a set A has outer measure 0, then we can make infinite profit by investing in the paths from A , without ever risking to lose more than the initial capital 1.

This motivates the following definition:

Definition 2.5. We say that a property (P) holds for *typical price paths* if the set A where (P) is violated is a null set.

The basic idea of Vovk, which we shall adopt in the following, is that we only need to concentrate on typical price paths. Indeed, “non-typical price path” can be excluded since they are in a certain sense “too good to be true”: they would allow investors to realize infinite profit while at the same time taking essentially no risk.

2.2 Arbitrage notions and link to classical mathematical finance

Before we continue, let us discuss different notions of arbitrage and argue that our outer measure is an interesting object to study. We start by observing that \bar{P} is an outer measure which simultaneously dominates all local martingale measures on Ω .

Proposition 2.6 ([Vov12], Lemma 6.3). *Let \mathbb{P} be a probability measure on (Ω, \mathcal{F}) , such that the coordinate process S is a \mathbb{P} -local martingale, and let $A \in \mathcal{F}$. Then $\mathbb{P}(A) \leq \overline{P}(A)$.*

Proof. Let $\lambda > 0$ and let $(H^n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_\lambda$ be such that $\liminf_n (\lambda + (H^n \cdot S)_T) \geq \mathbf{1}_A$. Then

$$\mathbb{P}(A) \leq \mathbb{E}_{\mathbb{P}}[\liminf_n (\lambda + (H^n \cdot S)_T)] \leq \liminf_n \mathbb{E}_{\mathbb{P}}[\lambda + (H^n \cdot S)_T] \leq \lambda,$$

where in the last step we used that $\lambda + (H^n \cdot S)$ is a nonnegative \mathbb{P} -local martingale and thus a \mathbb{P} -supermartingale. \square

This already indicates that \overline{P} -null sets are quite degenerate, in the sense that they are null sets under all local martingale measures. However, if that was the only reason for our definition of typical price paths, then a definition based on model free arbitrage opportunities would be equally valid. A map $X: \Omega \rightarrow [0, \infty)$ is a *model free arbitrage opportunity* if X is not identically 0 and if there exists $c > 0$ and a sequence $(H^n) \subseteq \mathcal{H}_c$ such that $\liminf_{n \rightarrow \infty} (H^n \cdot S)_T(\omega) = X(\omega)$ for all $\omega \in \Omega$. See [DH07, ABPS14] where (a similar) definition is used in the discrete time setting.

It might then appear more natural to say that a property holds for typical price paths if the indicator function of its complement is a model free arbitrage opportunity, rather than working with Definition 2.5. This “arbitrage definition” would also imply that any property which holds for typical price paths is almost surely satisfied under every local martingale measure. Nonetheless we decidedly claim that our definition is “the correct one”. First of all the arbitrage definition would make our life much more difficult, since it seems not very easy to work with. But of course this is only a convenience and cannot possibly serve as justification of our approach. Instead, we argue by relating the two notions to classical mathematical finance.

For that purpose recall the fundamental theorem of asset pricing [DS94]: If \mathbb{P} is a probability measure on (Ω, \mathcal{F}) under which S is a semimartingale, then there exists an equivalent measure \mathbb{Q} such that S is a \mathbb{Q} -local martingale if and only if S admits *no free lunch with vanishing risk (NFLVR)*. But (NFLVR) is equivalent to the two conditions *no arbitrage (NA)* (intuitively: no profit without risk) and *no arbitrage opportunities of the first kind (NA1)* (intuitively: no very large profit with a small risk). The (NA) property holds if for every $c > 0$ and every sequence $(H^n) \subseteq \mathcal{H}_c$ for which $\lim_{n \rightarrow \infty} (H^n \cdot S)_T(\omega)$ exists for all ω we have $\mathbb{P}(\lim_{n \rightarrow \infty} (H^n \cdot S)_T < 0) > 0$ or $\mathbb{P}(\lim_{n \rightarrow \infty} (H^n \cdot S)_T = 0) = 1$. The (NA1) property holds if $\{1 + (H \cdot S)_T : H \in \mathcal{H}_1\}$ is bounded in \mathbb{P} -probability, i.e. if

$$\lim_{c \rightarrow \infty} \sup_{H \in \mathcal{H}_1} \mathbb{P}(1 + (H \cdot S)_T \geq c) = 0.$$

Strictly speaking this is (NA1) with simple strategies, but as observed by [KP11] (NA1) and (NA1) with simple strategies are equivalent; see also [IP11]. In the case of continuous S , the equivalence of (NA1) and (NA1) with simple strategies had previously been shown by [Ank05], Corollary 8.3.2, although here the result is formulated in a slightly different language.

Now the arbitrage definition of typical price paths corresponds to (NA), while our definition corresponds to (NA1):

Proposition 2.7. *Let $A \in \mathcal{F}$ be a null set, and let \mathbb{P} be a probability measure on (Ω, \mathcal{F}) such that the coordinate process satisfies (NA1). Then $\mathbb{P}(A) = 0$.*

Proof. Let $(H^n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_1$ be such that $1 + \liminf_n (H^n \cdot S)_T \geq \infty \cdot \mathbf{1}_A$. Then for every $c > 0$

$$\mathbb{P}(A) = \mathbb{P}\left(A \cap \left\{ \liminf_{n \rightarrow \infty} (H^n \cdot S)_T > c \right\}\right) \leq \sup_{H \in \mathcal{H}_1} \mathbb{P}(\{(H \cdot S)_T > c\}).$$

By assumption, the right hand side converges to 0 as $c \rightarrow \infty$ and thus $\mathbb{P}(A) = 0$. \square

Remark 2.8. *Proposition 2.7 is actually a consequence of Proposition 2.6, because if S satisfies (NA1) under \mathbb{P} , then there exists a dominating measure $\mathbb{Q} \gg \mathbb{P}$, such that S is a \mathbb{Q} -local martingale. See [Ruf13] for the case of continuous S , and [IP11] for the general case.*

The crucial point is now that (NA1) is *the* essential property which every sensible market model has to satisfy, whereas (NA) is nice to have but not strictly necessary. Indeed, (NA1) is equivalent to the existence of an unbounded utility function such that the maximum expected utility is finite [KK07, IP11]. (NA) is what is needed in addition to (NA1) in order to obtain equivalent local martingale measures. But there are perfectly viable models which violate (NA), for example the three dimensional Bessel process. By working with the arbitrage definition of typical price paths, we would in a certain sense ignore these models.

2.3 A topology on path-dependent functionals

It will be very useful to introduce a topology on functionals on Ω . For that purpose let us identify $X, Y: \Omega \rightarrow \mathbb{R}$ if $X = Y$ for typical price paths. Clearly this defines an equivalence relation, and we write \bar{L}_0 for the space of equivalence classes. We then introduce the analog of convergence in probability in our context: (X_n) converges in outer measure to X if

$$\lim_{n \rightarrow \infty} \bar{P}(|X_n - X| > \varepsilon) = 0 \quad \text{for all } \varepsilon > 0.$$

We follow [Vov12] in defining an expectation operator. If $X: \Omega \rightarrow [0, \infty]$, then

$$\bar{E}[X] := \inf \left\{ \lambda > 0 : \exists (H^n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_\lambda \text{ s.t. } \liminf_{n \rightarrow \infty} (\lambda + (H^n \cdot S)_T(\omega)) \geq X(\omega) \forall \omega \in \Omega \right\}.$$

In particular, $\bar{P}(A) = \bar{E}[\mathbf{1}_A]$. The expectation \bar{E} is countably subadditive, monotone, and positively homogeneous. It is an easy exercise to verify that

$$d(X, Y) := \bar{E}[|X - Y| \wedge 1]$$

defines a metric on \bar{L}_0 .

Lemma 2.9. *The distance d metrizes the convergence in outer measure. More precisely, a sequence (X_n) converges to X in outer measure if and only if $\lim_n d(X_n, X) = 0$. Moreover, (\bar{L}_0, d) is a complete metric space.*

Proof. The arguments are the same as in the classical setting. Using subadditivity and monotonicity of the expectation operator, we have

$$\varepsilon \bar{P}(|X_n - X| \geq \varepsilon) \leq \bar{E}[|X_n - X| \wedge 1] \leq \bar{P}(|X_n - X| > \varepsilon) + \varepsilon$$

for all $\varepsilon \in (0, 1]$, showing that convergence in outer measure is equivalent to convergence with respect to d .

As for completeness, let (X_n) be a Cauchy sequence with respect to d . Then there exists a subsequence (X_{n_k}) such that $d(X_{n_k}, X_{n_{k+1}}) \leq 2^{-k}$ for all k , so that

$$\overline{E} \left[\sum_k (|X_{n_k} - X_{n_{k+1}}| \wedge 1) \right] \leq \sum_k \overline{E}[|X_{n_k} - X_{n_{k+1}}| \wedge 1] = \sum_k d(X_{n_k}, X_{n_{k+1}}) < \infty,$$

which means that (X_{n_k}) converges for typical price paths. Define $X := \liminf_k X_{n_k}$. Then we have for all n and k

$$d(X_n, X) \leq d(X_n, X_{n_k}) + d(X_{n_k}, X) \leq d(X_n, X_{n_k}) + \sum_{\ell \geq k} d(X_{n_\ell}, X_{n_{\ell+1}}) \leq d(X_n, X_{n_k}) + 2^{-k}.$$

Choosing n and k large, we see that $d(X_n, X)$ tends to 0. \square

2.4 Relation to Vovk's outer measure

Our definition of the outer measure \overline{P} is not exactly the same as Vovk's [Vov12]. We find our definition more intuitive and it also seems to be easier to work with. However, since we rely on some of the results established by Vovk, let us compare the two notions.

For $\lambda > 0$, Vovk defines the set of processes

$$\mathcal{S}_\lambda := \left\{ \sum_{k=0}^{\infty} H^k : H^k \in \mathcal{H}_{\lambda_k}, \lambda_k > 0, \sum_{k=0}^{\infty} \lambda_k = \lambda \right\}.$$

For every $G = \sum_{k \geq 0} H^k \in \mathcal{S}_\lambda$, every $\omega \in \Omega$ and every $t \in [0, T]$, the integral

$$(G \cdot S)_t(\omega) := \sum_{k \geq 0} (H^k \cdot S)_t(\omega) = \sum_{k \geq 0} (\lambda_k + (H^k \cdot S)_t(\omega)) - \lambda$$

is well defined and takes values in $[-\lambda, \infty]$. Vovk then defines for $A \subseteq \Omega$ the cheapest superhedging price as

$$\overline{Q}(A) := \inf \{ \lambda > 0 : \exists G \in \mathcal{S}_\lambda \text{ s.t. } \lambda + (G \cdot S)_T \geq \mathbf{1}_A \}.$$

This definition corresponds to the usual construction of an outer measure from an outer content (i.e. an outer measure which is only finitely subadditive and not countably subadditive); see [Fol99], Chapter 1.4, or [Tao11], Chapter 1.7. Here, the outer content is given by the cheapest superhedging price using only simple strategies. It is easy to see that \overline{P} is dominated by \overline{Q} :

Lemma 2.10. *Let $A \subseteq \Omega$. Then $\overline{P}(A) \leq \overline{Q}(A)$.*

Proof. Let $G = \sum_k H^k$, with $H^k \in \mathcal{H}_{\lambda_k}$ and $\sum_k \lambda_k = \lambda$, and assume that $\lambda + (G \cdot S)_T \geq \mathbf{1}_A$. Then $(\sum_{k=0}^n H^k)_{n \in \mathbb{N}}$ defines a sequence of simple strategies in \mathcal{H}_λ , such that

$$\liminf_{n \rightarrow \infty} \left(\lambda + \left(\left(\sum_{k=0}^n H^k \right) \cdot S \right)_T \right) = \lambda + (G \cdot S)_T \geq \mathbf{1}_A.$$

So if $\overline{Q}(A) < \lambda$, then also $\overline{P}(A) \leq \lambda$, and therefore $\overline{P}(A) \leq \overline{Q}(A)$. \square

Corollary 2.11. *For every $p > 2$, the set $A_p := \{\omega \in \Omega : \|S(\omega)\|_{p\text{-var}} = \infty\}$ has outer measure zero, that is $\overline{P}(A_p) = 0$.*

Proof. Theorem 1 of Vovk [Vov08] states that $\overline{Q}(A_p) = 0$, so $\overline{P}(A_p) = 0$ by Lemma 2.10. \square

It is a remarkable result of [Vov12] that if $\Omega = C([0, \infty), \mathbb{R})$ (i.e. if the asset price process is one-dimensional), and if $A \subseteq \Omega$ is “invariant under time changes” and such that $S_0(\omega) = 0$ for all $\omega \in A$, then $A \in \mathcal{F}$ and $\overline{Q}(A) = \mathbb{P}(A)$, where \mathbb{P} denotes the Wiener measure. This can be interpreted as a pathwise Dambis Dubins-Schwarz theorem.

3 Model free Itô integration

The present section is devoted to the construction of a model free Itô integral. The main ingredient is a (weak) type of model free Itô isometry, which allows us to estimate the integral against a step function in terms of the amplitude of the step function and the quadratic variation of the price path. Based on the topology introduced in Section 2.3, it is then easy to extend the integral to càdlàg integrands by a continuity argument.

Since we are in an unusual setting, let us spell out the following standard definitions:

Definition 3.1. A process $F: \Omega \times [0, T] \rightarrow \mathbb{R}^d$ is called *adapted* if the random variable $\omega \mapsto F_t(\omega)$ is \mathcal{F}_t -measurable for all $t \in [0, T]$.

The process F is said to be *càdlàg* if the sample path $t \mapsto F_t(\omega)$ is càdlàg for all $\omega \in \Omega$.

To prove our weak Itô isometry, we will need an appropriate sequence of stopping times: Let $\omega = (\omega^1, \dots, \omega^d) \in \Omega$ and $n \in \mathbb{N}$. For each $i = 1, \dots, d$ define inductively

$$\sigma_0^{n,i}(\omega) := 0, \quad \sigma_{k+1}^{n,i}(\omega) := \inf \{t \geq \sigma_k^{n,i} : |\omega^i(t) - \omega^i(\sigma_k^{n,i})| \geq 2^{-n}\}, \quad k \in \mathbb{N}.$$

Since we are working with continuous paths and we are considering entrance times into closed sets, the maps $(\sigma^{n,i})$ are indeed stopping times, despite the fact that (\mathcal{F}_t) is neither complete nor right-continuous. Denote $\pi^{n,i} := \{\sigma_k^{n,i} : k \in \mathbb{N}\}$. To obtain an increasing sequence of partitions, we take the union of the $(\pi^{n,i})$, that is we define $\sigma_0^n := 0$ and then

$$\sigma_{k+1}^n(\omega) := \min \left\{ t > \sigma_k^n(\omega) : t \in \bigcup_{i=1}^d \pi^{n,i}(\omega) \right\}, \quad k \in \mathbb{N}, \quad (3)$$

and we write $\pi^n := \{\sigma_k^n : k \in \mathbb{N}\}$ for the corresponding partition.

Lemma 3.2 ([Vov11], Theorem 4.1). *For typical price paths $\omega \in \Omega$, the quadratic variation along $(\pi^{n,i}(\omega))_{n \in \mathbb{N}}$ exists. That is,*

$$V_t^{n,i}(\omega) := \sum_{k=0}^{\infty} (\omega^i(\sigma_{k+1}^{n,i} \wedge t) - \omega^i(\sigma_k^{n,i} \wedge t))^2, \quad t \in [0, T], \quad n \in \mathbb{N},$$

converges uniformly to a function $\langle S^i \rangle(\omega) \in C([0, T], \mathbb{R})$ for all $i \in \{1, \dots, d\}$.

For later reference, let us estimate $N_t^n := \max\{k \in \mathbb{N} : \sigma_k^n \leq t\}$, the number of stopping times $\sigma_k^n \neq 0$ in π^n with values in $[0, t]$:

Lemma 3.3. For all $\omega \in \Omega$, $n \in \mathbb{N}$, and $t \in [0, T]$, we have

$$2^{-2n} N_t^n(\omega) \leq \sum_{i=1}^d V_t^{n,i}(\omega) =: V_t^n(\omega).$$

Proof. For $i \in \{1, \dots, d\}$ define $N_t^{n,i} := \max\{k \in \mathbb{N} : \sigma_k^{n,i} \leq t\}$. Since ω^i is continuous, we have $|\omega^i(\sigma_{k+1}^{n,i}) - \omega^i(\sigma_k^{n,i})| = 2^{-n}$ as long as $\sigma_{k+1}^{n,i} \leq T$. Therefore, we obtain

$$N_t^n(\omega) \leq \sum_{i=1}^d N_t^{n,i}(\omega) = \sum_{i=1}^d \sum_{k=0}^{N_t^{n,i}(\omega)-1} \frac{1}{2^{-2n}} (\omega(\sigma_{k+1}^{n,i}) - \omega(\sigma_k^{n,i}))^2 \leq 2^{2n} \sum_{i=1}^d V_t^{n,i}(\omega).$$

□

We will start by constructing the integral against step functions, which are defined similarly as simple strategies, except possibly unbounded: A process $F: \Omega \times [0, T] \rightarrow \mathbb{R}^d$ is called a *step function* if there exist stopping times $0 = \tau_0 < \tau_1 < \dots$, and \mathcal{F}_{τ_n} -measurable functions $F_n: \Omega \rightarrow \mathbb{R}^d$, such that for every $\omega \in \Omega$ we have $\tau_n(\omega) = \infty$ for all but finitely many n , and such that

$$F_t(\omega) = \sum_{n=0}^{\infty} F_n(\omega) \mathbf{1}_{[\tau_n(\omega), \tau_{n+1}(\omega))}(t).$$

For notational convenience we are now considering the interval $[\tau_n(\omega), \tau_{n+1}(\omega))$ which is closed on the right hand side. This allows us define the integral

$$(F \cdot S)_t := \sum_{n=0}^{\infty} F_n S_{\tau_n \wedge t, \tau_{n+1} \wedge t} = \sum_{n=0}^{\infty} F_{\tau_n} S_{\tau_n \wedge t, \tau_{n+1} \wedge t}, \quad t \in [0, T].$$

The following lemma will be the main building block in the construction of our integral.

Lemma 3.4 (Model free Itô isometry). *Let $a > 0$ and let F be a step function. Then for all $a, b, c > 0$ we have*

$$\bar{P}(\{\| (F \cdot S) \|_{\infty} \geq ab\sqrt{c}\} \cap \{\| F(\omega) \|_{\infty} \leq a\} \cap \{\langle S \rangle_T \leq c\}) \leq 2 \exp(-b^2/(2d)),$$

where the set $\{\langle S \rangle_T \leq c\}$ should be read as $\{\langle S \rangle_T = \lim_n V_T^n$ exists and satisfies $\langle S \rangle_T \leq c\}$.

Proof. Assume $F_t = \sum_{n=0}^{\infty} F_n \mathbf{1}_{[\tau_n, \tau_{n+1})}(t)$ and set $\tau_a := \inf\{t > 0 : |F_t| \geq a\}$. Let $n \in \mathbb{N}$ and define $\rho_0^n := 0$ and then for $k \in \mathbb{N}$

$$\rho_{k+1}^n := \min\{t > \rho_k^n : t \in \pi^n \cup \{\tau_m : m \in \mathbb{N}\}\},$$

where we recall that $\pi^n = \{\sigma_k^n : k \in \mathbb{N}\}$ is the n -th generation dyadic partition generated by S . For $t \in [0, T]$, we have $(F \cdot S)_{\tau_a \wedge t} = \sum_k F_{\rho_k^n} S_{\tau_a \wedge \rho_k^n \wedge t, \tau_a \wedge \rho_{k+1}^n \wedge t}$, and by the definition of $\pi^n(\omega)$ and τ_a we get

$$\sup_{t \in [0, T]} |F_{\rho_k^n} S_{\tau_a \wedge \rho_k^n \wedge t, \tau_a \wedge \rho_{k+1}^n \wedge t}| \leq a\sqrt{d}2^{-n}.$$

Hence, the pathwise Hoeffding inequality, Lemma A.1 in Appendix A, yields for every $\lambda \in \mathbb{R}$ the existence of a 1-admissible simple strategy $H^{\lambda, n} \in \mathcal{H}_1$ such that

$$1 + (H^{\lambda, n} \cdot S)_t \geq \exp\left(\lambda(F \cdot S)_{\tau_a \wedge t} - \frac{\lambda^2}{2}(N_t^{(\rho^n)} + 1)2^{-2n}a^2d\right) =: \mathcal{E}_{\tau_a \wedge t}^{\lambda, n}$$

for all $t \in [0, T]$, where

$$N_t^{(\rho^n)} := \max\{k : \rho_k^n \leq t\} \leq N_t^n + N_t^{(\tau)} := N_t^n + \max\{k : \tau_k \leq t\}.$$

By Lemma 3.3, we have $N_t^n \leq 2^{2n} V_t^n$, so that

$$\mathcal{E}_{\tau_a \wedge t}^{\lambda, n} \geq \exp\left(\lambda(F \cdot S)_t - \frac{\lambda^2}{2} V_T^n a^2 d - \frac{\lambda^2}{2} (N_T^{(\tau)} + 1) 2^{-2n} a^2 d\right).$$

If now $\|(F \cdot S)\|_\infty \geq ab\sqrt{c}$, $\|F(\omega)\|_\infty \leq a$ and $\langle S \rangle_T \leq c$, then

$$\liminf_{n \rightarrow \infty} \sup_{t \in [0, T]} \frac{\mathcal{E}_t^{\lambda, n} + \mathcal{E}_t^{-\lambda, n}}{2} \geq \frac{1}{2} \exp\left(\lambda ab\sqrt{c} - \frac{\lambda^2}{2} ca^2 d\right).$$

The argument inside the exponential is maximized for $\lambda = b/(a\sqrt{cd})$, in which case we obtain $1/2 \exp(b^2/(2d))$. The statement now follows from Remark 2.2. \square

Of course, we did not actually establish an isometry but only an upper bound for the integral. But this estimate is the key ingredient which allows us to extend the model free Itô integral to more general integrands, and it is this analogy to the classical setting that the term “model free Itô isometry” alludes to.

Let us extend the topology of Section 2.3 to processes: we identify $X, Y : \Omega \times [0, T] \rightarrow \mathbb{R}^m$ if for typical price paths we have $X_t = Y_t$ for all $t \in [0, T]$, and we write $\bar{L}_0([0, T], \mathbb{R}^m)$ for the resulting space of equivalence classes which we equip with the distance

$$d_\infty(X, Y) := \bar{E}[\|X - Y\|_\infty \wedge 1].$$

Ideally, we would like the stochastic integral on step functions to be continuous with respect to d_∞ . However, using Proposition 2.6 it is easy to see that $\bar{P}(\|((1/n) \cdot S)\|_\infty > \varepsilon) = 1$ for all $n \in \mathbb{N}$ and $\varepsilon > 0$. This is why we also introduce for $c > 0$ the pseudometric

$$d_c(X, Y) := \bar{E}[(\|X - Y\|_\infty \wedge 1) \mathbf{1}_{\langle S \rangle_T \leq c}] \leq d_\infty(X, Y),$$

and then

$$d_{\text{cpct}}(X, Y) := \sum_{n=1}^{\infty} 2^{-n} d_{2^n}(X, Y) \leq d_\infty(X, Y).$$

For step functions F, G , we get from Lemma 3.4

$$\begin{aligned} d_c((F \cdot S), (G \cdot S)) &\leq \bar{P}(\{\|((F - G) \cdot S)\|_\infty \geq ab\sqrt{c}\} \cap \{\|F - G\|_\infty \leq a\} \cap \{\langle S \rangle_T \leq c\}) \\ &\quad + \frac{d_c(F, G)}{a} + ab\sqrt{c} \\ &\leq 2 \exp\left(-\frac{b^2}{2d}\right) + \frac{d_c(F, G)}{a} + ab\sqrt{c} \end{aligned}$$

whenever $a, b > 0$. Setting $a = \sqrt{d_c(F, G)}$ and $b = \sqrt{d|\log a|}$, we deduce that

$$d_c((F \cdot S), (G \cdot S)) \lesssim (1 + \sqrt{c}) d_c(F, G)^{1/2-\varepsilon} \quad (4)$$

for all $\varepsilon > 0$, and in particular

$$d_{\text{cpct}}((F \cdot S), (G \cdot S)) \lesssim \sum_{n=1}^{\infty} 2^{-n/2} d_{2^n}(F, G)^{1/2-\varepsilon} \lesssim d_\infty(F, G)^{1/2-\varepsilon}.$$

Theorem 3.5. *Let F be an adapted, càdlàg process with values in \mathbb{R}^d . Then there exists $\int FdS \in \overline{L}_0([0, T], \mathbb{R})$ such that for every sequence of step functions (F^n) with $\lim_n d_\infty(F^n, F) = 0$ we have $\lim_n d_{\text{cpct}}((F^n \cdot S), \int FdS) = 0$. The integral process $\int FdS$ is continuous for typical price paths, and there exists a representative $\int FdS$ which is adapted, although it may take the values $\pm\infty$. We usually write $\int_0^t F_s dS_s := \int FdS(t)$, and we call $\int FdS$ the model free Itô integral of F with respect to S .*

The map $F \mapsto \int FdS$ is linear, satisfies

$$d_{\text{cpct}}\left(\int FdS, \int GdS\right) \lesssim d_\infty(F, G)^{1/2-\varepsilon}$$

for all $\varepsilon > 0$, and the model free Itô isometry extends to this setting:

$$\overline{P}\left(\left\{\left\|\int FdS\right\|_\infty \geq ab\sqrt{c}\right\} \cap \{\|F\|_\infty \leq a\} \cap \{\langle S \rangle_T \leq c\}\right) \leq 2\exp(-b^2/(2d))$$

for all $a, b, c > 0$.

Proof. Everything follows in a straightforward way from (4) in combination with Lemma 2.9. We have to use the fact that F is adapted and càdlàg in order to approximate it uniformly by step functions. \square

Another simple consequence of the model free Itô isometry is a strengthened version of Karandikar's [Kar95] pathwise Itô integral.

Corollary 3.6. *In the setting of Theorem 3.5, let $(F^m)_{m \in \mathbb{N}}$ be a sequence of step functions with $\|F^m(\omega) - F(\omega)\|_\infty \leq c_m$ for all $\omega \in \Omega$ and all $m \in \mathbb{N}$. Then for typical price paths ω there exists a constant $C(\omega) > 0$ such that*

$$\left\|\left(F^m \cdot S\right)(\omega) - \int FdS(\omega)\right\|_\infty \leq C(\omega)c_m\sqrt{\log m} \quad (5)$$

for all $m \in \mathbb{N}$. So, if $(c_m\sqrt{\log m})$ converges to 0, then for typical price paths $(F^m \cdot S)$ converges to $\int FdS$.

Proof. For $c > 0$ the model free Itô isometry gives

$$\overline{P}\left(\left\{\left\|\left(F^m \cdot S\right) - \int FdS\right\|_\infty \geq c_m\sqrt{4d\log m}\sqrt{c}\right\} \cap \{\langle S \rangle_T \leq c\}\right) \leq \frac{1}{m^2}.$$

Since this is summable in m , the claim follows from Borel Cantelli (which only requires countable subadditivity). \square

Remark 3.7. *The speed of convergence (5) is better than the one that can be obtained using the arguments in [Kar95], where the summability of (c_m) is needed.*

4 Rough path integration for typical price paths

Our second approach to model free stochastic integration is based on the rough path integral, which has the advantage of being a continuous linear operator between Banach spaces. The disadvantage is that we have to restrict the set of integrands to those “locally looking like S ”, modulo a smoother remainder. Our two main results in this section are that every typical price path has a naturally associated Itô rough path, and that the rough path integral can be constructed as limit of Riemann sums.

Let us start by recalling the basic definitions and results of rough path theory.

4.1 The Lyons-Gubinelli rough path integral

Here we follow more or less the lecture notes [FH14], to which we refer for a gentle introduction to rough paths. More advanced monographs are [LQ02, LCL07, FV10]. The main difference to [FH14] in the derivation below is that we use p -variation to describe the regularity, and not Hölder continuity, because it is not true that all typical price paths are Hölder continuous. Also, we make an effort to give reasonably sharp results, whereas in [FH14] the focus lies more on the pedagogical presentation of the material. We stress that in this subsection we are merely collecting classical results.

Definition 4.1. A *control function* is a continuous map $c: \Delta_T \rightarrow [0, \infty)$ with $c(t, t) = 0$ for all $t \in [0, T]$ and such that $c(s, u) + c(u, t) \leq c(s, t)$ for all $0 \leq s \leq u \leq t \leq T$.

Observe that if $f: [0, T] \rightarrow \mathbb{R}^d$ satisfies $|f_{s,t}|^p \leq c(s, t)$ for all $(s, t) \in \Delta_T$, then the p -variation of f is bounded from above by $c(0, T)$.

Definition 4.2. Let $p \in (2, 3)$. A p -rough path is a map $\mathbb{S} = (S, A): \Delta_T \rightarrow \mathbb{R}^d \times \mathbb{R}^{d \times d}$ such that *Chen's relation*

$$S^i(s, t) = S^i(s, u) + S^i(u, t) \quad \text{and} \quad A^{i,j}(s, t) = A^{i,j}(s, u) + A^{i,j}(u, t) + S^i(s, u)S^j(u, t)$$

holds for all $1 \leq i, j \leq d$ and $0 \leq s \leq u \leq t \leq T$ and such that there exists a control function c with

$$|S(s, t)|^p + |A(s, t)|^{p/2} \leq c(s, t)$$

(in other words S has finite p -variation and A has finite $p/2$ -variation). In that case we call A the *area* of S .

Remark 4.3. *Chen's relation simply states that S is the increment of a function, that is $S(s, t) = S(0, t) - S(0, s) =: S(t) - S(s)$, and that for all i, j there exists a function $f^{i,j}: [0, T] \rightarrow \mathbb{R}$ such that $A^{i,j}(s, t) = f^{i,j}(t) - f^{i,j}(s) - S^i(s)S^j_{s,t}$. Indeed, it suffices to set $f^{i,j}(t) = A^{i,j}(0, t) + S^i(0)S^j_{0,t}$.*

Remark 4.4. *The (strictly speaking incorrect) name "area" stems from the fact that if S is smooth and two-dimensional and if*

$$A^{i,j}(s, t) = \int_s^t \int_s^{r_2} dS^i(r_1) dS^j(r_2) = \int_s^t S^i_{s,r_2} dS^j(r_2),$$

then the antisymmetric part of $A(s, t)$ corresponds to the algebraic area enclosed by the curve $(S(r))_{r \in [s,t]}$. It is a deep insight of Lyons [Lyo98], proving a conjecture of Föllmer, that the area is exactly the additional information which is needed to solve differential equations driven by S in a pathwise continuous manner, and to construct stochastic integrals as continuous maps. Actually, [Lyo98] solves a much more general problem and proves that if the driving signal is of finite p -variation for some $p > 1$, then it has to be equipped with the iterated integrals up to order $\lfloor p \rfloor - 1$ to obtain a continuous integral map. The for us relevant case $p \in (2, 3)$ was already treated in [Lyo95a].

Example 4.5. If S is a continuous semimartingale and if we set $S(s, t) = S_{s,t}$ as well as

$$A^{i,j}(s, t) = \int_s^t \int_s^{r_2} dS^i(r_1) dS^j(r_2) = \int_s^t S^i_{s,r_2} dS^j(r_2),$$

where the integral can be understood either in the Itô or in the Stratonovich sense, then almost surely $\mathbb{S} = (S, A)$ is a p -rough path for all $p \in (2, 3)$. This is shown in [CL05], and we will give a simplified model free proof below (indeed we will show that every typical price path is a p -rough path for all $p \in (2, 3)$, from where the statement about continuous semimartingales easily follows).

From now on we fix $p > 2$ and we assume that \mathbb{S} is a p -rough path. Gubinelli [Gub04] observed that for every rough path there is a naturally associated Banach space of integrands, the space of *controlled paths*. Heuristically, a path F is controlled by S , if it locally “looks like S ”, modulo a smooth remainder. The precise definition is:

Definition 4.6. Let $q > 0$ be such that $2/p + 1/q > 1$. Let $F: [0, T] \rightarrow \mathbb{R}^n$ and $F': [0, T] \rightarrow \mathbb{R}^{n \times d}$. We say that the pair (F, F') is *controlled* by S if the *derivative* F' has finite q -variation, and the *remainder* $R_F: \Delta_T \rightarrow \mathbb{R}^n$, defined by

$$R_F(s, t) = F_{s,t} - F'_s S_{s,t},$$

has finite r -variation for $1/r = 1/p + 1/q$. In this case, we write $(F, F') \in \mathcal{C}_{\mathbb{S}}^q = \mathcal{C}_{\mathbb{S}}^q(\mathbb{R}^n)$, and define

$$\|(F, F')\|_{\mathcal{C}_{\mathbb{S}}^q} := \|F'\|_{q\text{-var}} + \|R_F\|_{r\text{-var}}.$$

Equipped with the norm $|F_0| + |F'_0| + \|(F, F')\|_{\mathcal{C}_{\mathbb{S}}^q}$, the space $\mathcal{C}_{\mathbb{S}}^q$ is a Banach space.

Naturally, the function F' should be interpreted as the derivative of F with respect to S . The reason for considering couples (F, F') and not just functions F is that the smoothness requirement on the remainder R_F usually does not determine F' uniquely for a given path F . For example, if F and S both have finite r -variation rather than just finite p -variation, then for every F' of finite q -variation we have $(F, F') \in \mathcal{C}_{\mathbb{S}}^q$.

Note that we do not require F or F' to be continuous. We will point out in Remark 4.10 below why this does not pose any problem.

To gain a more “quantitative” feeling for the condition on q , let us assume for the moment that we can choose $p > 2$ arbitrarily close to 2 (which is the case in the example of a continuous semimartingale rough path). Then $2/p + 1/q > 1$ as long as $q > 0$, so that the derivative F' may essentially be as irregular as we want. The remainder R_F has to be of finite r -variation for $1/r = 1/p + 1/q$, so in other words it should be of finite r -variation for some $r < 2$ and thus slightly more regular than the sample path of a continuous local martingale.

Example 4.7. Let $\varepsilon \in (0, 1]$ be such that $(2+\varepsilon)/p > 1$. Let $\varphi \in C_b^{1+\varepsilon}$ and define $F_s := \varphi(S_s)$ and $F'_s := \varphi'(S_s)$. Then $(F, F') \in \mathcal{C}_{\mathbb{S}}^{p/\varepsilon}$: Clearly F' has finite p/ε -variation. For the remainder, we have

$$|R_F(s, t)|^{p/(1+\varepsilon)} = |\varphi(S_t) - \varphi(S_s) - \varphi'(S_s)S_{s,t}|^{p/(1+\varepsilon)} \leq \|\varphi\|_{C_b^{1+\varepsilon}} c(s, t),$$

where c is a control function for S . As the image of the continuous path S is compact, it is not actually necessary to assume that φ is bounded. We may always consider a $C^{1+\varepsilon}$ function ψ of compact support, such that ψ agrees with φ on the image of S .

This example shows that in general $R_F(s, t)$ is not a path increment of the form $R_F(s, t) = G(t) - G(s)$ for some function G defined on $[0, T]$, but really a function of two variables.

Example 4.8. Let G be a path of finite r -variation for some r with $1/p + 1/r > 1$. Setting $(F, F') = (G, 0)$, we obtain a controlled path in \mathcal{C}_S^q , where $1/q = 1/r - 1/p$. In combination with Theorem 4.9 below, this example shows in particular that the controlled rough path integral extends the Young integral and the Riemann-Stieltjes integral.

The basic idea of rough path integration is that if we already know how to define $\int S dS$, and if F looks like S on small scales, then we should be able to define $\int F dS$ as well. The precise result is given by the following theorem:

Theorem 4.9 (Theorem 4.9 in [FH14], see also [Gub04], Theorem 1). *Let q be such that $2/p + 1/q > 1$. Let $(F, F') \in \mathcal{C}_S^q$. Then there exists a unique function $\int F dS \in C([0, T], \mathbb{R}^n)$ which satisfies*

$$\left| \int_s^t F_u dS_u - F_s S_{s,t} - F'_s A(s, t) \right| \lesssim \|S\|_{p\text{-var}, [s,t]} \|R_F\|_{r\text{-var}, [s,t]} + \|A\|_{p/2\text{-var}, [s,t]} \|F'\|_{q\text{-var}, [s,t]}$$

for all $(s, t) \in \Delta_T$. The integral is given as limit of the compensated Riemann sums

$$\int_0^t F_u dS_u = \lim_{m \rightarrow \infty} \sum_{[s_1, s_2] \in \pi^m} [F_{s_1} S_{s_1, s_2} + F'_{s_1} A(s_1, s_2)], \quad (6)$$

where (π^m) is any sequence of partitions of $[0, t]$ with mesh size going to 0.

The map $(F, F') \mapsto (G, G') := (\int F_u dS_u, F)$ is continuous from \mathcal{C}_S^q to \mathcal{C}_S^p and satisfies

$$\|(G, G')\|_{\mathcal{C}_S^p} \lesssim \|F\|_{p\text{-var}} + (\|F'\|_\infty + \|F'\|_{q\text{-var}}) \|A\|_{p/2\text{-var}} + \|S\|_{p\text{-var}} \|R_F\|_{r\text{-var}}.$$

Remark 4.10. *To the best of our knowledge, there is no publication in which the controlled path approach to rough paths is formulated using p -variation regularity. The references on the subject all work with Hölder continuity. But in the p -variation setting, all the proofs work exactly as in the Hölder setting, and it is a simple exercise to translate the proof of Theorem 4.9 in [FH14] (which is based on Young's maximal inequality which we will encounter below) to obtain Theorem 4.9.*

There is only one small pitfall: We did not require F or F' to be continuous. The rough path integral for discontinuous functions is somewhat tricky, see [Wil01]. But here we do not run into any problems, because the integrand $\mathbb{S} = (S, A)$ is continuous. The construction based on Young's maximal inequality works as long as integrand and integrator have no common discontinuities, see the Theorem on page 264 of [You36].

If now $\varphi \in C_b^{1+\varepsilon}$ for some $\varepsilon > 0$, then using a Taylor expansion one can show that there exist $p > 2$ and $q > 0$ with $2/p + 1/q > 0$, such that $(F, F') \mapsto (\varphi(F), \varphi'(F)F')$ is a locally bounded map from \mathcal{C}_S^p to \mathcal{C}_S^q . Combining this with the fact that the rough path integral is a bounded map from \mathcal{C}_S^q to \mathcal{C}_S^p , it is not hard to prove the *existence* of solutions to the rough differential equation

$$dX_t = \varphi(X_t) dS_t, \quad X_0 = x_0, \quad (7)$$

$t \in [0, T]$, where $X \in \mathcal{C}_S^p$, $\int \varphi(X_t) dS_t$ denotes the rough path integral, and S is a typical price path. Similarly, if $\varphi \in C_b^{2+\varepsilon}$, then the map $(F, F') \mapsto (\varphi(F), \varphi'(F)F')$ is a locally Lipschitz continuous from \mathcal{C}_S^p to \mathcal{C}_S^q , and this yields the *uniqueness* of the solution to (7) – at least among the functions in the Banach space \mathcal{C}_S^p . See Section 5.3 of [Gub04] for details.

A remark is in order about the stringent regularity requirements on φ . In the classical Itô theory of SDEs, the function φ is only required to be Lipschitz continuous. But to solve a Stratonovich SDE, we need better regularity of φ . This is natural, because the Stratonovich SDE can be rewritten as an Itô SDE with a Stratonovich correction term: the equations

$$\begin{aligned} dX_t &= \varphi(X_t) \circ dW_t \quad \text{and} \\ dX_t &= \varphi(X_t) dW_t + \frac{1}{2} \varphi'(X_t) \varphi(X_t) dt \end{aligned}$$

are equivalent (where W is a standard Brownian motion, dW_t denotes Itô integration, and $\circ dW_t$ denotes Stratonovich integration). To solve the second equation, we need $\varphi' \varphi$ to be Lipschitz continuous, which is always satisfied if $\varphi \in C_b^2$. But rough path theory cannot distinguish between Itô and Stratonovich integrals: If we define the area of W using Itô (respectively Stratonovich) integration, then the rough path solution of the equation will coincide with the Itô (respectively Stratonovich) solution. So in the rough path setting, the function φ should satisfy at least the same requirements as in the Stratonovich setting. The regularity requirements on φ are essentially sharp, see [Dav07], but the boundedness assumption can be relaxed, see [Lej12]. See also Section 10.5 of [FV10] for a slight relaxation of the regularity requirements in the Brownian case.

Of course, the most interesting result of rough path theory is that the solution to a rough differential equation depends continuously on the driving signal. This is a consequence of the following observation:

Proposition 4.11 (Proposition 9.1 of [FH14]). *Let $p > 2$ and $q > 0$ with $2/p + 1/q > 0$. Let $\mathbb{S} = (S, A)$ and $\tilde{\mathbb{S}} = (\tilde{S}, \tilde{A})$ be two rough paths of finite p -variation, let $(F, F') \in \mathcal{C}_{\mathbb{S}}^q$ and $(\tilde{F}, \tilde{F}') \in \mathcal{C}_{\tilde{\mathbb{S}}}^q$, and let $(s, t) \in \Delta_T$. Then for every $M > 0$ there exists $C_M > 0$ such that*

$$\begin{aligned} \left\| \int_0^{\cdot} F_s dS_s - \int_0^{\cdot} \tilde{F}_s d\tilde{S}_s \right\|_{p\text{-var}} &\leq C_M \left(|F_0 - \tilde{F}_0| + |F'_0 - \tilde{F}'_0| + \|F' - \tilde{F}'\|_{q\text{-var}} \right. \\ &\quad \left. + \|R_F - R_{\tilde{F}}\|_{r\text{-var}} + \|S - \tilde{S}\|_{p\text{-var}} + \|A - \tilde{A}\|_{p/2\text{-var}} \right), \end{aligned}$$

as long as

$$\max\{|F'_0| + \|(F, F')\|_{\mathcal{C}_{\mathbb{S}}^q}, |\tilde{F}'_0| + \|(\tilde{F}, \tilde{F}')\|_{\mathcal{C}_{\tilde{\mathbb{S}}}^q}, \|S\|_{p\text{-var}}, \|A\|_{p/2\text{-var}}, \|\tilde{S}\|_{p\text{-var}}, \|\tilde{A}\|_{p/2\text{-var}}\} \leq M.$$

In other words, the rough path integral depends on integrand and integrator in a locally Lipschitz continuous way, and therefore it is no surprise that the solutions to differential equations driven by rough paths depend continuously on the signal.

4.2 Typical price paths as rough paths

Our second approach to stochastic integration in model free financial mathematics is based on the rough path integral. Here we show that for every typical price path, the pair (S, A) is a p -rough path for all $p \in (2, 3)$, where

$$A(s, t) = \int_s^t S_{s,r} dS_r := \int_s^t S_{s,r} \otimes dS_r := \left(\int_s^t S_r^i dS_r^j - S_s^i S_{s,t}^j \right)_{1 \leq i, j \leq d}.$$

The main ingredient in the proof will be our speed of convergence (5).

Theorem 4.12. For $(s, t) \in \Delta_T$, $\omega \in \Omega$, and $i, j \in \{1, \dots, d\}$ define

$$A_{s,t}^{i,j}(\omega) := \int_s^t S_r^i dS_r^j(\omega) - S_s^i(\omega) S_{s,t}^j(\omega) := \int_0^t S_r^i dS_r^j(\omega) - \int_0^s S_r^i dS_r^j(\omega) - S_s^i(\omega) S_{s,t}^j(\omega).$$

Let $p > 2$. Then for typical price paths, $A = (A^{i,j})_{1 \leq i, j \leq d}$ has finite $p/2$ -variation, and in particular $\mathbb{S} = (S, A)$ is a p -rough path.

Proof. Define the dyadic stopping times $(\tau_k^n)_{n, k \in \mathbb{N}}$ by $\tau_0^n := 0$ and

$$\tau_{k+1}^n := \inf\{t \geq \tau_k^n : |S_t - S_{\tau_k^n}| = 2^{-n}\},$$

and set $S_t^n := \sum_k S_{\tau_k^n} \mathbf{1}_{[\tau_k^n, \tau_{k+1}^n)}(t)$, so that $\|S^n - S\|_\infty \leq 2^{-n}$. According to (5), for typical price paths ω there exists $C(\omega) > 0$ such that

$$\left\| (S^n \cdot S)(\omega) - \int S dS(\omega) \right\|_\infty \leq C(\omega) 2^{-n} \sqrt{\log n}.$$

Fix such a typical price path ω , which is also of finite q -variation for all $q > 2$ (recall from Corollary 2.11 that this is satisfied by typical price paths). Let us show that for such ω , the process A is of finite $p/2$ -variation for all $p > 2$.

We have for $(s, t) \in \Delta_T$, omitting the argument ω of the processes under consideration,

$$\begin{aligned} |A_{s,t}| &\leq \left| \int_s^t S_r dS_r - (S^n \cdot S)_{s,t} \right| + |(S^n \cdot S)_{s,t} - S_s S_{s,t}| \\ &\leq C(\omega) 2^{-n} \sqrt{\log n} + |(S^n \cdot S)_{s,t} - S_s S_{s,t}| \lesssim_\varepsilon C(\omega) 2^{-n(1-\varepsilon)} + |(S^n \cdot S)_{s,t} - S_s S_{s,t}| \end{aligned}$$

for every $n \in \mathbb{N}$, $\varepsilon > 0$. The second term on the right hand side can be estimated, using an argument based on Young's maximal inequality (see [LCL07], Theorem 1.16), by

$$\max\{2^{-n} c(s, t)^{1/q}, (\#\{k : \tau_k^n \in [s, t]\})^{1-2/q} c(s, t)^{2/q} + c(s, t)^{2/q}\}, \quad (8)$$

where $c(s, t)$ is a control function with $|S_{s,t}|^q \leq c(s, t)$ for all $(s, t) \in \Delta_T$. Indeed, if there exists no k with $\tau_k^n \in [s, t]$, then $|(S^n \cdot S)_{s,t} - S_s S_{s,t}| \leq 2^{-n} c(s, t)^{1/q}$, using that $|S_{s,t}| \leq c(s, t)^{1/q}$. This corresponds to the first term in the maximum in (8).

Otherwise, note that at the price of adding $c(s, t)^{2/q}$ to the right hand side, we may suppose that $s = \tau_{k_0}^n$ for some k_0 . Let now $\tau_{k_0}^n, \dots, \tau_{k_0+N-1}^n$ be those $(\tau_k^n)_k$ which are in $[s, t]$. Without loss of generality we may suppose $N \geq 2$, because otherwise $(S^n \cdot S)_{s,t} = S_s S_{s,t}$. Abusing notation, we write $\tau_{k_0+N}^n = t$. The idea is now to successively delete points $(\tau_{k_0+\ell}^n)$ from the partition, in order to pass from $(S^n \cdot S)$ to $S_s S_{s,t}$. By super-additivity of c , there must exist $\ell \in \{1, \dots, N-1\}$, for which

$$c(\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell+1}^n) \leq \frac{2}{N-1} c(s, t).$$

Deleting $\tau_{k_0+\ell}^n$ from the partition and subtracting the resulting integral from $(S^n \cdot S)_{s,t}$, we get

$$\begin{aligned} &|S_{\tau_{k_0+\ell-1}^n}^n S_{\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell}^n}^n + S_{\tau_{k_0+\ell}^n}^n S_{\tau_{k_0+\ell}^n, \tau_{k_0+\ell+1}^n}^n - S_{\tau_{k_0+\ell-1}^n}^n S_{\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell+1}^n}^n| \\ &= |S_{\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell}^n}^n S_{\tau_{k_0+\ell}^n, \tau_{k_0+\ell+1}^n}^n| \leq c(\tau_{k_0+\ell-1}^n, \tau_{k_0+\ell+1}^n)^{2/q} \leq \left(\frac{2}{N-1} c(s, t) \right)^{2/q}. \end{aligned}$$

Successively deleting all the points except $\tau_{k_0}^n = s$ and $\tau_{k_0+N}^n = t$ from the partition gives

$$|(S^n \cdot S)_{s,t} - S_s S_{s,t}| \leq \sum_{k=2}^N \left(\frac{2}{k-1} c(s,t) \right)^{2/q} \lesssim N^{1-2/q} c(s,t)^{2/q},$$

and therefore (8). Now it is easy to see that $\#\{k : \tau_k^n \in [s,t]\} \leq 2^{nq} c(s,t)$ (compare also the proof of Lemma 3.3), and thus

$$\begin{aligned} |A_{s,t}| &\lesssim_\varepsilon C(\omega) 2^{-n(1-\varepsilon)} + \max\{2^{-n} c(s,t)^{1/q}, (2^{nq} c(s,t))^{1-2/q} c(s,t)^{2/q} + c(s,t)^{2/q}\} \\ &= C(\omega) 2^{-n(1-\varepsilon)} + \max\{2^{-n} c(s,t)^{1/q}, 2^{-n(2-q)} c(s,t) + c(s,t)^{2/q}\}. \end{aligned} \quad (9)$$

This holds for all $n \in \mathbb{N}$, $\varepsilon > 0$, $q > 2$. Let us suppose for the moment that $c(s,t) \leq 1$ and let $\alpha > 0$ to be determined later. Then there exists $n \in \mathbb{N}$ for which $2^{-n-1} < c(s,t)^{1/\alpha(1-\varepsilon)} \leq 2^{-n}$. Using this n in (9), we get

$$\begin{aligned} |A_{s,t}|^\alpha &\lesssim_{\varepsilon,\omega,\alpha} c(s,t) + \max\left\{c(s,t)^{1/(1-\varepsilon)} c(s,t)^{\alpha/q}, c(s,t)^{(2-q)/(1-\varepsilon)+\alpha} + c(s,t)^{2\alpha/q}\right\} \\ &= c(s,t) + \max\left\{c(s,t)^{\frac{q+\alpha(1-\varepsilon)}{q(1-\varepsilon)}}, c(s,t)^{\frac{2-q+\alpha(1-\varepsilon)}{1-\varepsilon}} + c(s,t)^{2\alpha/q}\right\}. \end{aligned}$$

We would like all the exponents in the maximum on the right hand side to be larger or equal to 1. For the first term, this is satisfied as long as $\varepsilon < 1$. For the third term, we need $\alpha \geq q/2$. For the second term, we need $\alpha \geq (q-1-\varepsilon)/(1-\varepsilon)$. Since $\varepsilon > 0$ can be chosen arbitrarily close to 0, it suffices if $\alpha > q-1$. Now, since $q > 2$ can be chosen arbitrarily close to 2, we see that α can be chosen arbitrarily close to 1. In particular, we may take $\alpha = p/2$ for any $p > 2$, and we obtain

$$|A_{s,t}|^{p/2} \lesssim_{\omega,\delta} c(s,t)(1+c(s,t)^\delta) \leq c(s,t)(1+c(0,T)^\delta)$$

for a suitable $\delta > 0$.

It remains to treat the case $c(s,t) > 1$, for which we simply estimate

$$|A_{s,t}|^{p/2} \lesssim_p \left\| \int_0^\cdot S_r dS_r \right\|_\infty^{p/2} + \|S\|_\infty^p \leq \left(\left\| \int_0^\cdot S_r dS_r \right\|_\infty^{p/2} + \|S\|_\infty^p \right) c(s,t).$$

So for every interval $[s,t]$ we can estimate $|A_{s,t}|^{p/2} \lesssim_{\omega,p} c(s,t)$, and the proof is complete. \square

Remark 4.13. *To the best of our knowledge, this is one of the first times that a non-geometric rough path is constructed in a non-probabilistic setting, and certainly we are not aware of any works where rough paths are constructed using financial arguments.*

We also point out that, thanks to Proposition 2.6, we gave a simple, model free, and pathwise proof for the fact that a local martingale together with its Itô integral defines a rough path. While this seems intuitively clear, the only other proof that we are aware of is somewhat involved: it relies on a strong version of the Burkholder-Davis-Gundy inequality, a time change, and Kolmogorov's continuity criterion; see [CL05] or Chapter 14 of [FV10].

The following auxiliary result will allow us to obtain the rough path integral as a limit of Riemann sums, rather than compensated Riemann sums which are usually used to define it.

Lemma 4.14. *Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $(c_n^\varepsilon \sqrt{\log n})$ converges to 0 for all $\varepsilon > 0$. For $n \in \mathbb{N}$ define $\tau_0^n := 0$ and $\tau_{k+1}^n := \inf\{t \geq \tau_k^n : |S_t - S_{\tau_k^n}| = c_n\}$, $k \in \mathbb{N}$, and set $S_t^n = \sum_k S_{\tau_k^n} \mathbf{1}_{[\tau_k^n, \tau_{k+1}^n)}(t)$. Then for typical price paths, $((S^n \cdot S))$ converges uniformly to $\int \text{Sd}S$. Moreover, for $p > 2$ and for typical price paths there exists a control function $c = c(p, \omega)$ such that*

$$\sup_n \sup_{k < \ell} \frac{|(S^n \cdot S)_{\tau_k^n, \tau_\ell^n}(\omega) - S_{\tau_k^n}(\omega) S_{\tau_\ell^n}(\omega)|^{p/2}}{c(\tau_k^n, \tau_\ell^n)} \leq 1.$$

Proof. The uniform convergence of $((S^n \cdot S))$ to $\int \text{Sd}S$ follows from Corollary 3.6. For the second claim, fix $n \in \mathbb{N}$ and $k < \ell$ such that $\tau_\ell^n \leq T$. Then

$$\begin{aligned} |(S^n \cdot S)_{\tau_k^n, \tau_\ell^n} - S_{\tau_k^n} S_{\tau_\ell^n}| &\lesssim \left\| (S^n \cdot S) - \int_0^\cdot S_s \text{d}S_s \right\|_\infty + |A_{\tau_k^n, \tau_\ell^n}| \\ &\lesssim_\omega c_n \sqrt{\log n} + v_{p/2}(\tau_k^n, \tau_\ell^n)^{2/p} \lesssim_\varepsilon c_n^{1-\varepsilon} + v_{p/2}(\tau_k^n, \tau_\ell^n)^{2/p}, \end{aligned} \quad (10)$$

where $\varepsilon > 0$ and the last estimate holds by our assumption on the sequence (c_n) , and where $v_{p/2}(s, t) := \|A\|_{p/2\text{-var}, [s, t]}^{p/2}$ for $(s, t) \in \Delta_T$. Of course, this inequality only holds for typical price paths and not for all $\omega \in \Omega$.

On the other side, the same argument as in the proof of Theorem 4.12 (using Young's maximal inequality and successively deleting points from the partition) shows that

$$|(S^n \cdot S)_{\tau_k^n, \tau_\ell^n} - S_{\tau_k^n} S_{\tau_\ell^n}| \lesssim c_n^{2-q} v_q(\tau_k^n, \tau_\ell^n), \quad (11)$$

where $v_q(s, t) := \|S\|_{q\text{-var}, [s, t]}^q$ for $(s, t) \in \Delta_T$.

Let us define the control function $\tilde{c} := v_q + v_{p/2}$. Take $\alpha > 0$ to be determined below. If $c_n > \tilde{c}(s, t)^{1/\alpha(1-\varepsilon)}$, then we use (11) and the fact that $2 - q < 0$, to obtain

$$|(S^n \cdot S)_{\tau_k^n, \tau_\ell^n} - S_{\tau_k^n} S_{\tau_\ell^n}|^\alpha \lesssim (\tilde{c}(\tau_k^n, \tau_\ell^n))^{\frac{2-q}{1-\varepsilon}} v_q(\tau_k^n, \tau_\ell^n)^\alpha \leq \tilde{c}(\tau_k^n, \tau_\ell^n)^{\frac{2-q+\alpha(1-\varepsilon)}{(1-\varepsilon)}}.$$

The exponent is larger or equal to 1 as long as $\alpha \geq (q - 1 - \varepsilon)/(1 - \varepsilon)$. Since q and ε can be chosen arbitrarily close to 2 and 0 respectively, we can take $\alpha = p/2$, and get

$$|(S^n \cdot S)_{\tau_k^n, \tau_\ell^n} - S_{\tau_k^n} S_{\tau_\ell^n}|^{p/2} \lesssim \tilde{c}(\tau_k^n, \tau_\ell^n) (1 + \tilde{c}(0, T)^\delta)$$

for a suitable $\delta > 0$.

On the other side, if $c_n \leq \tilde{c}(s, t)^{1/\alpha(1-\varepsilon)}$, then we use (10) to obtain

$$|(S^n \cdot S)_{\tau_k^n, \tau_\ell^n} - S_{\tau_k^n} S_{\tau_\ell^n}|^\alpha \lesssim \tilde{c}(\tau_k^n, \tau_\ell^n) + \tilde{c}(\tau_k^n, \tau_\ell^n)^{2\alpha/p},$$

so that also in this case we may take $\alpha = p/2$, and thus we have in both cases

$$|(S^n \cdot S)_{\tau_k^n, \tau_\ell^n} - S_{\tau_k^n} S_{\tau_\ell^n}|^{p/2} \leq c(\tau_k^n, \tau_\ell^n),$$

where c is a suitable (ω -dependent) multiple of \tilde{c} . \square

4.3 The rough path integral as limit of Riemann sums

Theorem 4.12 shows that we can apply the controlled rough path integral in model free financial mathematics, since every typical price path is a rough path. But there remains a philosophical problem: As we have seen in Theorem 4.9, the rough path integral $\int F dS$ is given as limit of the compensated Riemann sums

$$\int_0^t F_s dS_s = \lim_{m \rightarrow \infty} \sum_{[r_1, r_2] \in \pi^m} [F_{r_1} S_{r_1, r_2} + F'_{r_1} A(r_1, r_2)],$$

where (π^m) is an arbitrary sequence of partitions of $[0, t]$ with mesh size going to 0. The term $F_{r_1} S_{r_1, r_2}$ has an obvious financial interpretation as profit made by buying F_{r_1} units of the traded asset at time r_1 and by selling them at time r_2 . However, for the ‘‘compensator’’ $F'_{r_1} A(r_1, r_2)$ there seems to be no financial interpretation, and therefore it is not clear whether the rough path integral can be understood as profit obtained by investing in S .

However, we observed in Section 3 that along suitable stopping times $(\tau_k^n)_{n,k}$, we have

$$\int_0^t S_s dS_s = \lim_{n \rightarrow \infty} \sum_k S_{\tau_k^n} S_{\tau_k^n \wedge t, \tau_{k+1}^n \wedge t}.$$

By the philosophy of controlled paths, we expect that also for F which looks like S on small scales we should obtain

$$\int_0^t F_s dS_s = \lim_{n \rightarrow \infty} \sum_k F_{\tau_k^n} S_{\tau_k^n \wedge t, \tau_{k+1}^n \wedge t},$$

without having to introduce the compensator $F'_{\tau_k^n} A(\tau_k^n \wedge t, \tau_{k+1}^n \wedge t)$ in the Riemann sum. With the results we have at hand, this statement is actually relatively easy to prove. Nonetheless, it seems not to have been observed before.

For the remainder of this section we fix $S \in C([0, T], \mathbb{R}^d)$, and we work under the following assumption:

Assumption (RIE). Let $\pi^n = \{0 = t_0^n < t_1^n < \dots < t_{N_n}^n = T\}$, $n \in \mathbb{N}$, be a given sequence of partitions such that $\sup\{|S_{t_k^n, t_{k+1}^n}^n| : k = 0, \dots, N_n - 1\}$ converges to 0, and let $p \in (2, 3)$. Set

$$S_t^n := \sum_{k=0}^{N_n-1} S_{t_k^n} \mathbf{1}_{[t_k^n, t_{k+1}^n)}(t).$$

We assume that the Riemann sums $(S^n \cdot S)$ converge uniformly to $\int S dS$, and that there exists a control function c for which

$$\sup_{(s,t) \in \Delta_T} \frac{|S_{s,t}|^p}{c(s,t)} + \sup_n \sup_{0 \leq k < \ell \leq N_n} \frac{|(S^n \cdot S)_{t_k^n, t_\ell^n} - S_{t_k^n} S_{t_k^n, t_\ell^n}|^{p/2}}{c(t_k^n, t_\ell^n)} \leq 1. \quad (12)$$

Remark 4.15. The ‘‘coarse-grained’’ regularity condition (12) has recently been independently rediscovered in [Kel14]; see also [GIP14].

Our proof that the rough path integral is given as limit of Riemann sums is somewhat indirect. We translate everything from Itô type integrals to related Stratonovich type integrals, for which the convergence follows from the continuity of the rough path integral, Proposition 4.11. Then we translate everything back to our Itô type integrals. To go from Itô to Stratonovich, we need the quadratic variation:

Lemma 4.16. *Under Assumption (RIE), let $1 \leq i, j \leq d$, and define*

$$\langle S^i, S^j \rangle_t := S_t^i S_t^j - S_0^i S_0^j - \int_0^t S_r^i dS_r^j - \int_0^t S_r^j dS_r^i.$$

Then $\langle S^i, S^j \rangle$ is a continuous function and

$$\langle S^i, S^j \rangle_t = \lim_{n \rightarrow \infty} \langle S^i, S^j \rangle_t^n = \lim_{n \rightarrow \infty} \sum_{k=0}^{N_n-1} (S_{t_{k+1}^n \wedge t}^i - S_{t_k^n \wedge t}^i)(S_{t_{k+1}^n \wedge t}^j - S_{t_k^n \wedge t}^j). \quad (13)$$

The sequence $(\langle S^i, S^j \rangle^n)_n$ is of uniformly bounded total variation, and in particular $\langle S^i, S^j \rangle$ is of bounded variation. We write $\langle S \rangle = \langle S, S \rangle = (\langle S^i, S^j \rangle)_{1 \leq i, j \leq d}$, and call $\langle S \rangle$ the quadratic variation of S .

Proof. The function $\langle S^i, S^j \rangle$ is continuous by definition. The specific form (13) of $\langle S^i, S^j \rangle$ follows from two simple observations:

$$S_t^i S_t^j - S_0^i S_0^j = \sum_{k=0}^{N_n-1} \left(S_{t_{k+1}^n \wedge t}^i S_{t_{k+1}^n \wedge t}^j - S_{t_k^n \wedge t}^i S_{t_k^n \wedge t}^j \right)$$

for every $n \in \mathbb{N}$, and

$$S_{t_{k+1}^n \wedge t}^i S_{t_{k+1}^n \wedge t}^j - S_{t_k^n \wedge t}^i S_{t_k^n \wedge t}^j = S_{t_k^n \wedge t}^i S_{t_k^n \wedge t, t_{k+1}^n \wedge t}^j + S_{t_k^n \wedge t}^j S_{t_k^n \wedge t, t_{k+1}^n \wedge t}^i + S_{t_k^n \wedge t, t_{k+1}^n \wedge t}^i S_{t_k^n \wedge t, t_{k+1}^n \wedge t}^j + S_{t_k^n \wedge t, t_{k+1}^n \wedge t}^j S_{t_k^n \wedge t, t_{k+1}^n \wedge t}^i,$$

so that the convergence in (13) is a consequence of the convergence of $(S^n \cdot S)$ to $\int S dS$.

To see that $\langle S^i, S^j \rangle$ is of bounded variation, note that

$$S_{t_k^n \wedge t, t_{k+1}^n \wedge t}^i S_{t_k^n \wedge t, t_{k+1}^n \wedge t}^j = \frac{1}{4} \left(\left((S^i + S^j)_{t_k^n \wedge t, t_{k+1}^n \wedge t} \right)^2 - \left((S^i - S^j)_{t_k^n \wedge t, t_{k+1}^n \wedge t} \right)^2 \right)$$

(read $\langle S^i, S^j \rangle = 1/4(\langle S^i + S^j \rangle - \langle S^i - S^j \rangle)$). In other words, the n -th approximation of $\langle S^i, S^j \rangle$ is the difference of two increasing functions, and its total variation is bounded from above by

$$\sum_{k=0}^{N_n-1} \left(\left((S^i + S^j)_{t_k^n, t_{k+1}^n} \right)^2 + \left((S^i - S^j)_{t_k^n, t_{k+1}^n} \right)^2 \right) \lesssim \sup_m \sum_{k=0}^{N_m-1} \left((S_{t_k^m, t_{k+1}^m}^i)^2 + (S_{t_k^m, t_{k+1}^m}^j)^2 \right).$$

Since the right hand side is finite, also the limit $\langle S^i, S^j \rangle$ is of bounded variation. \square

Given the quadratic variation, the existence of the Stratonovich integral is straightforward:

Lemma 4.17. *Under Assumption (RIE), define $\tilde{S}^n|_{[t_k^n, t_{k+1}^n]}$ as the linear interpolation of $S_{t_k^n}$ and $S_{t_{k+1}^n}$ for $k = 0, \dots, N_n - 1$. Then $(\int \tilde{S}^n d\tilde{S}^n)$ converges uniformly to*

$$\int_s^t S_r \circ dS_r := \int_s^t S_r dS_r + \frac{1}{2} \langle S \rangle_{s,t}. \quad (14)$$

Moreover, setting $\tilde{A}^n(s, t) := \int_s^t \tilde{S}_{s,r}^n d\tilde{S}_r^n$ for $(s, t) \in \Delta_T$, we have $\sup_n \|\tilde{A}^n\|_{p/2\text{-var}} < \infty$.

Proof. Let $n \in \mathbb{N}$ and $k \in \{0, \dots, N_n - 1\}$. Then for $t \in [t_k^n, t_{k+1}^n]$ we have

$$\tilde{S}_t^n = S_{t_k^n} + \frac{t - t_k^n}{t_{k+1}^n - t_k^n} S_{t_k^n, t_{k+1}^n},$$

so that

$$\int_{t_k^n}^{t_{k+1}^n} \tilde{S}_r^n d\tilde{S}_r^n = S_{t_k^n} S_{t_k^n, t_{k+1}^n} + \frac{1}{2} S_{t_k^n, t_{k+1}^n} S_{t_k^n, t_{k+1}^n}, \quad (15)$$

from where the uniform convergence and the representation (14) follow by Lemma 4.16.

To prove that \tilde{A}^n has uniformly bounded $\frac{p}{2}$ -variation, consider $(s, t) \in \Delta_T$. If there exists k such that $t_k^n \leq s < t \leq t_{k+1}^n$, then we estimate

$$\begin{aligned} |\tilde{A}^n(s, t)|^{p/2} &= \left| \int_s^t \tilde{S}_{s,r}^n d\tilde{S}_r^n \right|^{p/2} \leq \left| \int_s^t (r - s) \frac{|S_{t_k^n, t_{k+1}^n}|^2}{|t_{k+1}^n - t_k^n|^2} dr \right|^{p/2} \\ &= \frac{1}{2^{p/2}} |t - s|^p \frac{|S_{t_k^n, t_{k+1}^n}|^p}{|t_{k+1}^n - t_k^n|^p} \leq \frac{|t - s|}{|t_{k+1}^n - t_k^n|} \|S\|_{p\text{-var}, [t_k^n, t_{k+1}^n]}^p. \end{aligned} \quad (16)$$

Otherwise, let k_0 be the smallest k such that $t_k^n \in (s, t)$, and let k_1 be the largest such k . We decompose

$$\tilde{A}^n(s, t) = \tilde{A}^n(s, t_{k_0}^n) + \tilde{A}^n(t_{k_0}^n, t_{k_1}^n) + \tilde{A}^n(t_{k_1}^n, t) + \tilde{S}_{s, t_{k_0}^n}^n \tilde{S}_{t_{k_0}^n, t_{k_1}^n}^n + \tilde{S}_{s, t_{k_1}^n}^n \tilde{S}_{t_{k_1}^n, t}^n.$$

We get from (15) that

$$|\tilde{A}^n(t_{k_0}^n, t_{k_1}^n)|^{p/2} \lesssim |(S^n \cdot S)_{t_{k_0}^n, t_{k_1}^n} - S_{t_{k_0}^n} S_{t_{k_0}^n, t_{k_1}^n}|^{p/2} + (\langle S \rangle_{t_{k_0}^n, t_{k_1}^n}^n)^{p/2},$$

where $\langle S \rangle^n$ denotes the n -th approximation of the quadratic variation. By (12) and Lemma 4.16, there exists a control function \tilde{c} so that the right hand side is bounded from above by $\tilde{c}(t_{k_0}^n, t_{k_1}^n)$. Combining this with (16) and a simple estimate for the terms $\tilde{S}_{s, t_{k_0}^n}^n \tilde{S}_{t_{k_0}^n, t_{k_1}^n}^n$ and $\tilde{S}_{s, t_{k_1}^n}^n \tilde{S}_{t_{k_1}^n, t}^n$, we deduce that $\|\tilde{A}^n\|_{p/2\text{-var}} \lesssim \tilde{c}(0, T) + \|S\|_{p\text{-var}}^2$, and the proof is complete. \square

We are now ready to prove the main result of this section.

Theorem 4.18. *Under Assumption (RIE), let $q > 0$ be such that $2/p + 1/q > 1$. Let $(F, F') \in \mathcal{C}_S^q$ be a controlled path such that F is continuous. Then the rough path integral $\int F dS$ which was defined in Theorem 4.9 is given by*

$$\int_0^t F_s dS_s = \lim_{n \rightarrow \infty} \sum_{k=0}^{N_n-1} F_{t_k^n} S_{t_k^n \wedge t, t_{k+1}^n \wedge t},$$

where the convergence is uniform in t .

Proof. For $n \in \mathbb{N}$ define \tilde{F}^n as the linear interpolation of F between the points in π^n . Then (\tilde{F}^n, F') is controlled by \tilde{S}^n : Clearly $\|\tilde{F}^n\|_{q\text{-var}} \leq \|F\|_{q\text{-var}}$. The remainder $\tilde{R}_{\tilde{F}^n}^n$ of \tilde{F}^n with respect to \tilde{S}^n is given by $\tilde{R}_{\tilde{F}^n}^n(s, t) = \tilde{F}_{s,t}^n - F'_s \tilde{S}_{s,t}^n$ for $(s, t) \in \Delta_T$. We need to show that $\tilde{R}_{\tilde{F}^n}^n$ has finite r -variation for $1/r = 1/p + 1/q$.

If $t_k^n \leq s \leq t \leq t_{k+1}^n$, we have

$$\begin{aligned}
|\tilde{R}_{\tilde{F}^n}^n(s, t)|^r &= \left| \frac{t-s}{t_{k+1}^n - t_k^n} F_{t_k^n, t_{k+1}^n} - F'_s \frac{t-s}{t_{k+1}^n - t_k^n} S_{t_k^n, t_{k+1}^n} \right|^r \\
&\leq \left| \frac{t-s}{t_{k+1}^n - t_k^n} \right|^r \left(\|R_F\|_{r\text{-var}, [t_k^n, t_{k+1}^n]} + \|F'\|_{q\text{-var}, [t_k^n, s]}^{r/q} \|S\|_{p\text{-var}, [t_k^n, t_{k+1}^n]}^{r/p} \right) \\
&\leq \frac{|t-s|}{|t_{k+1}^n - t_k^n|} \left(\|R_F\|_{r\text{-var}, [t_k^n, t_{k+1}^n]} + \|F'\|_{q\text{-var}, [t_k^n, t_{k+1}^n]} + \|S\|_{p\text{-var}, [t_k^n, t_{k+1}^n]} \right), \quad (17)
\end{aligned}$$

where in the last step we used that $1/r = 1/p + 1/q$, and thus $r/q + r/p = 1$.

Otherwise, there exists $k \in \{1, \dots, N_n - 1\}$ with $t_k^n \in (s, t)$. Let k_0 and k_1 the smallest and largest such k , respectively. Then

$$|\tilde{R}_{\tilde{F}^n}^n(s, t)|^r \lesssim_r |\tilde{R}_{\tilde{F}^n}^n(s, t_{k_0}^n)|^r + |\tilde{R}_{\tilde{F}^n}^n(t_{k_0}^n, t_{k_1}^n)|^r + |\tilde{R}_{\tilde{F}^n}^n(t_{k_1}^n, t)|^r + |F'_{s, t_{k_0}^n} S_{t_{k_0}^n, t_{k_1}^n}|^r + |F'_{s, t_{k_1}^n} S_{t_{k_1}^n, t}|^r.$$

Now $\tilde{R}_{\tilde{F}^n}^n(t_{k_0}^n, t_{k_1}^n) = R_F(t_{k_0}^n, t_{k_1}^n)$, and therefore we can use (17), the assumption on R_F , and the fact that $1/r = 1/p + 1/q$ (which is needed to treat the last two terms on the right hand side), to obtain

$$\|\tilde{R}_{\tilde{F}^n}^n\|_{r\text{-var}} \lesssim_r \|R_F\|_{r\text{-var}} + \|F'\|_{q\text{-var}} + \|S\|_{p\text{-var}}.$$

On the other side, since F and R_F are continuous, $(\tilde{F}^n, \tilde{R}_{\tilde{F}^n}^n)$ converges uniformly to (F, R_F) . Now for continuous functions, uniform convergence with uniformly bounded p -variation implies convergence in p' -variation for every $p' > p$. See Exercise 2.8 in [FH14] for the case of Hölder continuous functions.

Thus, using Lemma 4.17, we see that if $p' > p$ and $q' > q$ are such that $2/p' + 1/q' > 0$, then $((\tilde{S}^n, \tilde{A}^n)_n)$ converges in $(p', p'/2)$ -variation to (S, A°) , where $A^\circ(s, t) = A(s, t) + 1/2 \langle S \rangle_{s,t}$. Similarly, $((\tilde{F}^n, F', \tilde{R}_{\tilde{F}^n}^n)_n)$ converges in (q', p', r') -variation to (F, F', R_F) , where $1/r' = 1/p' + 1/q'$.

Proposition 4.11 now yields the uniform convergence of $\int \tilde{F}^n d\tilde{S}^n$ to $\int F \circ dS$, by which we denote the rough path integral of the controlled path (F, F') against the rough path (S, A°) . But for every $t \in [0, T]$ we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \int_0^t \tilde{F}^n d\tilde{S}^n &= \lim_{n \rightarrow \infty} \sum_{k: t_{k+1}^n \leq t} \frac{1}{2} (F_{t_k^n} + F_{t_{k+1}^n}) S_{t_k^n, t_{k+1}^n} \\
&= \lim_{n \rightarrow \infty} \left(\sum_{k: t_{k+1}^n \leq t} F_{t_k^n} S_{t_k^n, t_{k+1}^n} + \frac{1}{2} \sum_{k: t_{k+1}^n \leq t} F_{t_k^n, t_{k+1}^n} S_{t_k^n, t_{k+1}^n} \right).
\end{aligned}$$

Using that F is controlled by S , it is easy to see that the second term on the right hand side converges uniformly to $1/2 \int_0^t F'_s d\langle S \rangle_s$, $t \in [0, T]$. Thus, the Riemann sums $\sum_k F_{t_k^n} S_{t_k^n \wedge, t_{k+1}^n \wedge}$ converge uniformly to $\int F \circ dS - 1/2 \int F' d\langle S \rangle$, and from the representation of the rough path integral as limit of compensated Riemann sums (6), it is easy to see that $\int F \circ dS = \int F dS + 1/2 \int F' d\langle S \rangle$, which completes the proof. \square

Theorem 4.18 is reminiscent of Föllmer's pathwise Itô integral [Föll81]. Föllmer assumes that the quadratic variation $\langle S \rangle$ of S exists along a given sequence of partitions and is continuous, and uses this to prove an Itô formula for S : if $F \in C^2$, then

$$F(S_t) = F(S_0) + \int_0^t \nabla F(S_s) dS_s + \frac{1}{2} \int_0^t D^2 F(S_s) d\langle S \rangle_s, \quad (18)$$

where the integral $\int_0^\cdot \nabla F(S_s) dS_s$ is given as limit of Riemann sums along that same sequence of partitions.

Friz and Hairer [FH14] observe that if for $p \in (2, 3)$ the function S is of finite p -variation and $\langle S \rangle$ is an arbitrary continuous function of finite $p/2$ -variation, then setting

$$\text{Sym}(A)(s, t) := \frac{1}{2}(S_{s,t}^i S_{s,t}^j + \langle S \rangle_{s,t})$$

one obtains a “reduced rough path” $(S, \text{Sym}(A))$. They continue to show that if F is controlled by S with *symmetric* derivative F' , then it is possible to define the rough path integral $\int F dS$. This is not surprising since then we have $F'_s A_{s,t} = F'_s \text{Sym}(A)_{s,t}$ for the compensator term in the definition of the rough path integral. They also derive an Itô formula for reduced rough paths, which takes the same form as (18), except that now $\int \nabla F(S) dS$ is a rough path integral (and therefore defined as limit of compensated Riemann sums).

So both the assumption and the result of [FH14] are slightly different from the ones in [Föl81], and while it seems intuitively clear, it is still not shown rigorously that Föllmer’s pathwise Itô integral is a special case of the rough path integral. We will now show that Föllmer’s result is a special case of Theorem 4.18. For that purpose we only need to prove that Föllmer’s condition on the convergence of the quadratic variation is a special case of the assumption in Theorem 4.18, at least as long as we only need the symmetric part of the area.

Definition 4.19. Let $f \in C([0, T], \mathbb{R})$ and let $\pi^n = \{0 = t_0^n < t_1^n < \dots < t_{N_n}^n = T\}$, $n \in \mathbb{N}$ be such that $\sup\{|f_{t_k^n, t_{k+1}^n}| : k = 0, \dots, N_n - 1\}$ converges to 0. We say that f has *quadratic variation along (π^n) in the sense of Föllmer* if the sequence of discrete measures (μ^n) on $([0, T], \mathcal{B}[0, T])$, defined by

$$\mu_n := \sum_{k=0}^{N_n-1} |f_{t_k^n, t_{k+1}^n}|^2 \delta_{t_k^n}, \quad (19)$$

converges weakly to a non-atomic measure μ . We write $[f]_t$ for the “distribution function” of μ (in general μ will not be a probability measure). The function $f = (f^1, \dots, f^d) \in C([0, T], \mathbb{R}^d)$ has *quadratic variation along (π^n) in the sense of Föllmer* if this holds for all f^i and $f^i + f^j$, $1 \leq i, j \leq d$. In this case, we set

$$[f^i, f^j]_t := \frac{1}{2}([f^i + f^j]_t - [f^i]_t - [f^j]_t), \quad t \in [0, T].$$

Lemma 4.20 (see also [Vov11], Proposition 6.1). *Let $p \in (2, 3)$, and let $S = (S^1, \dots, S^d) \in C([0, T], \mathbb{R}^d)$ have finite p -variation. Let $\pi^n = \{0 = t_0^n < t_1^n < \dots < t_{N_n}^n = T\}$, $n \in \mathbb{N}$, be a sequence of partitions such that $\sup\{|S_{t_k^n, t_{k+1}^n}^i| : k = 0, \dots, N_n - 1\}$ converges to 0. Then the following conditions are equivalent:*

1. *The function S has quadratic variation along (π^n) in the sense of Föllmer.*
2. *For all $1 \leq i, j \leq d$, the discrete quadratic variation*

$$\langle S^i, S^j \rangle_t^n := \sum_{k=0}^{N_n-1} S_{t_k^n \wedge t, t_{k+1}^n}^i \wedge t S_{t_k^n \wedge t, t_{k+1}^n}^j \wedge t$$

converges uniformly in $C([0, T], \mathbb{R})$ to a limit $\langle S^i, S^j \rangle$.

3. For $S^{n,i} := \sum_{k=0}^{N_n-1} S_{t_k^n}^i \mathbf{1}_{[t_k^n, t_{k+1}^n)}$, $i \in \{1, \dots, d\}$, $n \in \mathbb{N}$, the Riemann sums $(S^{n,i} \cdot S^j) + (S^{n,j} \cdot S^i)$ converge uniformly to a limit $\int S^i dS^j + \int S^j dS^i$. Moreover, the symmetric part of the approximate area,

$$\text{Sym}(A^n)^{i,j}(s,t) = \frac{1}{2}((S^{n,i} \cdot S^j)_{s,t} + (S^{n,j} \cdot S^i)_{s,t} - S_s^i S_{s,t}^j - S_s^j S_{s,t}^i), \quad 1 \leq i, j \leq d, (s,t) \in \Delta_T,$$

has uniformly bounded $p/2$ -variation along (π^n) , in the sense of (12).

If these conditions hold, then $[S^i, S^j] = \langle S^i, S^j \rangle$ for all $1 \leq i, j \leq d$.

Proof. Assume 1. and note that

$$S_{t_k^n \wedge t, t_{k+1}^n \wedge t}^i S_{t_k^n \wedge t, t_{k+1}^n \wedge t}^j = \frac{1}{2}(((S^i + S^j)_{t_k^n \wedge t, t_{k+1}^n \wedge t})^2 - (S_{t_k^n \wedge t, t_{k+1}^n \wedge t}^i)^2 - (S_{t_k^n \wedge t, t_{k+1}^n \wedge t}^j)^2).$$

Thus, the uniform convergence of $\langle S^i, S^j \rangle^n$ and the fact that $\langle S^i, S^j \rangle = [S^i, S^j]$ follow once we show that Föllmer's weak convergence of the measures (19) implies the uniform convergence of their distribution functions. But since the limiting distribution is continuous by assumption, this is a standard result.

Next, assume 2. The uniform convergence of the Riemann sums $(S^{n,i} \cdot S^j) + (S^{n,j} \cdot S^i)$ is shown as in Lemma 4.16. To see that $\text{Sym}(A^n)$ has uniformly bounded $p/2$ -variation along (π^n) , note that for $0 \leq k \leq \ell \leq N_n$ and $1 \leq i, j \leq d$ we have

$$\begin{aligned} |(S^{n,i} \cdot S^j)_{t_k^n, t_\ell^n} + (S^{n,j} \cdot S^i)_{t_k^n, t_\ell^n} - S_s^i S_{t_k^n, t_\ell^n}^j - S_s^j S_{t_k^n, t_\ell^n}^i|^{p/2} &= |S_{t_k^n, t_\ell^n}^i S_{t_k^n, t_\ell^n}^j - \langle S^i, S^j \rangle_{t_k^n, t_\ell^n}^n|^{p/2} \\ &\leq \|S\|_{p\text{-var}, [t_k^n, t_\ell^n]} + \|\langle S^i, S^j \rangle^n\|_{1\text{-var}, [t_k^n, t_\ell^n]}. \end{aligned}$$

That $\|\langle S^i, S^j \rangle^n\|_{1\text{-var}}$ is uniformly bounded in n is shown in Lemma 4.16.

That 3. implies 1. is also shown in Lemma 4.16. \square

Remark 4.21. *With Theorem 4.18 we can only derive an Itô formula for $F \in C^{2+\varepsilon}$, since we are only able to integrate $\nabla F(S)$ if $\nabla F \in C^{1+\varepsilon}$. But this only seems to be due to the fact that our analysis is not sharp. We expect that typical price paths have an associated rough path of finite 2-variation, up to logarithmic corrections. For such rough paths, the integral extends to integrands $F \in C^1$, see Chapter 10.5 of [FV10]. For typical price paths (but not for the area), it is shown in [Vov12], Section 4.3, that they are of finite 2-variation up to logarithmic corrections.*

A Pathwise Hoeffding inequality

In the construction of the pathwise Itô integral for typical price processes we needed the following result, a pathwise formulation of the Hoeffding inequality which is due to Vovk. Here we present a slightly adapted version.

Lemma A.1 ([Vov12], Theorem A.1). *Let $(\tau_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of stopping times with $\tau_0 = 0$, such that for every $\omega \in \Omega$ we have $\tau_n(\omega) = \infty$ for all but finitely many $n \in \mathbb{N}$. Let for $n \in \mathbb{N}$ the function $h_n: \Omega \rightarrow \mathbb{R}^d$ be \mathcal{F}_{τ_n} -measurable, and suppose that there exists a \mathcal{F}_{τ_n} -measurable bounded function $b_n: \Omega \rightarrow \mathbb{R}$, such that*

$$\sup_{t \in [0, T]} |h_n(\omega) S_{\tau_n \wedge t, \tau_{n+1} \wedge t}(\omega)| \leq b_n(\omega) \quad (20)$$

for all $\omega \in \Omega$. Then for every $\lambda \in \mathbb{R}$ there exists a simple strategy $H^\lambda \in \mathcal{H}_1$ such that

$$1 + (H^\lambda \cdot S)_t \geq \exp \left(\lambda \sum_{n=0}^{\infty} h_n S_{\tau_n \wedge t, \tau_{n+1} \wedge t} - \frac{\lambda^2}{2} \sum_{n=0}^{N_t} b_n^2 \right)$$

for all $t \in [0, T]$, where $N_t := \max\{n \in \mathbb{N} : \tau_n \leq t\}$.

Proof. Let $\lambda \in \mathbb{R}$. The proof is based on the following deterministic inequality: if (20) is satisfied, then for all $\omega \in \Omega$ and all $t \in [0, T]$ we have that

$$\begin{aligned} & \exp \left(\lambda h_n(\omega) S_{\tau_n \wedge t, \tau_{n+1} \wedge t}(\omega) - \frac{\lambda^2}{2} b_n^2(\omega) \right) - 1 \\ & \leq \exp \left(-\frac{\lambda^2}{2} b_n^2(\omega) \right) \frac{e^{\lambda b_n(\omega)} - e^{-\lambda b_n(\omega)}}{2b_n(\omega)} h_n(\omega) S_{\tau_n \wedge t, \tau_{n+1} \wedge t}(\omega) \\ & =: f_n(\omega) S_{\tau_n \wedge t, \tau_{n+1} \wedge t}(\omega). \end{aligned} \tag{21}$$

This inequality is shown in (A.1) of [Vov12]. We define $H_t^\lambda := \sum_n F_n 1_{(\tau_n, \tau_{n+1}]}(t)$, with F_n that have to be specified. We choose $F_0 := f_0$, which is bounded and \mathcal{F}_{τ_0} -measurable, and on $[0, \tau_1]$ we obtain

$$1 + (H^\lambda \cdot S)_t \geq \exp \left(\lambda h_0 S_{\tau_0 \wedge t, \tau_1 \wedge t} - \frac{\lambda^2}{2} b_0^2 \right).$$

Observe also that $1 + (H^\lambda \cdot S)_{\tau_1} = 1 + f_0 S_{\tau_0, \tau_1}$ is bounded, because by assumption $h_0 S_{\tau_0, \tau_1}$ is bounded by the bounded random variable b_0 .

Assume now that F_k has been defined for $k = 0, \dots, m-1$, that

$$1 + (H^\lambda \cdot S)_t \geq \exp \left(\lambda \sum_{n=0}^{\infty} h_n S_{\tau_n \wedge t, \tau_{n+1} \wedge t} - \frac{\lambda^2}{2} \sum_{n=0}^{N_t} b_n^2 \right)$$

for all $t \in [0, \tau_m]$, and that $1 + (H^\lambda \cdot S)_{\tau_m}$ is bounded. We define $F_m := (1 + (H^\lambda \cdot S)_{\tau_m}) f_m$, which is \mathcal{F}_{τ_m} -measurable and bounded. From (21) we obtain for $t \in [\tau_m, \tau_{m+1}]$

$$\begin{aligned} 1 + (H^\lambda \cdot S)_t &= 1 + (H^\lambda \cdot S)_{\tau_m} + (1 + (H^\lambda \cdot S)_{\tau_m}) f_m S_{\tau_m \wedge t, \tau_{m+1} \wedge t} \\ &\geq (1 + (H^\lambda \cdot S)_{\tau_m}) \exp \left(\lambda h_m S_{\tau_m \wedge t, \tau_{m+1} \wedge t} - \frac{\lambda^2}{2} b_m^2 \right) \\ &\geq \exp \left(\lambda \sum_{n=0}^{\infty} h_n S_{\tau_n \wedge t, \tau_{n+1} \wedge t} - \frac{\lambda^2}{2} \sum_{n=0}^{N_t} b_n^2 \right), \end{aligned}$$

where the last step follows from the induction hypothesis. From the first line of the last equation we also obtain that $1 + (H^\lambda \cdot S)_{\tau_{m+1}}$ is bounded because $f_m S_{\tau_m, \tau_{m+1}}$ is bounded for the same reason that $f_0 S_{\tau_0, \tau_1}$ is bounded. \square

B Davie's criterion

It was already observed by Davie [Dav07] that in certain situations the rough path integral can be constructed as limit of Riemann sums and not just compensated Riemann sums. Davie shows that under suitable conditions, the usual Euler scheme (without ‘‘area compensation’’)

converges to the solution of a given rough differential equation. But from there it is easily deduced that then also the rough path integral is given as limit of Riemann sums. Here we show that Davie's criterion implies our assumption (RIE).

Let $p \in (2, 3)$ and let $\mathbb{S} = (S, A)$ be a $1/p$ -Hölder continuous rough path, that is $|S_{s,t}| \lesssim |t - s|^{1/p}$ and $|A(s, t)| \lesssim |t - s|^{2/p}$. Write $\alpha := 1/p$ and let $\beta \in (1 - \alpha, 2\alpha)$. Davie assumes that there exists $C > 0$ such that the area process A satisfies

$$\left| \sum_{j=k}^{\ell-1} A(jh, (j+1)h) \right| \leq C(\ell - k)^\beta h^{2\alpha}, \quad (22)$$

whenever $0 < k < \ell$ are integers and $h > 0$ such that $\ell h \leq T$. Under these conditions, Theorem 7.1 of [Dav07] implies that for $F \in C^\gamma$ with $\gamma > p$ and for $t_k^n = kT/n$, $n, k \in \mathbb{N}$, the Riemann sums

$$\sum_{k=0}^{n-1} F(S_{t_k^n}) S_{t_k^n \wedge t, t_{k+1}^n \wedge t}, \quad t \in [0, T],$$

converge uniformly to the rough path integral. But it can be easily deduced from (22) that the area process A is given as limit of non-anticipating Riemann sums along $(t^n)_n$. Indeed, letting $h = T/n$,

$$\begin{aligned} \left| \int_0^t S_s dS_s - \sum_{k=0}^{n-1} S_{t_k^n} S_{t_k^n \wedge t, t_{k+1}^n \wedge t} \right| &= \left| \sum_{k=0}^{n-1} \left(\int_{t_k^n \wedge t}^{t_{k+1}^n \wedge t} S_s dS_s - S_{t_k^n \wedge t} S_{t_k^n \wedge t, t_{k+1}^n \wedge t} \right) \right| \\ &= \left| \sum_{k=0}^{n-1} A(t_k^n \wedge t, t_{k+1}^n \wedge t) \right| \leq \left| \sum_{k=0}^{\lfloor t/h \rfloor - 1} A_{kh, (k+1)h} \right| + |A(\lfloor t/h \rfloor h, t)| \\ &\lesssim C \lfloor t/h \rfloor^\beta h^{2\alpha} + h^{2\alpha} \|A\|_{2\alpha} \lesssim C t h^{2\alpha - \beta} + h^{2\alpha} \|A\|_{2\alpha}. \end{aligned}$$

Since $\beta < 2\alpha$, the right hand side converges to 0 as n goes to ∞ (and thus h goes to 0). Futhermore, (22) implies the “uniformly bounded $p/2$ -variation” condition (12):

$$\begin{aligned} \left| (S^n \cdot S)_{t_k^n, t_\ell^n} - S_{t_k^n} S_{t_k^n, t_\ell^n} \right| &\leq \left| \int_{t_k^n}^{t_\ell^n} S_s dS_s - S_{t_k^n} S_{t_k^n, t_\ell^n} \right| + \left| \sum_{j=k}^{\ell-1} \left(\int_{t_j^n}^{t_{j+1}^n} S_s dS_s - S_{t_j^n} S_{t_j^n, t_{j+1}^n} \right) \right| \\ &\leq \|A\|_{2\alpha} |t_\ell^n - t_k^n|^{2\alpha} + \left| \sum_{j=k}^{\ell-1} A_{t_j^n, t_{j+1}^n} \right| \leq \|A\|_{2\alpha} |t_\ell^n - t_k^n|^{2\alpha} + C(\ell - k)^\beta h^{2\alpha} \\ &\leq \|A\|_{2\alpha} |t_\ell^n - t_k^n|^{2\alpha} + C |t_\ell^n - t_k^n|^{2\alpha}. \end{aligned}$$

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