Topology of the Grünbaum–Hadwiger–Ramos hyperplane mass partition problem

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1. History

In 1960 Branko Grünbaum [4] asked whether any convex body in $\mathbb{R}^d$ could be cut into 2^d parts of equal volume/measure by d suitably chosen affine hyperplanes. He proved this for $d = 2$ via a simple application of the intermediate value theorem.

In 1966 Hugo Hadwiger [5] proved this for $d = 3$, while answering a problem by Jaworowski posed at the 1963 Topology meeting in Oberwolfach. As an intermediate step, he considered *two* masses in $\mathbb{R}^3$ and showed that they can simultaneously be cut into four pieces of equal measure by suitable hyperplanes. This was reproved later by Yao et al. [10] using the Borsuk–Ulam theorem.

In 1984 David Avis [1] noted that the answer is "no" for $d > 4$, by considering a measure concentrated on the moment curve. This also follows from a parameter count, as the space of d affine hyperplanes has dimension d^2, while $2^d - 1$ independent conditions have to be met. See [10] for still a different argument.

In 1996 Edgar A. Ramos [9] posed the general question: For $j, k \geq 1$, determine the smallest dimension $d = \Delta(j, k)$ such that any j masses in $\mathbb{R}^d$ can be simultaneously cut into 2^k orthants of equal measure by k suitably chosen affine hyperplanes. In view of the history just sketched, we call this the "Grünbaum–Hadwiger–Ramos hyperplane mass partition problem."

As a special case, this problem contains the ham-sandwich theorem: j measures can be halved in $\mathbb{R}^d$ by a single hyperplane if and only if $j \leq d$: That is, $\Delta(j, 1) = j$. Hadwiger's results quoted above amount to $\Delta(2, 2) = \Delta(1, 3) = 3$.

2. Summary

In [2] we provide a detailed status report about the results obtained for the Grünbaum–Hadwiger–Ramos problem up to now. This in particular includes

$$\left\lceil \tfrac{2^k-1}{k} j \right\rceil \;\leq\; \Delta(j,k) \;\leq\; j + (2^{k-1} - 1) 2^{\lfloor \log_2 j \rfloor} \qquad \text{for all } j, k \geq 1.$$

The lower bound was derived for $j = 1$ by Avis [1] and for general j by Ramos [9] from measures concentrated on the moment curve. The upper bound was obtained by Mani-Levitska, Vrećica & Živaljević [8] from a Fadell–Husseini index [3] calculation. All the available evidence is consistent with the belief that the lower bound is tight for all j and k; we refer to this as the "Ramos conjecture."

In particular, the lower and upper bounds coincide for the cases $k = 1$ (the ham-sandwich theorem) and for $k = 2$, $j = 2^t - 1$. A number of further values and bounds have been claimed in the literature up to now; however, as documented in [2], the proofs for all of these (except for the values listed above) seem to be incomplete. The last remaining case $\Delta(4, 1)$ of Grnbaum's problem was the subject of Blagojević's lecture at this Oberwolfach workshop.

3. Ansatz

Affine halfspaces (or oriented hyperplanes) are naturally parameterized by S^d, if one includes the special cases of all of $\mathbb{R}^d$ (for the north pole) and the empty subset of $\mathbb{R}^d$ (given by the south pole of $\mathbb{R}^d$). Whence we get $(S^d)^k$ as the configuration space for k affine half-spaces on $\mathbb{R}^d$. Thus the natural configuration space/test map scheme provides the following criterion:

Proposition 1 (Product Scheme)**.** *If there is no continuous map*

$$(S^d)^k \quad \longrightarrow_{\mathfrak{S}_k^\pm} \quad S(\{X \in \mathbb{R}^{j\times 2^k} : \text{sum of columns} = 0\})$$

that is equivariant with respect to the natural actions of the hyperoctahedral group $\mathfrak{S}_k^\pm$*, then there is no counterexample, i.e. "simultaneous mass partition works" for* j *measures and* k *hyperplanes in* $\mathbb{R}^d$*, that is,* $\Delta(j,k) \le d$*.*

The target space for this scheme, which records the values of measures in the orthants, is a sphere of dimension $(2^k-1)j-1$.

In the special case $k=1$ this says that if there is no map $(S^d) \to_{\mathbb{Z}/2} S(\mathbb{R}^j)$, which by Borsuk–Ulam happens when $j \le d$, then $\Delta(j,1) \le d$. This yields the ham-sandwich theorem, as discussed above.

We note that this scheme fails already for $k=1$ if we do not use the full configuration space S^d, e.g. by deleting north and south pole, as then the equivariant map *does exist*.

We also note that for $k>1$ the group action of $\mathfrak{S}_k^\pm$ is not free on $(S^d)^k$, which makes the treatment of the equivariant problem more difficult; however, the action of the subgroup $(\mathbb{Z}/2)^k$ is free—this is used for the upper bound quoted above.

4. A Theorem

Theorem 2. $\Delta(2^t+1,2) = \frac{3}{2}2^t + 2$ *for* $t \ge 1$*.*

This theorem was previously claimed by Živaljević in [11], but we concluded in [2] that his proof technique, which employs a newly-designed "equivariant algebraic obstruction theory," is not valid.

As a corollary (using a reduction detailed in [2]) we obtain that $\frac{3}{2}2^t \le \Delta(2^t,2) \le \frac{3}{2}2^t+1$ for $t \ge 1$. It was previously claimed by Ramos in [9] that indeed the lower bound holds, but we concluded in [2] that Ramos' proofs for this are not valid.

Sketch of proof. This is known to hold for $t=1$, so we may assume $t \ge 2$.
(i) For parameters $2d = 3j+1$ we try to prove that there is no equivariant map

$$S^d \times S^d \quad \longrightarrow_{\mathfrak{S}_2^\pm} \quad S^{3j-1}$$

induced by a test map. By restricting to the equator spheres (which parameterize linear hyperplanes in $\mathbb{R}^d$), we obtain

$$\Psi : S^{d-1} \times S^{d-1} \quad \longrightarrow_{\mathfrak{S}_2^\pm} \quad S^d \times S^d \quad \longrightarrow_{\mathfrak{S}_2^\pm} \quad S^{3j-1},$$

which is a map between orientable manifolds of the same dimensions, which has to have degree 0 if $d > 2$.

(ii) Let $N \subset S^{d-1} \times S^{d-1}$ be the non-free subset (that is, all points where the 8-element dihedral group $\mathfrak{S}_2^{\pm}$ has non-trivial stabilizer).

If Ψ, Ψ' are test maps for different families of measures, then we observe that $\Psi|_N \simeq \Psi'|_N$ as on the non-free part the test maps go to an affine subspace that is not linear (does not contain the origin) and thus they can be connected by a linear equivariant homotopy. From this we get $\deg \Psi \equiv \deg \Psi' \pmod 8$ from a generalized equivariant Hopf theorem [7].

(iii) We compute $\deg \Psi'$ for some specific measures on the moment curve without support at the origin, by looking at zeros of the extended map $\hat{\Psi}' : S_+^d \times S^{d-1} \to \mathbb{R}^{3j}$, where S_+^d denotes the upper hemisphere. When d is odd these zeros come in pairs with opposite orientations, so they cancel and $\deg \Psi' = 0$. When d is even the zeros come in pairs with the same orientations, and we count solutions to get

$$\deg \Psi' = 2\binom{j}{\frac{j-1}{2}}.$$

This is nonzero mod 8 if and only if $j = 2^t + 1$ by Kummer's criterion [6] from 1852. □

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