Simplicial complex models for arrangement complements

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1. Why do we care?

1.1. Arrangements and configuration spaces. The configuration space of \( n \) labeled distinct points on a manifold

\[
F(X, n) := \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for } i < j \}
\]

appears in diverse contexts in topology (providing, for example, embedding invariants and models for loop spaces), knot theory, and physics (KZ equation, renormalization); see e.g. Vassiliev [20] and Fadell & Husseini [8]. In particular, the space

\[
F(\mathbb{R}^d, n) := \{(x_1, \ldots, x_n) \in \mathbb{R}^{d \times n} : x_i \neq x_j \text{ for } i < j \}
\]

has been studied in great detail. It is the complement of an arrangement of linear subspaces in \( \mathbb{R}^d \) whose intersection lattice (with the customary ordering by reversed inclusion) is the partition lattice \( \Pi_n \) of rank \( n - 1 \). The cohomology is free, with Poincaré polynomial

\[
\prod_{i=1}^{n-1}(1 + t(d-1));
\]

see e.g. Björner [4], Goresky & MacPherson [13, Part III].

In particular, for \( d = 2 \) the space \( F(\mathbb{C}, n) = F(\mathbb{R}^2, n) \) appears in work by Arnol’d related to Hilbert’s 13th problem, continued in papers by works by Fuks, Deligne, Orlik & Solomon, and many others. It is a key example for the theory of (complex) hyperplane arrangements; see e.g. Orlik & Terao [17].

The space \( F(\mathbb{R}^d, n) \) is the complement of a codimension \( d \) subset in \( \mathbb{R}^{d \times n} \), so in particular it provides a \( (d - 2) \)-connected space on which the symmetric group \( \mathfrak{S}_n \) acts freely. Thus the inclusions \( F(\mathbb{R}^2, n) \subset F(\mathbb{R}^3, n) \subset \cdots \) can be used to compute the cohomology of \( \mathfrak{S}_n \); see Giusti & Sinha [12] for recent work, which is based on the Fox–Neuwirth stratification [11] of the configuration spaces \( F(\mathbb{R}^d, n) \).

1.2. Cell complex models. A problem by R. Nandakumar and N. Ramana Rao [16] asks whether every bounded convex set \( P \) in the plane can be divided into \( n \) convex pieces that have equal area and equal perimeter. In [18] the same authors prove this for \( n = 2k \) in the case where \( P \) is a convex polygon. Blagojević, Bárány & Szűcs [2] established the problem for \( n = 3 \).

Karasev [15] and Hubard & Aronov [14] observed that a positive solution for the problem would — via optimal transport (cf. Villani [21]) and generalized Voronoi diagrams (cf. Aurenhammer et al. [1]) — follow from the non-existence of an equivariant map

\[
F(\mathbb{R}^2, n) \rightarrow \mathfrak{S}_n \quad S(\{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_1 + \cdots + y_n = 0\}) \simeq S^{n-2}.
\]

A \( d \)-dimensional and more general version of the problem, to partition any sufficiently continuous measure on \( \mathbb{R}^d \) into \( n \) pieces of equal measure that also equalize \( d - 1 \) further continuous functions, could be solved by the non-existence of

\[
F(\mathbb{R}^d, n) \rightarrow \mathfrak{S}_n \quad S(\{(y_1, \ldots, y_n) \in \mathbb{R}^{d-1 \times n} : y_1 + \cdots + y_n = 0\}) \simeq S^{(n-1)(d-1)-1}.
\]
In the cited works by Karasev and Hubard & Aronov this is approached via (twisted) Euler class computations on the one-point compactification of \( F(\mathbb{R}^d, n) \) (which is not a manifold), which leads to the non-existence of these maps if \( n \) is a prime power.

Here we report about an alternative approach, via Equivariant Obstruction Theory (as developed by tom Dieck [7, Sect. II.3]). For this we need an equivariant cell complex model for \( F(\mathbb{R}^d, n) \).

2. A method

We rely on a method developed in Björner & Ziegler [5] to obtain a compact cell complex model for the complements of linear subspace arrangements. For this let \( \mathcal{A} \) be a finite arrangement of linear subspaces in a real vector space \( \mathbb{R}^N \). Each \( k \)-dimensional subspace \( F \) of the arrangement is embedded into a complete flag of linear subspaces \( F = F_k \subset F_{k+1} \subset \cdots \subset \mathbb{R}^N \). The union of these flags yields a stratification of \( \mathbb{R}^N \) into relative-open convex cones. These cones are not usually pointed, but their faces are unions of strata. The barycentric subdivision of the stratification yields a triangulation of a star-shaped neighborhood of the origin in \( \mathbb{R}^N \), which as a subcomplex contains a triangulation of the link of the arrangement. Its Alexander dual is the barycentric subdivision of a regular CW complex, realized as a geometric simplicial complex that is a strong deformation retract of the complement. Moreover, if the arrangement has a symmetry, and the flags are chosen to be compatible with the symmetry, then the resulting complex carries the symmetry of the arrangement.

3. Examples

Implementing the construction from [5] yields the Fox–Neuwirth stratification on the complement \( F(\mathbb{R}^d, n) \) of the arrangement, but indeed our construction and proof uses the stratification on the full ambient space \( \mathbb{R}^{d \times n} \).

**Theorem 3.1.** There is a regular cell complex \( F(d, n) \) of dimension \( (n-1)(d-1) \) that has \( n! \) vertices and \( n! \) facets (maximal cells), with a free cellular action of the symmetric group \( \mathfrak{S}_n \), that is transitive on the vertices as well as on the facets.

The barycentric subdivision \( \text{sd} F(d, n) \) has a geometric realization in \( F(\mathbb{R}^d, n) \) as an equivariant strong deformation retract.

Based on this model, our Equivariant Obstruction Theory calculation gives a complete answer to the equivariant map problem, and thus a simple combinatorial proof for the prime power case of the Nandakumar & Ramana Rao conjecture:

**Theorem 3.2 ([6]).** An equivariant continuous map

\[
F(\mathbb{R}^d, n) \to \mathfrak{S}_n, S(\{(y_1, \ldots, y_n) \in \mathbb{R}^{(d-1) \times n} : y_1 + \cdots + y_n = 0\}) \simeq S^{(n-1)(d-1)-1}. 
\]

does not exist if and only if \( n \) is a prime power.

At the combinatorial core of our calculation lies the fact, apparently first proved by B. Ram in 1909 [19], that \( \gcd\{\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}\} \) equals \( p \) for any prime power \( n = p^k \), and equals 1 otherwise.
4. Further Examples

In view of further applications to geometric measure partition problems, there is interest in constructing and analyzing cell complex models for spaces such as $F^d(S^d, n)$ (see Feichtner & Ziegler [9] and Basabe et al. [3]) as well as $F_{\pm}(S^d, n)$ (see [10]).

References

[19] B. Ram, Common factors of $\frac{n^m}{m(m-1)}$, $(m = 1, 2, \ldots, n-1)$, J. Indian Math. Club (Madras) 1 (1909), 39–43.