Ten Problems in Geometry

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Abstract

We describe ten problems that wait to be solved.

Introduction

Geometry is a field of knowledge, but it is at the same time an active field of research — our understanding of space, about shapes, about geometric structures develops in a lively dialog, where problems arise, new questions are asked every day. Some of the problems are settled nearly immediately, some of them need years of careful study by many authors, still others remain as challenges for decades.

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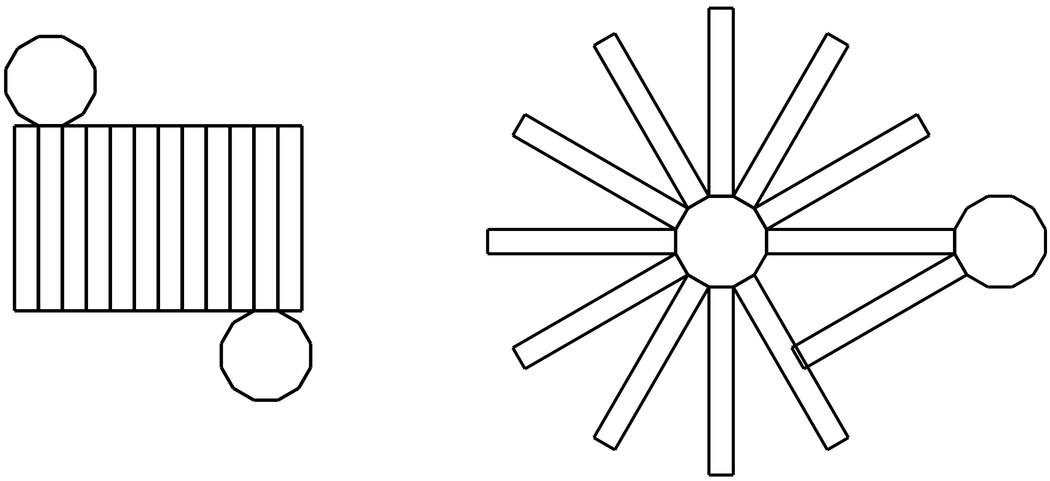
1 Unfolding polytopes

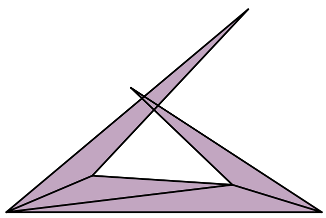
In 1525 Albrecht Dürer's famous geometry masterpiece "Underweysung der Messung mit dem Zirckel und Richtscheyt"[1] was published in Nuremberg. Its fourth part contains many drawings of nets of 3-dimensional polytopes. Implicitly it contains the following conjecture:

Every 3-dimensional convex polytope can be cut open along a spanning tree of its graph and then unfolded into the plane without creating overlaps.

Explicitly, this conjecture was posed by the British mathematician Geoffrey C. Shephard in 1975. It has captured many geometers' attention since then — and led to many interesting results and insights in this area. One of these insights is that the spanning tree has to be chosen with care. Given any polytope, it is easy to find some spanning tree in its graph. After cutting the boundary along the edges of this tree, there is a unique way to unfold it into a planar figure. However, the problem is that overlaps could occur, and indeed they do occur. The following drawing shows an unfolded prism, once cut open along a good spanning tree, once cut open along a bad spanning tree.



Moreover, perhaps surprisingly, Makoto Namiki has observed that even an unfolding of a tetrahedron can result in an overlap:



The conjecture has been verified for certain somewhat narrow classes of polytopes. For example, it holds for so-called *prismoids*. These are built by taking the convex hull of two polygons that lie in parallel planes, have the same number of sides and the same angles and are positioned in such a way that their corresponding edges are parallel. Another class of polytopes for which the conjecture has been established are the *domes*. A dome has a base face and all its other faces share an edge with this base.

[1] The title of the English translation is: "The Painter's Manual: Instructions for Measuring with Compass and Ruler"

One approach to the problem is algorithmic. For a proof that all polytopes can be unfolded, we need a good strategy to choose a suitable spanning tree. One could look, for example, for a shortest or a longest spanning tree, which minimizes or maximizes the sum of the lengths of the edges, respectively. Or one can place the polytope in space such that no edge is horizontal and the highest vertex is unique, and then from any other vertex choose the steepest edge or the "rightmost" edge that points upwards. Such rules are motivated by and derived from various "pivot rules" of linear programming, which describe local strategies to move from any given vertex of a polyhedron along edges to the highest vertex. Extensive experiments with such rules were performed by Wolfram Schlickenrieder for his 1997 diploma thesis. None of the strategies tested by him worked for all examples, but for all examples *some* of his strategies worked.

Further interesting studies motivated by the unfolding conjecture concern relaxations of the problem. For example, there are unfolding techniques that do not cut only along edges, but may cut into faces, such as the *source unfolding* and the *star unfolding* discovered by Alexandrov. Here we only sketch the latter technique: For a star unfolding one picks one point on the boundary of the polytope such that it has a unique shortest path to every vertex. The union of theses paths form a tree that connects all the vertices. If one cuts the polytope boundary open along this tree then this an unfolding that provably has no overlaps.

Notes on Problem 1

Dürer's geometry [2] book mentioned in the beginning is an exciting piece of art and science. The original source for the unfolding polytopes problem is Shephard [6]. The example of an overlapping unfolding of a tetrahedron is reported by Fukuda [3]. The first figure in our presentation is taken from [5], with kind permission of the author. For more detailed treatments of nets and unfolding and for rich sources of related material, see O'Rourke's chapter in this volume, as well as Demaine and O'Rourke [1] and Pak [4, Sect. 40], where also the source and star unfoldings are presented and discussed.

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2 Almost disjoint triangles

How complicated can polyhedral structures be in 3-dimensional space? For example, we are interested in triangulated surfaces on n vertices in $\mathbb{R}^3$, such as the boundary of a tetrahedron, which has $n = 4$ vertices and $n = 6$ edges, of an octahedron, with $n = 6$ vertices and $n = 12$ edges, or of an icosahedron, with $n = 12$ vertices and $e = 30$ edges.

But what is the maximal number of edges for a triangulated surface in $\mathbb{R}^3$ on n vertices? Certainly it cannot have more than $\binom{n}{2}$ edges. This bound is not tight for all n, since for a triangulated surface the number of edges is divisible by 3. Indeed, every triangle is bounded by three edges, while each edge is contained in two triangles, the number e of edges and the number f of triangles satisfy the equation $3f = 2e$ and thus f is even and e is divisible by three. Another constraint comes from the fact that the surfaces we look at are embedded in $\mathbb{R}^3$. They have an "inside" and an "outside", so they are *orientable*, which implies that the *Euler characteristic* $n - e + f$ is even (it equals $2 - 2g$, where g is known as the *genus* of the surface). Nevertheless, this leads only slight improvements of the upper bounds. If n is congruent to $0, 3, 4$, or 7 modulo 12 then it seems entirely possible that a surface with $e = \binom{n}{2}$ exists; its face numbers would be given by $(n, e, f) = (n, \binom{n}{2}, \frac{2}{3}\binom{n}{2})$.

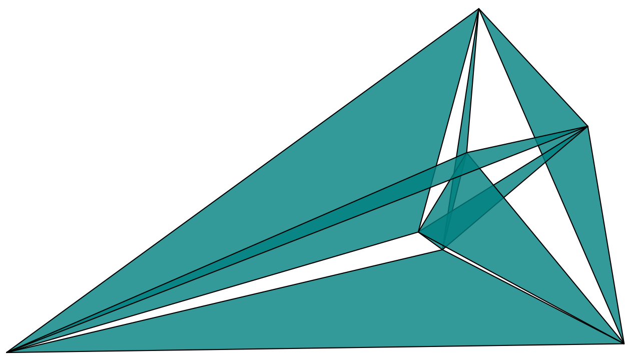
Is there such a "neighborly" triangulated surface for all these parameters? For small values of n it seems so: For $n = 4$ we have the tetrahedron, and for $n = 7$ there is a triangulated surface, known as the *Császár torus*, which consists of 14 triangles and $\binom{7}{2} = 21$ edges. At the next n where we get integer parameters, $n = 12$ and $e = 66$ and $g = 6$, it is known that combinatorial schemes for suitable triangulated surfaces exist; indeed, this was established for all $n \equiv 0, 3, 4$, or 7 (mod 12) as part of the so-called *Map Color Theorem* by Ringel et al. (1974). However, *none* of the 59 combinatorial schemes for such a surface can be realized as a triangulated surface in $\mathbb{R}^3$. This was established by Jürgen Bokowski, Antonio Guedes de Oliveira, and finally Lars Schewe quite recently.

But let us get beyond the small parameters. What can we expect when n gets large? Will the maximal number of edges in a triangulated surface with n vertices grow quadratically with n, or much slower? All we know at the moment is that the maximal e grows at least as fast as $n \log n$; this can be seen from surfaces that were constructed by Peter McMullen, Christoph Schulz, and Jörg Wills in 1983.

However, Gil Kalai has proposed to study a closely-related problem that is even easier to state, and may be similarly fundamental:

Given n points in 3-dimensional space, how many triangles could they span that are disjoint, except that they are allowed to share vertices?

So for Kalai's problem the triangles are not allowed to share an edge, and they are not allowed to intersect in any other way. Let us call this *almost disjoint triangles.*



Without loss of generality we may assume that the n points that we use as vertices lie in general position, no three of them on a line and no four of them in a plane. Clearly the maximal number $T(n)$ of vertex disjoint triangles on n points is not larger than $\frac{1}{3}\binom{n}{2}$. But is

$$T(n) \;\leq\; \tfrac{1}{3}\binom{n}{2}$$

a tight upper bound for infinitely many values of n? Does $T(n)$ grow quadratically when n gets large? All we know is that there is a lower bound that grows like $n^{3/2}$: Gyula Károlyi and Jozsef Solymosi in 2002 presented a very simple and elegant method to position $n = m^2 + \binom{m}{2}$ points in $\mathbb{R}^3$ that span $m\binom{m}{2}$ almost-disjoint triangles.

Notes on Problem 2

The neighborly triangulation of the torus with 7 vertices (with $\binom{7}{2} = 21$ edges) was described by Möbius in 1861, but the first polytopal realization without self-intersections was only provided by Császár in 1948 [2]; see also [4]. Neighborly triangulations of orientable surfaces for all possible parameters were provided by Ringel et al. as part of the *Map Color Theorem* [6]. Beyond $n = 4$ (the boundary of a tetrahedron) and $n = 7$ (the Császár torus) the next possible value is $n = 12$: But it was shown by Bokowski and Guedes de Oliveira [1] and Schewe [7] that indeed there is no realization of any of the 59 combinatorial types of a neighborly surface with 12 vertices and $\binom{12}{2} = 66$ edges (of genus 6) without self-intersections in $\mathbb{R}^3$. The McMullen–Schulz–Wills surfaces "with unusually large genus" were constructed in [5]; see also [8]. The question about almost disjoint triangles was posed by Gil Kalai; see Károlyi and Solymosi [3] for the problem and for the lower bound of $n^{3/2}$.

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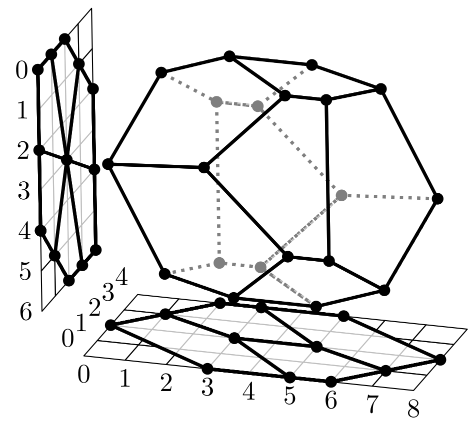
3 Representing polytopes with small coordinates

A famous theorem by Steinitz from 1922 characterizes the graphs of 3-dimensional convex polytopes. It states:

Theorem. *A finite graph G is the edge graph of a polytope P if and only if G is planar and 3-connected.*

Obtaining a polytope from a given such graph is a construction problem. Here one is especially interested in nice realizations. Of course, the meaning of "nice" depends on the context. One possibility is to ask for a polytope that has all its edges tangent to a sphere. Such a realization exists and it is essentially unique. This can be derived from the Koebe–Andreev–Thurston circle packing theorem. However, the edge tangent realizations are not combinatorial, and in general they have irrational vertex coordinates. One can also ask for rational realizations, such that all vertex coordinates are rational, or equivalently (after multiplication with a common denominator) for integral realizations. The existence of such realizations can be derived from Steinitz' original proofs. Just how large would the integers have to be?

The following drawing shows a dodecahedron realized with very small integer coordinates due to Francisco Santos.



The big open problem is:

Can every 3-dimensional convex polytope with n vertices be realized with its vertex coordinates in the integer grid $\{0, 1, \ldots, f(n)\}^3$, where f is a polynomial?

All we know at the moment are exponential upper bounds on $f(n)$. The first such bounds were derived by Shmuel Onn and Bernd Sturmfels in 1994; they were subsequently improved to $f(n) < 148^n$. But indeed we know of no lower bounds that would exclude that all combinatorial types can be realized with $f(n) < n^2$. A recent result by Erik Demaine and André Schulz from 2010 is that for the very special case of stacked polytopes (that is, obtained from a tetrahedron by repeatedly stacking a flat pyramid onto a facet), realizations with polynomially-bounded integer coordinates exist. But do these exist for graphs of 3-polytopes?

Notes on Problem 3

Steinitz' theorem [8, 9] is a fundamental result. See Grünbaum [3, Chap. 13] and Ziegler [12, Lect. 3] for modern treatments. Steinitz' proofs imply that a realization with integer vertex coordinates exists for every combinatorial type. Furthermore, there are only finitely many different combinatorial types for each n, so $f(n)$ exists and is finite. The first explicit upper bounds on $f(n)$ were derived by Onn and Sturmfels [4] from the rubber band realization method of Tutte [11]. After the exposition by Richter-Gebert [7] there has been a lot of research to improve the upper bounds; see in particular Ribó Mor, Rote & Schulz [5, 6]. The upper bound of $f(n) < 148^n$ can be found in the paper by Buchin and Schulz [1]. The result about stacked polytopes was achieved by Demaine and Schulz in the [2]. A lower bound of type $f(n) \geq n^{3/2}$ follows from the fact that grids of such a size are needed to realize a convex n-gon; compare [10]. The theorem about edge-tangent realizations of polytopes via circle packings is detailed in [13, Sect. 1]; we refer to that exposition also for further references.

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4 Polyhedra that tile space

An innocent-sounding question is

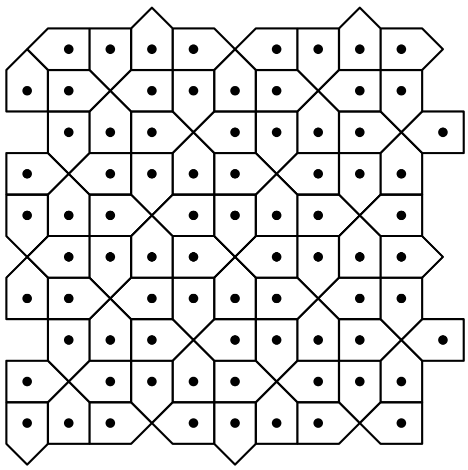
Which convex polytopes can be used to tile 3-dimensional space?

Unfortunately, answering this question seems to be quite difficult. Indeed, not even the 2-dimensional version of this problem has been solved completely, although this has been claimed and believed several times, starting with Reinhardt's contribution from 1918. Nevertheless, for tilings of the plane it is not hard to see that any convex polygon that admits a tiling of the plane — that is, such that the plane can be completely covered by congruent copies of this polygon, without gaps and without overlapping interiors — can have at most six faces. The reason for this is topological and can be connected to Euler's polyhedron formula. Clearly the regular hexagon can be used to tile the plane (any bee knows that), but many other types of convex hexagons admit such a tiling as well.

One dimension higher, we all know the tiling of space by congruent cubes, which have 8 faces. However, it is also not hard to see that translates of the so-called permutahedron with its 14 faces and 24 vertices tile space face-to-face. So this begs the question: What is the maximal number of faces for a convex polytope that allows for a tiling of 3-space by congruent copies? In 1980, the crystallographer Peter Engel from Bern has produced two types of polytopes with 38 faces that tile space, and up to now this record apparently has not been topped. On the other hand, no finite upper bound is known, and the answer may as well be that there is no finite upper bound. The problem seems to be that the only effective method to produce such tilings is to look at dot-patterns (discrete point sets) in $\mathbb{R}^3$ that have a transitive symmetry group, that is, such that for any two points in the pattern there is a symmetry of space that moves one point to any other one. For such a point configuration $S \subset \mathbb{R}^3$ all the *Voronoi domains*

$$V_s \;:=\; \left\{x \in \mathbb{R}^3 : \|x - s\| \leq \|x - s'\| \text{ for } s' \in S \backslash \{s\}\right\}$$

for $s \in S$ are congruent. The Voronoi domain of s collects all points in space for which no other point in S is closer. The following drawing shows an excerpt of a symmetric dot pattern with its Voronoi cells.



The Voronoi construction applied to symmetric dot patterns is very effective in producing tilings by congruent polytopes. Indeed, Engel's two examples with 38 faces were produced this way. However, it is also known that for tilings of this type the number of faces is bounded.

So symmetry helps to construct tilings. We should, however, not rely on this too much. In his famous list of 23 problems from the 1900 International Congress of Mathematicians in Paris, David Hilbert had asked as part of his 18th problem whether there could be a convex polytope that tiles 3-dimensional space, but such that there is no tiling that would have a symmetry group that moves tiles to tiles. The answer has long been known to be *yes*: such tilings exist. For example, various types of quasicrystals demonstrate this. This shows that even though symmetry helps a lot in constructing tilings, it should not be used as our only resource.

Notes on Problem 4

A survey of the theory of tilings can be found in Schulte's article in [6, Chap. 3.5]. For tilings with congruent polytopes, we refer to the survey by Grünbaum & Shephard [4]. The book by Grünbaum & Shephard [5] is a rich source of information on planar tilings. For the problem about the maximal number of faces, see also Brass et al. [1, Sect. 4.3]. Engel presented his tilings by congruent polytopes with up to 38 faces in [2] and [3].

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5 Fatness

Whereas *regular* convex polytopes (the "Platonic solids") have been studied since antiquity, *general* convex polytopes came into the focus of attention much later. To Descartes and Euler we owe the "Euler polyhedron formula". In modern notation, where we write f_i for the number of i-dimensional faces of a convex polytope, it states that every 3-dimensional convex polytope satisfies

$$f_0 - f_1 + f_2 \;=\; 2.$$

In 1906 Ernst Steinitz characterized the set $\mathcal{F}_3$ of all possible triples (f_0, f_1, f_2) for convex polytopes:

$$\mathcal{F}_3 \;=\; \{(f_0, f_1, f_2) \in \mathbb{Z}^3 : f_0 - f_1 + f_2 \;=\; 2, \quad f_2 \le 2f_0 - 4, \quad f_0 \le 2f_2 - 4\}.$$

More than one-hundred years later, no similarly complete description is available for the possible sequences of face numbers, or *f-vectors*, of d-dimensional polytopes for any $d > 3$. Indeed, we know that the f-vectors of d-dimensional convex polytopes satisfy essentially only one linear *equation*, the so-called *Euler–Poincaré equation*:

$$f_0 - f_1 + f_2 - \;\cdots\; + (-1)^{d-1} f_{d-1} \;=\; 1 - (-1)^d.$$

However, we do not know all the linear *inequalities*. In particular, we would be interested in linear inequalities that hold with equality for the d-dimensional simplex, which has $f_i = \binom{d+1}{i+1}$, as these special inequalities describe the "cone of f-vectors".

To make this concrete, let us concentrate on the case $d = 4$. Here everything boils down to the question whether the parameter called *fatness*,

$$\Phi \;:=\; \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10},$$

can be arbitrarily large for 4-polytopes. Can it be, say, larger than 10? Or is it true that

$$f_1 + f_2 - 20 \;\le\; 10(f_0 + f_3 - 10)$$

for all convex 4-dimensional polytopes?

Indeed, it is not hard to show that the fatness parameter Φ ranges between 2.5 and 3 for simple and simplicial polytopes. However, it is $\Phi = 4.52$ for a fascinating 4-dimensional regular polytope known as the "24-cell", which has $f_0 = 24$ vertices and $f_3 = 24$ facets, which are regular octahedra; its complete f-vector is $(f_0, f_1, f_2, f_3) = (24, 96, 96, 24)$. An even higher value of $\Phi = 5.021$ is achieved for the "dipyramidal 720-cell" constructed in 1994 by Gabor Gévay, which has f-vector $(720, 3600, 3600, 720)$. Finally, a class of polytopes named "projected products of polygons", constructed by the second author in 2004, get arbitrarily close to $\Phi = 9$. That's where we stand at the time of writing. But is there a finite upper bound at all?

This may read like a problem of 4-dimensional geometry and thus outside our range of visualization, but it isn't really, since the boundary of a 4-dimensional polytope is of dimension 3. Thus one can relate the question to problems about polytopal tilings in 3-space. Here is one such problem:

Are there normal face-to-face tilings of 3*-space by convex polytopes in which*

(1) *all tiles have many vertices,* ***and***

(2) *each vertex is in many tiles?*

For example, in the usual tiling of space by unit cubes all tiles have 8 vertices and each vertex is in 8 tiles. For a *normal* tiling we require that there is a lower bound for the inradius and an upper bound for the circumradius of the tiles. This is satisfied, for example, if there are only finitely many types of tiles. It is not too hard to show that either of the two conditions (1) and (2) can be satisfied. But can they be satisfied by the same tiling, at the same time? If no, then fatness Φ for 4-polytopes is bounded.

Notes on Problem 5

The fascinating history of the original work by Descartes — which was lost, reconstructed and rediscovered several times — is reported in a book by Federico [1]. The paper by Steinitz from 1906 that describes the f-vectors (f_0, f_1, f_2) of 3-polytopes completely is [4]. The fatness parameter first appears (with a slightly different definition) in [5]; see also [7, Sect. 4]. The 720-cell was apparently first found and presented by Gévay [2]. The "projected deformed products of polygons" were introduced in [6]; a complete combinatorial analysis is in Sanyal & Ziegler [3].

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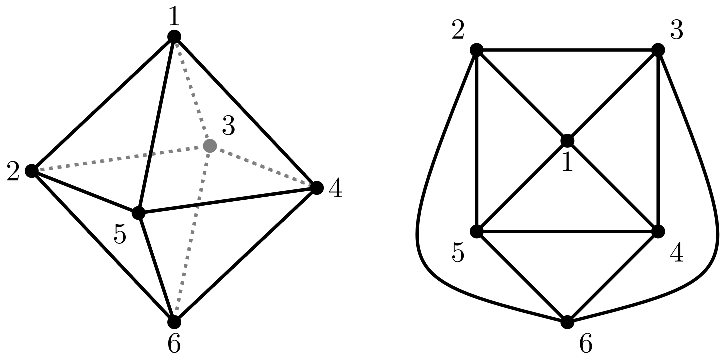
6 The Hirsch conjecture

One of the biggest mysteries in convex geometry is about the graphs of convex polytopes and their diameters. The *graph* of a polytope is a combinatorial model which captures vertex-edge incidences. Such a graph has the vertices of the polytope as nodes and two nodes are adjacent in the graph if they are connected by an edge as vertices in the polytope. The following picture shows the octahedron and its graph.



The *diameter* of a graph is the greatest distance of two vertices in the graph, where the *distance* of two vertices is the length of the shortest path connecting them. In 1957 Warren Hirsch raised the question:

What is the maximal diameter of the graph of a d-polytope with n facets?

He conjectured that

$$\Delta(d, n) \leq n - d,$$

where $\Delta(d, n)$ denotes the above maximal diameter. Even though decades of research went into a solution of this problem, for over 50 years little progress was made. Finally, in May 2010, Francisco Santos announced a counter-example. By an explicit construction he could demonstrate that $\Delta(43, 86) > 43$.

While this certainly was a breakthrough, it is merely a first step in answering the above question. Santos' construction does not even rule out a diameter linear in $n - d$. Many researchers in discrete geometry believe that the real question is whether there is a polynomial bound in n and d. The best upper bound for general d-polytopes was derived by Gil Kalai and Daniel Kleitman in 1992. By using a strictly combinatorial approach they were able to prove that

$$\Delta(d, n) \ \leq \ n^{2+\log_2 d},$$

but of course this is still very far away from a polynomial bound. Furthermore, by a result of Larman it follows that if one fixes the dimension d, the bound is linear in n. In general it is sufficient to prove an upper bound for simple polytopes: The facets of a non-simple polytope can be perturbed such that one gets a simple polytope. This new polytope has a graph whose diameter is at least as large as for the original graph.

Besides its importance for polyhedral geometry, the question also relates closely to linear programming. The maximal graph diameter $\Delta(d, n)$ is a lower bound for the number of steps that the simplex algorithm would need on a problem with n constraints in d variables for *any* pivot rule that would select the edges. Thus researchers from Operations Research and Mathematical Optimization are interested in the Hirsch Conjecture as well.

Notes on Problem 6

The Hirsch conjecture appears in Dantzig's classic 1963 book on linear programming [1, p. 168]. For surveys see Klee [2, Chap. 16], Klee & Kleinschmidt [7], Ziegler [11, Sect. 3.3], and most recently Kim & Santos [6]. The Kim–Santos paper in particular explains very nicely many bad examples for the Hirsch conjecture. Santos' long-awaited counter-example to the Hirsch conjecture appears in [10]. The Kalai–Kleitman quasipolynomial upper bound [5] also can be found in [11, Sect. 3.3]. The result by Larman was published in [8]. For the connection to Linear Programming we refer to [11, Lect. 3], Matoušek, Sharir & Welzl [9], Kalai [4], and Kaibel et al. [3].

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7 Unimodality for cyclic polytopes

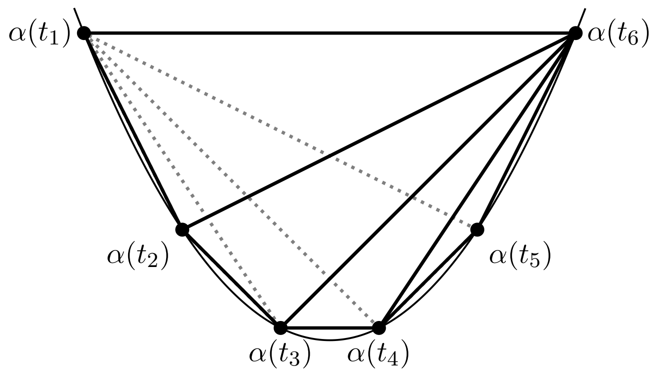
In 1970 McMullen settled the long-standing open question "What is the maximal number of k-faces of a d-polytope on n vertices?" He confirmed that neighborly simplicial polytopes are extremal with regard to their f-vectors: A *neighborly* polytope is a polytope such that every subset of the vertices of cardinality at most $\lfloor d/2 \rfloor$ is the vertex set of a face; the *f-vector* of a polytope is the sequence of numbers of vertices, edges, 2-faces, 3-faces, and so on. One well-known class of such polytopes are the cyclic polytopes. These can be defined as the convex hull of finitely many points on the *moment curve*

$$\alpha : \mathbb{R} \longrightarrow \mathbb{R}^d, \quad t \longmapsto (t, t^2, \ldots, t^d).$$

That is, one chooses n different reals, $t_1 < \cdots < t_n$, and calls

$$C_d(n) = \operatorname{conv}(\alpha(t_1), \ldots, \alpha(t_n))$$

the d-dimensional *cyclic polytope* on n vertices. (A simple analysis, using the Vandermonde determinant, shows that cyclic polytopes are simplicial, and that the combinatorial type does not depend on the particular parameters t_i chosen.) Our drawing shows a realization of $C_3(6)$.



Since the cyclic polytopes play such a distinct role in polytope theory it is of considerable interest to understand them completely. Unfortunately, even quite basic questions about their f-vector are still open:

Is it true that the f-vectors of cyclic polytopes are unimodal for all $n > d \geq 2$?

A sequence is called *unimodal* if it first increases and then decreases with no dips in-between. While all f-vectors of polytopes of dimension $d \leq 5$ are unimodal, polytopes with non-unimodal f-vector exist for all $d \geq 8$. For simplicial polytopes one has only unimodal f-vectors up to $d = 19$ but for each $d > 19$, there exist a simplicial polytope with non-unimodal f-vector. Nevertheless, it is generally expected that unimodality holds for the f-vectors of cyclic polytopes. For example, the face vector of $C_9(16)$ is

$$f(C_9(16)) = (1, 16, 120, 560, 1820, 4116, 6160, 5720, 2970, 660).$$

This open problem is especially tantalizing because the face numbers $f_k(C_d(n))$ are known explicitly. If d is odd then

$$f_{k-1}(C_d(n)) = \sum_{i=1}^{(d-1)/2} \left(\binom{d-i}{k-i} + \binom{i}{k-d+i}\right)\binom{n-d-1+i}{i}, \tag{1}$$

and similarly for even d. Numerical tests carried out on a computer support the conjecture. All cyclic polytopes with less than 1000 vertices or with $d \leq 125$ and $n \leq 10000$ have unimodal f-vectors. The conjecture also holds asymptotically: For fixed d and $n \gg d$ the f-vector $f(C_d(n))$ is unimodal. A natural approach to the problem is to try to bring the sum (1) into a closed form and then try to show unimodality. However, the sum under consideration cannot be represented as a fixed number of hypergeometric terms. If one splits the sum (1) into two, then the second one has a closed form (with $e = (d-1)/2$):

$$\sum_{i=0}^{e} \binom{i}{k-d+i}\binom{n-d-1+i}{i} = \frac{k-e}{n-k}\binom{e+1}{k-e}\binom{n-e-1}{e+1}.$$

So the troublemaker which prevents us from simplifying $f_k(C_d(n))$ is the first sum.

Notes on Problem 7

The upper-bound theorem was proved by McMullen in [6]. Very high-dimensional simplicial polytopes with non-unimodal f-vectors were apparently first constructed by Ludwig Danzer in the 1960s. The results quoted about non-unimodal d-polytopes, $d \geq 8$ and about non-unimodal simplicial d-polytopes, $d \geq 20$, are due to Björner [1] [2], Lee [5], and Eckhoff [3] [4]. For a current survey concerning general (non-simplicial) polytopes see Werner [8, Sect. 6.2]. A detailed discussion of cyclic polytopes and their properties can be found in [9, pp. 14-15]. The results about closed forms for the f-vectors of cyclic polytopes and their unimodality are due to Schmitt [7].

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8 Decompositions of the cube

Consider the d-dimensional cube $I_d = [0,1]^d$ and define the following parameters:

- Let $C(I_d)$ be the minimal number of d-dimensional simplices needed for a cover of I_d. A *cover* of I_d is a collection of simplices such that the union of all simplices is I_d. The interiors of simplices are allowed to intersect.
- If all vertices of a cover are also vertices of I_d, we speak of a *vertex cover*. The minimal cardinality of a vertex cover will be denoted by $C^v(I_d)$.
- The minimal number of dissections of I_d will be abbreviated by $D(I_d)$. A *dissection* is a decomposition of I_d into d-dimensional simplices whose interiors are pairwise disjoint but that do not necessarily intersect in a common face. So simplices are allowed to touch but the interiors must not intersect.
- $D^v(I_d)$ is the same as $D(I_d)$, except that we again require the vertices of the simplices to be vertices of I_d — such a dissection is called a *vertex dissection*.
- We define $T(I_d)$ to be the size of the minimal triangulation of I_d, where *triangulation* means decomposition of I_d into pairwise disjoint d-simplices which intersect in a common face or not at all.
- Finally, $T^v(I_d)$ is defined analogously to $D^v(I_d)$.

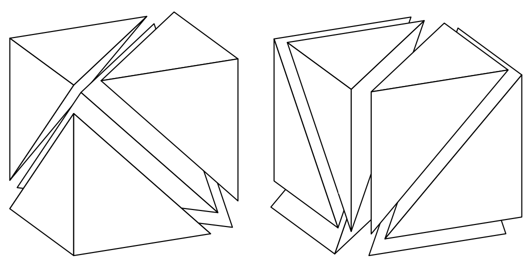
The obvious three questions are:

1. Given the dimension, what are the values for the parameters above?
2. Can we give good estimates of the parameters for large d?
3. And what is their relationship among each other?

For the rest of this description we will write C instead of $C(I_d)$, etc. With regard to the last question, we can easily sum up what is currently known:

$$C \leq C^v, D \leq T, D^v \leq T^v.$$

The only non-trivial relation is $C^v \leq T$, but this follows from a result by Bliss and Su. The status of the first two questions cannot be summarized so concisely. Best studied seems to be the parameter T^v. The case $d = 2$ is straightforward but already $d = 3$ allows vertex triangulations of different cardinality, as the following picture demonstrates:



To get an upper bound on T^v one considers the so-called *standard triangulation*. It is of size $d!$ and one constructs it by linking a simplex to each permutation $\pi \in S_d$ by using the following description

$$\Delta_\pi = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_{\pi(1)} \leq \cdots \leq x_{\pi(d)} \leq 1\right\}.$$

This triangulation is maximal among those that only use vertices of the cube, but minimal only for $d = 2$. For lower bounds one directly looks at the more general case of C^v. To get asymptotic estimates for example, the following idea is applied: If $V(d)$ denotes the maximal determinant of a 0/1-matrix, then $V(d)/d!$ is an upper bound of the volume of the largest simplex in I_d and we get

$$C^v \geq \frac{d!}{V(d)}.$$

Determining $V(d)$ is not easy but one can use the Hadamard inequality to bound it. By using hyperbolic volume instead of Euclidean volume, Smith obtained in 2000 the following asymptotic bound

$$C^v, D, T, D^v, T^v \geq \frac{6^{d/2} d!}{2(d+1)^{(d+1)/2}}.$$

Glazyrin recently improved this bound for T^v:

$$T^v \geq \frac{d!}{(\sqrt{d}/2)^d}.$$

The following table sums up lower bounds which are results of several research articles. Bold entries denote optimal bounds.

Dimension	D	C^v, T	D^v	T^v
3	**5**	**5**	**5**	**5**
4	15	**16**	**16**	**16**
5	48	60	61	**67**
6	174	252	270	**308**
7	681	1143	1175	**1493**
8	2863	5104	5522	5522
9	12811	22616	26593	26593
10	60574	98183	131269	131269
11	300956	520865	665272	665272
12	1564340	$2.9276 \cdot 10^6$		

Notes on Problem 8

The relation $C^v \leq T$ was proved by Bliss and Su in [1, Thm. 1]. The results by Smith can be found in [4] and those proved by Glazyrin in [3]. A lot more on triangulations in general can be found in the book by De Loera, Rambau and Santos [2], while Zong's book [5, Chap. 4] is taking an in-depth look at I_d.

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9 A problem concerning the ball and the cube

Consider the d-dimensional ball

$$B_d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$$

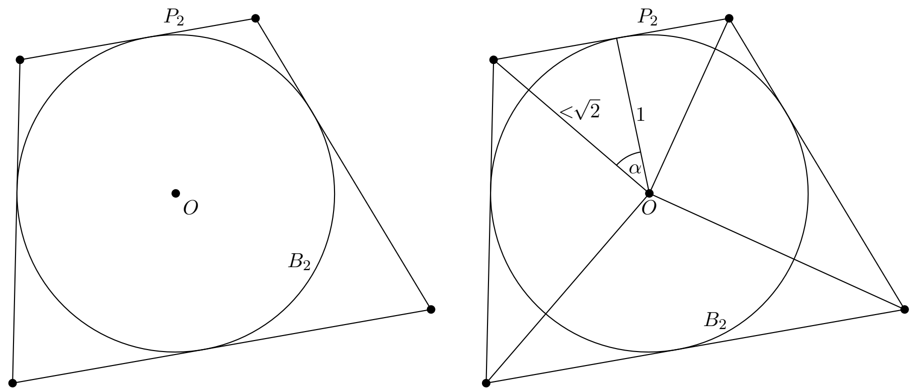
and let $P_d \subseteq \mathbb{R}^d$ be a convex polytope of dimension d with $2d$ facets that contains B_d. One example of such a polytope is the d-dimensional unit cube

$$C_d = \{(x_1, \dots, x_d) \in \mathbb{R}^d : |x_i| \leq 1 \text{ for } i = 1, \dots, d\}.$$

Furthermore, for such a polytope P_d let $\sigma(P_d)$ be the maximal distance between some point of P_d and the origin $0 \in \mathbb{R}^d$. Then a conjecture of Chuanming Zong from 1994 states that

$$\sigma(P_d) \geq \sqrt{d},$$

where equality is supposed to hold if and only if P_d is congruent to C_d. It is not difficult to verify this conjecture for $d = 2$: Assume that it is not correct, i.e., there exists a trapezoid P_2 that contains the unit disc but has $\sigma(P_2) < \sqrt{2}$. Using the diagonals of the trapezoid, one can dissect it into four triangles whose vertices are the corners of the trapezoid and the origin. Our assumption in particular means that the edge length of an origin-corner edge is strictly less than $\sqrt{2}$.



Since $\arccos : [-1, 1] \to [0, \pi]$ is strictly monotonically decreasing, we have for the angle α in the drawing

$$\cos \alpha > \frac{1}{\sqrt{2}} \quad \Longrightarrow \quad \alpha < \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

In total we have an angle sum of strictly less than $8 \cdot \frac{\pi}{4} = 2\pi$ for a whole circulation around the origin — clearly a contradiction. Besides this easy case not much is known. L. Fejes Tóth was able to prove an equivalent conjecture for $d = 3$ and Dalla et al. verified the statement for $d = 4$. All higher-dimensional cases are still open.

Notes on Problem 9

Zong's conjecture was proposed at a Geometry meeting in Vienna and it appears in [3, Prob. 8.1]. The book by Tóth [2] contains the solutions for the case $d = 3$. See Dalla et al. [1] for the proof of Zong's conjecture in dimension 4.

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10 The 3^d conjecture

In the last century there has been amazing progress in the understanding of face numbers of convex polytopes. For example, the case $d = 3$ was solved by Steinitz in 1906: The possible f-vectors are

$$\{(f_0, f_1, f_2) \in \mathbb{Z}^3 : f_0 - f_1 + f_2 = 2,\ f_2 \leq 2f_0 - 4,\ f_0 \leq 2f_2 - 4\}.$$

The first condition is Euler's equation, the first inequality is satisfied by equality for polytopes where all faces are triangles, while the second inequality characterizes polytopes where all vertices have degree 3 as the extreme case. In the case $d = 4$ one basic problem that remains is the fatness problem discussed above. A complete answer for d-dimensional simple or simplicial polytopes is available via the so-called g-Theorem proved by Billera–Lee and Stanley in 1980.

In contrast to this, it is amazing how little we know about *centrally-symmetric* convex polytopes, that is, polytopes that are left unchanged by a reflection in the origin in $\mathbb{R}^d$. Let's first look at the 3-dimensional case again. Here the possible f-vectors can be described as

$$\{(f_0, f_1, f_2) \in (2\mathbb{Z})^3 : f_0 - f_1 + f_2 = 2,\ f_2 \leq 2f_0 - 4,\ f_0 \leq 2f_2 - 4,\ f_0 + f_2 \geq 14\}.$$

All the face numbers of a centrally-symmetric polytope are even, thus we have $(f_0, f_1, f_2) \in (2\mathbb{Z})^3$. We recognize Euler's equation and the two inequalities from above. And then there is an additional relation, which for $d = 3$ is easy to prove: A centrally-symmetric 3-polytope has at least 6 vertices, and if it has only 6 vertices, then it must be an octahedron which has 8 facets. If, however, the centrally-symmetric 3-polytope has at least 8 vertices, then it also has at least 6 facets with the only extreme case of an affine cube. In summary, this yields the third inequality, which by Euler's equation we can rewrite as

$$f_0 + f_1 + f_2 + f_3 \geq 27,$$

with equality if and only if the polytope is either a cube or an octahedron. In 1989 Gil Kalai asked whether a similar statement was true in all dimensions:

Does every d-dimensional centrally-symmetric polytopes they all have at least 3^d non-empty faces?

Kalai's question fits into a series of three basic conjectures:

The 3^d conjecture (Kalai 1989)
Every centrally-symmetric d-dimensional polytope satisfies $f_0 + f_1 + \cdots + f_d \geq 3^d$.

The flag conjecture (Kalai 2008)
Every centrally-symmetric d-dimensional polytope satisfies $f_{0,1,2,\ldots,d-1} \geq 2^d d!$.

The Mahler conjecture (Mahler 1939)
Every centrally-symmetric convex body K satisfies $\mathrm{Vol}(K) \cdot \mathrm{Vol}(K^*) \geq 4^d/d!$, where K^* is the polar of K.

These three conjectures are remarkable since they seem basic, they have been around for quite a while, but we know so little about them. The 3^d conjecture was proved for $d \leq 4$ by Sanyal et al. in 2009, but is open beyond this. The flag conjecture is not even known for $d = 4$. Yet worse, the Mahler conjecture has been an object of quite some scrutiny, but it seems open even for $d = 3$.

The three conjectures belong together since we believe we know the answer — the same answer for all of them. Indeed, the class of *Hanner polytopes* introduced by Olof Hanner in 1956 is obtained by starting with a single centrally-symmetric interval such as $[-1, 1] \subset \mathbb{R}$ and then taking products and polars — or equivalent, taking products and direct sums of polytopes. It is easy to compute that all d-dimensional Hanner polytopes have exactly 3^d non-empty faces, they have exactly $2^d d!$ complete flags of faces, and they have exactly Mahler Volume $\mathrm{Vol}(P)\cdot\mathrm{Vol}(P^*) = 4^d/d!$. But are they the only centrally-symmetric polytopes with these properties? And can't there be any other polytopes with even smaller values? This is not known.

Notes on Problem 10

The Steinitz 1906 paper that solves the f-vector problem for 3-polytopes is [6]. The case $d = 4$ is surveyed in [9]. For the g-Theorem we refer to [8, Sect. 8.6]. Kalai's 3^d conjecture is "Conjecture A" in [3]; the conjecture is verified for $d \leq 4$ in [5]. The Hanner polytopes were introduced by Hanner [1]; the Hansen polytopes appear in [2]. Data for the Hansen polytope of the 4-vertex path are given in the paper by Sanyal et al. just quoted. The Mahler conjecture goes back to 1939 — see Mahler's original paper [4]; a detailed current discussion of the Mahler conjecture with recent related references appears in Tao's blog [7].

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