On the number of simplicial 3-spheres and 4-polytopes with N facets

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(joint work with Bruno Benedetti)

1. Question

Is the number of combinatorial types of simplicial 3*-spheres on* N *facets bounded by an exponential function* C^N*?*

This question is fundamental for the construction of a partition functions for quantum gravity [1], where space is modelled by a 3-sphere glued from regular tetrahedra of edge lengths ε, and one is interested in the limit if $N \to \infty$, which corresponds to modelling space by triangulations by regular tetrahedra of edge lengths $\varepsilon \to 0$.

2. Related

A related question asks for the number of simplicial 3-spheres and 4-polytopes on n vertices. Here it is long known that there are only exponentially many polytopes [5], while there are more than exponentially many spheres [11].

3. Local constructibility

In the lower-dimensional case of simplicial 2-spheres, we have the same count for 2-spheres and for 3-polytopes with N facets, due to Steinitz' theorem. The answer is asymptotically of the order of $(\frac{256}{27})^{N/2}$, according to Tutte [12].

An elementary approach to this case, which also gives an exponential upper bound and invites for generalization to higher dimensions, first counts plane "trees of N triangles" (which correspond to triangulations of an $(2N+1)$-gon, so there are less than 2^{2N} of these), and then gluings on the boundary, which amounts to planar matchings in the exterior (which again yields a factor of 2^{2N}).

In 1995 Durhuus and Jonsson [3] introduced a concept that generalizes this approach: A simplicial 3-sphere is *locally constructible* (LC) if it can be obtained from a tree of tetrahedra by successive gluings of adjacent (!) boundary triangles. They showed that there are only exponentially-many LC 3-spheres.

4. Hierarchy

We link the LC concept with the notions of shellability and constructibility that were established in combinatorial topology [2], and thus obtain the following hierarchy for simplicial 3-spheres:

$$\text{polytopal} \Rightarrow \text{shellable} \Rightarrow \text{constructible} \Rightarrow \text{LC}.$$

5. Main Results

Theorem 1. *Every constructible simplicial sphere is LC.*

This result establishes the hierarchy above. We also have an extension to simplicial d-spheres, $d \geq 2$. It depends on a simple lemma, according to which gluing two LC d-pseudomanifolds along a common strongly-connected pure $(d-1)$-complex in the boundary yields an LC d-complex.

Theorem 2. *There are less than* 2^{8N} *LC simplicial* 3*-spheres on* N *facets.*

This result slightly sharpens an estimate by Durhuus and Jonsson. We also extend it to LC d-spheres.

Combination of Theorem 2 with the hierarchy (Theorem 1) yields that there are only exponentially-many simplicial 4-polytopes with a given number of facets. (This answers a question by Kalai; as pointed out by Fukuda at the workshop, this may as well be derived from the fact that there are only exponentially many simplicial 4-polytopes on n vertices by [5].)

More generally, for fixed d we get that there are only exponentially-many shellable d-spheres on N facets. This is interesting when compared with the studies of Kalai [7] and Lee [8], which showed that for $d \geq 4$, there are more than exponentially many shellable d-spheres on n vertices.

Theorem 3. *If a simplicial* 3*-sphere* S *contains a triangle* L *that is knotted such that the fundamental group of its complement in* S *has no presentation with* 3 *generators, then* S *is not LC.*

This result is derived from the fact that if S is an LC 3-sphere and Δ is a facet of S, then $S \setminus \Delta$ is collapsible. By a result by Lickorish [9] this implies that the fundamental group of $S \setminus L$ has a presentation with at most 3 generators.

Combined with the known constructions of simplicial 3-spheres with badly-knotted triangles (which go back to Furch [4]), this yields that not all simplicial 3-spheres are locally constructible. This solves a problem by Durhuus and Jonsson. More precisely, spheres with a knotted triangle are not constructible by [6], but if the knot is not complicated, they can be LC (this we derive from [10]).

The basic question about the number of simplicial 3-spheres with N facets remains, as far as we know, open.

References

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