On the number of simplicial 3-spheres and 4-polytopes with \( N \) facets

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(joint work with Bruno Benedetti)

1. Question

Is the number of combinatorial types of simplicial 3-spheres on \( N \) facets bounded by an exponential function \( C^N \)?

This question is fundamental for the construction of a partition functions for quantum gravity [1], where space is modelled by a 3-sphere glued from regular tetrahedra of edge lengths \( \varepsilon \), and one is interested in the limit if \( N \to \infty \), which corresponds to modelling space by triangulations by regular tetrahedra of edge lengths \( \varepsilon \to 0 \).

2. Related

A related question asks for the number of simplicial 3-spheres and 4-polytopes on \( n \) vertices. Here it is long known that there are only exponentially many polytopes [5], while there are more than exponentially many spheres [11].

3. Local constructibility

In the lower-dimensional case of simplicial 2-spheres, we have the same count for 2-spheres and for 3-polytopes with \( N \) facets, due to Steinitz’ theorem. The answer is asymptotically of the order of \( \left( \frac{27}{256} \right)^{N/2} \), according to Tutte [12].

An elementary approach to this case, which also gives an exponential upper bound and invites for generalization to higher dimensions, first counts plane “trees of \( N \) triangles” (which correspond to triangulations of an \( (2N + 1) \)-gon, so there are less than \( 2^{2N} \) of these), and then gluings on the boundary, which amounts to planar matchings in the exterior (which again yields a factor of \( 2^{2N} \)).

In 1995 Durhuus and Jonsson [3] introduced a concept that generalizes this approach: A simplicial 3-sphere is locally constructible (LC) if it can be obtained from a tree of tetrahedra by successive gluings of adjacent (!) boundary triangles. They showed that there are only exponentially-many LC 3-spheres.

4. Hierarchy

We link the LC concept with the notions of shellability and constructibility that were established in combinatorial topology [2], and thus obtain the following hierarchy for simplicial 3-spheres:

\[
\text{polytopal} \Rightarrow \text{shellable} \Rightarrow \text{constructible} \Rightarrow \text{LC}.
\]
5. Main Results

**Theorem 1.** Every constructible simplicial sphere is LC.

This result establishes the hierarchy above. We also have an extension to simplicial $d$-spheres, $d \geq 2$. It depends on a simple lemma, according to which gluing two LC $d$-pseudomanifolds along a common strongly-connected pure $(d-1)$-complex in the boundary yields an LC $d$-complex.

**Theorem 2.** There are less than $2^{8N}$ LC simplicial 3-spheres on $N$ facets.

This result slightly sharpens an estimate by Durhuus and Jonsson. We also extend it to LC $d$-spheres.

Combination of Theorem 2 with the hierarchy (Theorem 1) yields that there are only exponentially-many simplicial 4-polytopes with a given number of facets. (This answers a question by Kalai; as pointed out by Fukuda at the workshop, this may as well be derived from the fact that there are only exponentially many simplicial 4-polytopes on $n$ vertices by [5].)

More generally, for fixed $d$ we get that there are only exponentially-many shellable $d$-spheres on $N$ facets. This is interesting when compared with the studies of Kalai [7] and Lee [8], which showed that for $d \geq 4$, there are more than exponentially many shellable $d$-spheres on $n$ vertices.

**Theorem 3.** If a simplicial 3-sphere $S$ contains a triangle $L$ that is knotted such that the fundamental group of its complement in $S$ has no presentation with 3 generators, then $S$ is not LC.

This result is derived from the fact that if $S$ is an LC 3-sphere and $\Delta$ is a facet of $S$, then $S \setminus \Delta$ is collapsible. By a result by Lickorish [9] this implies that the fundamental group of $S \setminus L$ has a presentation with at most 3 generators.

Combined with the known constructions of simplicial 3-spheres with badly-knotted triangles (which go back to Furch [4]), this yields that not all simplicial 3-spheres are locally constructible. This solves a problem by Durhuus and Jonsson. More precisely, spheres with a knotted triangle are not constructible by [6], but if the knot is not complicated, they can be LC (this we derive from [10]).

The basic question about the number of simplicial 3-spheres with $N$ facets remains, as far as we know, open.

References


