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Decompositions of simplicial balls and spheres with knots consisting of few edges

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Abstract Constructibility is a condition on pure simplicial complexes that is weaker than shellability. In this paper we show that non-constructible triangulations of the d-dimensional sphere exist for every $d \geq 3$. This answers a question of Danaraj and Klee [10]; it also strengthens a result of Lickorish [16] about non-shellable spheres.

Furthermore, we provide a hierarchy of combinatorial decomposition properties that follow from the existence of a non-trivial knot with "few edges" in a 3-sphere or 3-ball, and a similar hierarchy for 3-balls with a knotted spanning arc that consists of "few edges."

1 Introduction

From the hierarchy of conditions on simplicial complexes given by

$$\text{vertex decomposable} \implies \text{shellable} \implies \text{constructible},$$

that is,

$$\text{not vertex decomposable} \impliedby \text{non-shellable} \impliedby \text{non-constructible},$$

(non-)shellability is probably the most intensively studied one [4] [5]. All the boundary complexes of simplicial polytopes are shellable [7] [23, Chap. 8], but not all of them are vertex decomposable [15,

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Sect. 6]. A mysterious fact about shellability is that there exist triangulations of d-balls and also of d-spheres which are not shellable if $d \geq 3$, though all triangulations of 2-balls and 2-spheres are shellable. Non-shellable triangulations of balls are reviewed in [24].

An explicit construction of non-shellable triangulations of spheres was given by Lickorish [16]. Lickorish's result was that triangulations of 3-spheres which contain a knotted triangle are not shellable, *provided that the knot is complicated enough.* (Their $(d-3)$-fold suspensions give non-shellable triangulations of d-spheres for $d \geq 3$.) In [16], the added condition of complexity on the knot could not be deleted since for simple knots such as a single trefoil or the sum of two trefoils, Lickorish's technique fails and cannot determine whether the corresponding triangulated spheres are shellable or not.

Constructibility, a concept from combinatorial topology [22] that can be viewed as a relaxation of shellability, appears in different combinatorial contexts in [5], [10], [14], and [20]. In [13], two classes of non-constructible triangulations of 3-balls were identified, but the existence of non-constructible triangulations of spheres was left open. This problem dates back at least to the 1978 survey of Danaraj and Klee [10, Sect. 4]. Here we answer this question:

Theorem 1 *If a* 3*-ball or* 3*-sphere contains* ***any*** *knotted triangle, then it is not constructible.*

In particular, the above-mentioned triangulations of 3-spheres considered by Lickorish, where some triangle forms a trefoil or the sum of two trefoil knots, are non-shellable.

We will show that also the existence of a non-trivial knot consisting of 4 or 5 edges has "bad effects" on the decomposition properties of a triangulated 3-sphere. The results and examples provided in this paper may be summarized in the following remarkable hierarchy:

Theorem 2 *A* 3*-ball with a knotted spanning arc consisting of*

$$\left\{\begin{array}{rl} \textit{at most } 2 \textit{ edges} & \textit{is not constructible,} \\ 3 \textit{ edges} & \textit{can be shellable, but not vertex decomposable,} \\ 4 \textit{ edges} & \textit{can be vertex decomposable.} \end{array}\right.$$

A 3*-sphere or* 3*-ball with a knot consisting of*

$$\left\{\begin{array}{rl} 3 \textit{ edges} & \textit{is not constructible,} \\ 4 \textit{ or } 5 \textit{ edges} & \textit{can be shellable, but not vertex decomposable,} \\ 6 \textit{ edges} & \textit{can be vertex decomposable.} \end{array}\right.$$

Our results about non-shellable triangulated spheres are dual to those of Armentrout [2] (see also [1]), who considers (shellability of) the "cell partitionings" which also may be viewed as the dual block complexes of triangulations of 3-spheres. There is no obvious relation between Armentrout's results and ours, since there is no direct connection between the shellability of a triangulation and that of its dual cell partition. (Otherwise Armentrout's results would also imply non-shellability of Lickorish's spheres.) We think that our approach has the virtue of being very simple exactly *because* we head for the stronger property of non-constructibility. On the other hand, Armentrout's very interesting paper [2] suggests an extension of Theorem 2 for *number of edges vs. complexity* of a knot or spanning arc, where the complexity of the knot is measured in terms of its bridge number [2] [19]. Some results in this direction were achieved in [11].

2 Definitions and Notation

A *simplicial complex* is a finite set C of simplices (the *faces* of C) in some Euclidean space $\mathbb{R}^n$ satisfying that (i) if $\sigma \in C$ then all the faces of σ are members of C, and (ii) if $\sigma, \tau \in C$ then $\sigma \cap \tau$ is a face of both σ and τ. The 0-dimensional simplices in C are the *vertices*, the 1-dimensional simplices are the *edges* of C. The inclusion-maximal faces are called *facets*. The dimension of C is the largest dimension of a facet. A *d-complex* is short for a d-dimensional simplicial complex. If all the facets of C have the same dimension, then C is *pure*. (In particular, the simplicial complex which has only the empty set as a face, is a pure complex of dimension -1, with a single facet.) For a set of simplices $C' \subseteq C$, the simplicial complex $\overline{C'}$ consists of the simplices in C' together with all their faces. The union $|C|$ of the simplices of C is called the *underlying space* of C. If $|C|$ is homeomorphic to a manifold M, then C is a *triangulation* of M. If C is a triangulation of a d-ball or of a d-sphere, respectively, then C will be simply called a *d-ball* or a *d-sphere*. For any triangulation C of a manifold, the *boundary complex* ∂C is the collection of all simplices of C which lie in the boundary of the manifold. A d-dimensional pure simplicial complex is *strongly connected* if for any two of its facets F and F', there is a sequence of facets $F = F_1, F_2, \ldots, F_k = F'$ such that $F_i \cap F_{i+1}$ is a face of dimension $d-1$, for $1 \leq i \leq k-1$. If a d-dimensional pure simplicial complex is strongly connected and each $(d-1)$-dimensional face belongs to at most two facets, then it is called a *pseudomanifold*. Every triangulation of a connected manifold is a pseudomanifold.

A pure d-dimensional complex is *shellable* if its facets can be ordered $F_1, F_2, \ldots, F_t$ so that $(\bigcup_{i=1}^{j-1} \overline{F_i}) \cap \overline{F_j}$ is a pure $(d-1)$-complex for $2 \leq j \leq t$. This ordering of the facets is called a *shelling*.

Constructibility of pure simplicial complexes is defined recursively as follows:

(i) Every simplex (i. e., a complex with one single facet) is constructible.
(ii) A d-complex C which is not a simplex is constructible if and only if it can be written as $C = C_1 \cup C_2$, where C_1 and C_2 are constructible d-complexes and $C_1 \cap C_2$ is a constructible $(d-1)$-complex.

If we restrict this definition such that C_2 must be a simplex, then we get a characterization of shellability; thus constructibility is a relaxation of shellability.

For a simplicial complex C and a face σ, $\mathrm{star}_C\sigma$ is the simplicial complex that contains all faces of facets of C that contain σ, and $\mathrm{link}_C\sigma$ is the subcomplex of those simplices of $\mathrm{star}_C\sigma$ that do not intersect σ. For a simplex σ and a vertex $v \notin \sigma$, the *join* $v * \sigma$ is a simplex whose vertices are those of σ plus the extra vertex v. The join $v * C$ of a complex C with a new vertex v is defined such that $v * C = \{v * \tau : \tau \in C\}$. The *deletion* $C \backslash v$ is the subcomplex of C formed by all the faces of C that do not contain the vertex v.

A pure d-complex C is *vertex decomposable* if it is a simplex or there is a vertex x such that

(i) $\mathrm{link}_C x$ is $(d-1)$-dimensional and vertex decomposable, and
(ii) $C \backslash x$ is d-dimensional and vertex decomposable.

The vertex x is called a *shedding vertex.* Vertex decomposable simplicial complexes were introduced and shown to be shellable by Provan and Billera [18].

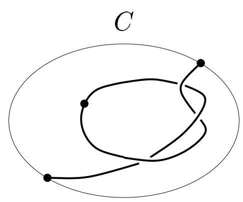
For a 3-ball, a *spanning arc* is a tame arc contained in the interior of the ball except for its two endpoints lying on the boundary. When joining the two endpoints by a second tame arc that is contained in the boundary of the ball, one always gets a knot of the same type. So one can say that a spanning arc is *knotted* if the spanning arc together with any added arc contained in the boundary forms a nontrivial knot embedded in the 3-ball. In fact, the same is also true if the relative interior of the spanning arc is not fully contained in the interior of the ball, provided that the spanning arc is contained in the ball and the added arc does not intersect with it. So in this paper we require of a spanning arc only that it is contained in the ball and that both ends of it are on the boundary, and allow for the case that some parts of the relative interior of the spanning arc are on the boundary.

3 Non-constructible 3-balls and 3-spheres

In the following, we use the simple fact that if all the $(d-1)$-dimensional faces of a constructible d-complex C are contained in at most two facets, then C must be a d-ball or a d-sphere [22] [5, Th. 11.4]. Since pseudomanifolds satisfy the condition, we get that every constructible pseudomanifold is a d-ball or a d-sphere, and that

- if C is a constructible d-sphere, then the complexes C_1 and C_2 in the definition of constructibility are constructible d-balls and $C_1 \cap C_2$ is a constructible $(d-1)$-sphere, and
- if C is a constructible d-ball, then C_1 and C_2 are constructible d-balls and $C_1 \cap C_2$ is a constructible $(d-1)$-ball.

Lemma 1 *If a triangulation C of a 3-ball has a* ***knotted spanning arc which consists of at most two edges of C****, then C is not constructible.*



This lemma is the crucial new observation of this paper. It extends a lemma from [13], namely that if a triangulation C of a 3-ball has a knotted spanning arc which consists of just *one* edge of C, then C is not constructible.

The fact that a ball C with a knotted spanning arc consisting of just one edge cannot be shellable is old, and can be traced back to Furch's 1924 paper [12] [23,24]. Furthermore, such balls exist:

Lemma 2 (Furch [12]) *Triangulations C of the 3-dimensional ball B^3 with a knotted spanning arc that consists of a single edge of C exist.*

To obtain Furch's "knotted hole ball," one "drills a hole" into a finely triangulated ball by removing tetrahedra along a knotted spanning curve; if one stops drilling one step before destroying the property of having a triangulated ball, then one arrives at a ball with a knotted spanning edge. (See also [12], [21], [23,24].)

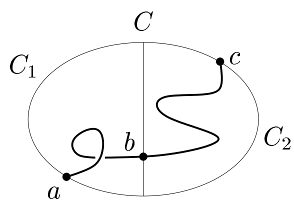
From any such ball with a knotted spanning edge one obtains a triangulated 3-sphere that has a knot that consists of only three edges — a *knotted triangle*, as needed below — by adding a cone over the boundary, that is, by forming $C \cup (v * \partial C)$ [16].

Proof (of Lemma 1) We show by induction on the number of facets of C that in a constructible triangulation C of a 3-ball, a spanning arc that consists of only two edges ab and bc cannot be knotted. (We may assume that the arc in question has exactly two edges, since an arc consisting of a single edge can be extended by an edge on the boundary. Recall for this that we allow parts of spanning arcs to lie in the boundary of the ball.)

If C is a single simplex (tetrahedron), then the arc cannot be knotted. Otherwise C decomposes into two constructible complexes C_1 and C_2 as in the definition of constructibility; both C_1 and C_2 are triangulated 3-balls. There are two cases to consider.

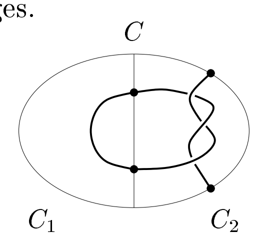
Case 1: The two edges ab and bc are both contained in C_1. They form a spanning arc ab-bc of C_1, which by induction cannot be knotted.

Case 2: One edge ab is contained in C_1 and the other one bc is contained in C_2. C_1 is constructible, so by induction ab is an unknotted spanning arc of C_1, and similarly for the arc bc in C_2.



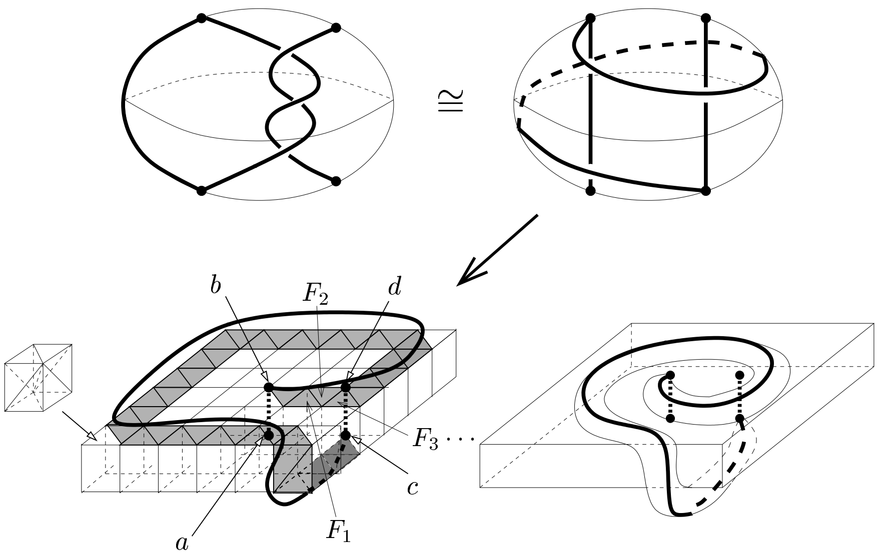
Now the fact that ab-bc is not knotted in C follows from a known fact from combinatorial topology: if two unknotted ball pairs meet in a common face, then their union an unknotted ball pair (see [22, Lemma 19] or [17, Chap. 2]). □

The existence of a knotted spanning arc with k edges, for any $k \geq 3$, does not assure non-constructibility in general. The proof technique of Lemma 1 breaks down for $k = 3$: Our figure shows a situation where a 3-ball contains a knotted spanning arc with $k = 3$ edges, but neither C_1 nor C_2 necessarily contains a knotted spanning arc with less than 4 edges.



In fact, we can construct a shellable 3-ball with a knotted spanning arc consisting of 3 edges, as follows.

Example 1 (A shellable 3*-ball with a knotted spanning arc consisting of* 3 *edges.)* Let C_1 be a pile of $6 \times 6 \times 1$ cubes in which each cube is split into 6 tetrahedra. Then $C := C_1 \cup (b * \text{(gray faces)}) = C_1 \cup (b * F_1) \cup (b * F_2) \cup \cdots$ is a shellable 3-ball because C_1 is shellable, and the arc *ab*-*bc*-*cd* is a knotted spanning arc of the 3-ball as is indicated in the upper figures.



$C = \text{(pile of cubes)} \cup (b * F_1) \cup (b * F_2) \cup \cdots$

Now we can show the following result, which includes Theorem 1.

Theorem 3 *In a constructible* 3*-ball or* 3*-sphere, every knot that consists of three edges and three vertices (a "triangle") is trivial.*

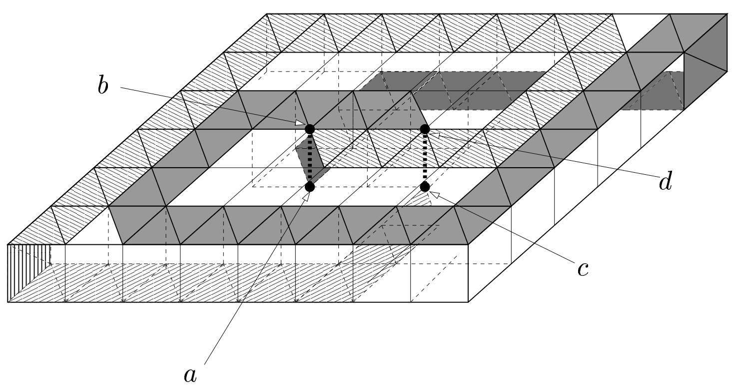
Proof We use Lemma 1 and induction on the number of facets. The case of a simplex C is clear. Otherwise the complex C can be divided into two constructible complexes C_1 and C_2. As noted in the beginning of this section, both C_1 and C_2 must be 3-balls. If one of them contains all the three edges of a triangle κ, then κ is trivial by induction. If not, then one of them, say C_1, has two edges ab and bc of κ, and the other one C_2 has the third edge ca of κ. Now *ab*-*bc* is a spanning arc of C_1 and ca is a spanning arc of C_2, and both spanning arcs are not knotted from Lemma 1. This implies that κ is trivial because the connected sum of two trivial knots is trivial. (This is based on the following combinatorial topology fact: if two unknotted ball pairs are joined by their boundary, then this yields an unknotted sphere pair; see [22, Lemma 18].) □

Corollary 1 *If a triangulation of a 3-sphere contains **any** knotted triangle, then it is not shellable.*

Remark 1 Lickorish's result was that if a triangulation C of a 3-sphere contains a complicated knotted triangle, then $C \backslash \sigma$ is not collapsible for any facet σ of C, and the non-shellability of C was a corollary to this statement. The property "$C \backslash \sigma$ is not collapsible for any facet σ" is stronger than non-shellability, and to get this Lickorish needed the condition that the knot must be complicated enough (specifically, the fundamental group of the complement of the knot may have no presentation with less than 4 generators), which is not needed here.

The number "3" of edges of a knot in Theorem 3 is best possible, as is shown in the following example.

Example 2 (A shellable 3-ball and 3-sphere with a knot consisting of 4 edges.) This example arises in the same line of construction as Example 1. Let C_1 be a pile of $8 \times 6 \times 1$ cubes in which each cube is split into 6 tetrahedra as before. Then the 3-ball $C_2 = C_1 \cup (b *$ (slashed faces)$) \cup (d *$ (gray faces)$)$ has a knot *ab-bc-cd-da*. This knot *ab-bc-cd-da* is not trivial because *ab-bc-cd* is a non-trivial knotted spanning arc. (It makes a trefoil knot.) Its shellability is easily seen as in Example 1. To get a 3-sphere with a knot consisting of 4 edges, we have only to take a cone over the boundary of C_2, that is, $C := C_2 \cup (v * \partial C_2)$. The shelling of C_2 can be trivially extended to that of C because ∂C_2 is shellable since it is a 2-sphere.



4 Removing a facet from a 3-sphere

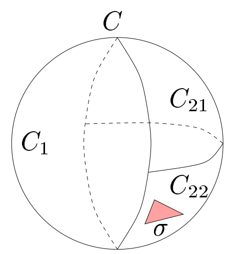
The following result reduces the constructibility question from 3-spheres to 3-balls. It leads to a different proof of Theorem 3 from

Lemma 1, where we remove from a 3-sphere any facet that contains an edge of the "knotted triangle." No similar result for the case of shellable 3-spheres seems to be available. (For a shellable 3-sphere, is every facet the last facet of some shelling?)

Theorem 4 *Let C be a triangulation of a 3-sphere and σ any facet of C. Then C is constructible if and only if $C \backslash \sigma$ is constructible.*

Proof The "if" part is trivial, so we show the "only if" part. Let C be constructible. Then by definition there are two constructible 3-balls C_1 and C_2 such that $C_1 \cup C_2 = C$, and $C_1 \cap C_2$ is a constructible 2-sphere. We may assume that σ is contained in C_2. If $C_2 = \overline{\sigma}$, then we are done. Otherwise C_2 is the union of two constructible 3-balls C_{21} and C_{22} that satisfy the conditions for constructibility. We may assume that C_{22} contains σ. We define $C_1' := C_1 \cup C_{21}$ and $C_2' := C_{22}$. Then

(i) C_2' is a constructible 3-ball by definition.
(ii) $C_1' \cap C_2' = \partial C_2' = \partial C_{22}$ is constructible because it is a 2-sphere.



(iii) $C_1' = C_1 \cup C_{21}$, where both C_1 and C_{21} are constructible 3-balls by definition. Their intersection $C_1 \cap C_{21} = \overline{\partial C_{21} \backslash (C_{21} \cap C_{22})}$ is a constructible 2-ball, since removal of a 2-ball from a 2-sphere always leaves a 2-ball, and all 2-balls are constructible. Thus C_1' is a constructible 3-ball.

So C_1' and C_2' instead of C_1 and C_2 satisfy the definition of constructibility. Continuing this argument, the number of facets of C_2 is reduced until C_2 has only the one facet σ, showing that $C \backslash \sigma$ is constructible. $\square$

5 Non-constructible *d*-spheres

The following lemma produces non-constructible triangulations of the d-sphere for all $d \geq 3$.

Lemma 3 (see Björner [3, Appendix] [5, p. 1855])
All links of a constructible simplicial complex are constructible.

Proof Let C be a constructible simplicial complex and τ a face of C. We use an induction on the number of facets of C. The case of a simplex C is trivial, so we write C as a union of two constructible complexes C_1 and C_2. If τ is contained in only one of C_1 and C_2, say in C_1, then $\mathrm{link}_C\tau = \mathrm{link}_{C_1}\tau$ is constructible by induction. If τ is contained in $C_1 \cap C_2$, then

(i) $(\mathrm{link}_C\tau) \cap C_1 = \mathrm{link}_{C_1}\tau =: L_1$,
(ii) $(\mathrm{link}_C\tau) \cap C_2 = \mathrm{link}_{C_2}\tau =: L_2$,
(iii) $L_1 \cap L_2 = (\mathrm{link}_{C_1}\tau) \cap (\mathrm{link}_{C_2}\tau) = \mathrm{link}_{C_1\cap C_2}\tau$, and
(iv) $L_1 \cup L_2 = \mathrm{link}_C\tau$.

These observations imply by induction that $\mathrm{link}_C\tau$ is constructible. $\square$

Corollary 2 *All d-spheres S^d, $d \geq 3$, have non-constructible triangulations.*

Proof Let C be a non-constructible triangulation of a $(d-1)$-sphere, and let v_1 and v_2 be two vertices not contained in C. Then the suspension $\Sigma C := (v_1 * C) \cup (v_2 * C) \cup C$ of C is a triangulation of the d-sphere. It is not constructible by Lemma 3, since $\mathrm{link}_{\Sigma C} v_1 = C$. $\square$

Remark 2 The double suspension $\Sigma^2 H^d$ of *any* homology d-sphere H^d is homeomorphic to S^{d+2}, according to Cannon [8]. Already Danaraj and Klee [10] pointed out that for $H^d \not\cong S^d$ this yields examples of non-PL, and hence non-shellable, spheres. (See Curtis and Zeeman [9] for a related much earlier discussion.) For this we note the following known [22] hierarchy for spheres (equivalently, for pseudomanifolds without boundary):

$$\text{shellable} \implies \text{constructible} \implies \text{PL},$$

that is,

$$\text{non-shellable} \impliedby \text{non-constructible} \impliedby \text{not PL}.$$

Thus Cannon's theorem assures the existence of non-constructible triangulations of d-spheres for $d \geq 5$, and Theorem 3 improves this to $d \geq 3$ and also to PL cases.

Recently, Björner and Lutz [6] constructed triangulations of non-PL d-spheres with $13 + d$ vertices, for $d \geq 5$. Their 18 vertex triangulation of a non-PL 5-sphere currently seems to have the smallest number of vertices known for a non-constructible sphere.

6 Knots and vertex decomposability

In Example 1 we constructed an example of shellable 3-ball which has a knotted spanning arc with 3 edges. The example, however, is not vertex decomposable. This can be observed directly from the figure, but we prove a more general fact: *no* 3-ball with a knotted spanning arc that consists of only three edges is vertex decomposable.

Lemma 4 *If a 3-ball C has a knotted spanning arc consisting of at most 3 edges, then C is not vertex decomposable.*

Proof First we observe that if x is a shedding vertex of a vertex decomposable d-ball, then x lies in the boundary. Furthermore, every vertex y adjacent to x is either in the interior of C, or the edge xy is contained in the boundary of C. This is because the deletion $C \backslash x$ must be a 3-ball, and the link of x is a 2-ball.

Again we use induction on the number of facets. If the spanning arc is made of 1 or 2 edges, then it is not knotted by Lemma 1. So we can assume that the spanning arc is made of 3 edges, where the first and last edge do not lie in the boundary of the ball. Thus if the arc is *ab*-*bc*-*cd*, the edges ab and cd lie in the interior of C. In particular, b and c are not shedding vertices.

The vertex a also cannot be a shedding vertex: otherwise *bc*-*cd* is a 2-edge knotted spanning arc in the 3-ball $C \backslash a$ (to verify this we use an argument as in the proof of Lemma 1), and thus $C \backslash a$ is not constructible (not even shellable) by Lemma 1. Similarly d cannot be a shedding vertex.

Thus x must be taken to be different from $\{a, b, c, d\}$. In this case, however, $C \backslash x$ has a knotted spanning arc with 3 edges and has a smaller number of facets than C, contradicting the induction hypothesis. $\square$

The number "3" of edges in the knotted spanning arc is best possible, because there are vertex decomposable 3-balls that have a knotted spanning arc with 4 edges.

Example 3 A vertex decomposable 3-ball with a knotted spanning arc made of 4 edges.] In the figure of Example 1, $C' = C_1 \cup (v *$ (gray faces)$)$, where v is a newly introduced vertex, has a knotted spanning arc *ab*-*bv*-*vc*-*cd* with 4 edges. This 3-ball C' is vertex decomposable. (One can take v as the first shedding vertex.)

As in the case of constructibility in Section 3, from Lemma 4 we get a result for knots in vertex decomposable 3-spheres resp. 3-balls.

Theorem 5 *If a 3-sphere or a 3-ball C has a knot which consists of at most 5 edges, then C is not vertex decomposable.*

Proof We use Lemma 4 and induction on the number of facets.

If C is a simplex, the statement obviously holds. Let C be vertex decomposable, let x be a shedding vertex of C and let κ be a knot with at most 5 edges. If x is a vertex of κ, then $C\backslash x$ has a knotted spanning arc with at most 3 edges, contradicting to Lemma 4. Otherwise $C\backslash x$ has a knot κ with at most 5 edges, contradicting to the induction hypothesis. □

The number of edges in this theorem is again best possible, as is shown in the following example.

Example 4 (A vertex decomposable 3-ball and 3-sphere with a knot consisting of 6 edges.) In the figure of Example 2, $C_2' = C_1 \cup (v *$ (slashed faces)) $\cup\, (w *$ (gray faces), where v and w are newly introduced vertices, has a knot *ab-bv-vc-cd-dw-wa* with 6 edges, and this 3-ball is vertex decomposable. From this 3-ball, we can construct a vertex decomposable 3-sphere by taking a cone over its boundary, namely, $C' = C_2' \cup (u * \partial C_2')$.

Thus we have established the complete hierarchy of Theorem 2.

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