Neighborly cubical polytopes*

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Abstract

Neighborly cubical polytopes exist: for any $n \geq d \geq 2r + 2$, there is a cubical convex d-polytope C_d^n whose r-skeleton is combinatorially equivalent to that of the n-dimensional cube. This solves a problem of Babson, Billera & Chan.

Kalai conjectured that the boundary ∂C_d^n of a neighborly cubical polytope C_d^n maximizes the f-vector among all cubical $(d-1)$-spheres with 2^n vertices. While we show that this is true for polytopal spheres for $n \leq d + 1$, we also give a counter-example for the non-polytopal case.

Further, the existence of neighborly cubical polytopes shows that the graph of the n-dimensional cube, where $n \geq 5$, is "dimensionally ambiguous" in the sense of Grünbaum. We also show that the graph of the 5-cube is "strongly 4-ambiguous".

In the special case $d = 4$, neighborly cubical polytopes have $f_3 = \frac{f_0}{4} \log_2 \frac{f_0}{4}$ vertices, so the facet-vertex ratio f_3/f_0 is not bounded; this solves a problem of Kalai, Perles and Stanley studied by Jockusch.

1 Introduction.

In Chapter 12 of his famous book [10] Grünbaum discusses the concept of *k-equivalence* of polytopes. A d-polytope P is *k-equivalent* to a d'-polytope P' if the k-skeleta of P and P' are combinatorially equivalent. An interesting case occurs when $\dim P \neq \dim P'$. In this situation Grünbaum calls the k-skeleton $\mathcal{S}$ of either polytope *dimensionally ambiguous.* Assume $d < d'$. Then $\mathcal{S}$ is called *strongly d-ambiguous* if there is another d-polytope Q,

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not combinatorially equivalent to P, whose k-skeleton is also combinatorially equivalent to $\mathcal{S}$. From the existence of neighborly polytopes, such as the cyclic polytopes, it follows that the $(\lfloor \frac{d}{2} \rfloor - 1)$-skeleton of the d-simplex is dimensionally ambiguous for $d \geq 5$.

Motivated by a problem of Kalai [15, Problem 3.4(iv)*], we first constructed a 4-polytope with the graph of the 5-cube, thus showing that the graph of the 5-dimensional cube is dimensionally ambiguous. This turns out to be a special case of a d-polytope with the r-skeleton of the $(d+1)$-cube, for $r = \lfloor \frac{d}{2} \rfloor - 1$. We describe a construction in Section 4. The resulting polytopes are cubical, and hence they appear in the combinatorial classification of the cubical polytopes with 2^{d+1} polytopes due to Blind & Blind [6, 7].

More generally, in this paper we construct *neighborly cubical polytopes* in the sense of Babson, Billera & Chan [3]: for every $n \geq d \geq 2$ there is a cubical d-polytope with the r-skeleton of the n-cube, for $r = \lfloor \frac{d}{2} \rfloor - 1$. In particular, this yields a 4-polytope with the graph of the n-cube for every $n \geq 4$. The neighborly cubical polytopes are constructed as linear projections of "deformed" cubes, see Section 7. The combinatorics of the moment curve is involved in an essential way; this is reminiscent of the construction of neighborly (simplicial) polytopes.

A result of Grünbaum [10, 12.2.1] implies that no d-polytope can have the r-skeleton of the n-cube for $r > \lfloor \frac{d}{2} \rfloor - 1$ where $n > d$; this is reviewed in Section 2.

Our construction specializes to a known phenomenon for $d = 2$: There are n-cubes whose 2-dimensional "shadows" have 2^n vertices ("projections that preserve the 0-skeleton"). These were first constructed by Murty [11] and more explicitly by Goldfarb [9]. They amount to linear programs for which the shadow boundary pivot rule takes an exponential number of steps. In Amenta & Ziegler [2] the Goldfarb cubes were interpreted as a special case of a construction of "deformed cubes", and indeed the neighborly cubical polytopes constructed in this paper are easily seen to be projections of deformed cubes $C^n(\varepsilon)$ as well.

An interesting new phenomenon occurs in the case $d = 4$: Jockusch [12] had constructed examples of cubical 4-polytopes for which the facet/vertex ratio f_3/f_0 was higher than previously expected, namely arbitrarily close to 5/4. The neighborly cubical polytopes show that indeed the ratio f_3/f_0 is not bounded for cubical 4-polytopes: these polytopes have $f_0 = 2^n$ vertices and $f_3 = (n-2)2^{n-2} = f_0 \log(f_0/4)/4$ facets.

Kalai's Cubical Upper Bound Conjecture [3, Conj. 4.2] claimed that among all cubical $(d-1)$-spheres with 2^n vertices, the boundaries of cubical neighborly polytopes simultaneously maximize all components of the f-vector. In Section 4 we give a counter-example, but we prove the claim in the special case $n = d+1$ for *polytopal* $(d-1)$-spheres. Further extremal properties of the neighborly cubical polytopes constructed in this paper are briefly discussed in Section 8.

A major part of the research on cubical polytopes is guided by the comparison with simplicial polytopes. In this paper, we extend this analogy by providing cubical analogs to the cyclic (neighborly simplicial) polytopes. But we also offer a suprising instance where the cubical case differs from the simplicial case: While even-dimensional neighborly polytopes are always simplicial, it is not true that a 4-polytope with the graph of the 5-cube must necessarily be cubical — we construct an explicit example in Section 5. Together with the existence of the unique cubical 4-polytope with the graph of the 5-cube this implies that the graph of the 5-cube is strongly 4-ambiguous.

2 Neighborly cubical polytopes.

We refer to [10, 15] for general introductions to polytopes and polytopal complexes. Two polytopes or polytopal complexes are *combinatorially equivalent* if their posets of faces are isomorphic. In the following, a *d-cube* is any polytope that is combinatorially isomorphic to the standard d-cube $C_d = [0,1]^d \subseteq \mathbb{R}^d$. A *combinatorial cube* is such a d-cube, for any d. A *cubical polytope* is any polytope all of whose proper faces are combinatorial cubes. The *k-skeleton* of a polytope is the polytopal complex given by all faces of dimension k or less. We say that *P has the k-skeleton of a cube* if its k-skeleton is combinatorially equivalent to that of a combinatorial cube. A *neighborly cubical polytope* is a cubical d-polytope (with 2^n vertices for some $n \geq d$) which has the $(\lfloor \frac{d}{2} \rfloor - 1)$-skeleton of a cube. This notion was introduced in [3], where neighborly cubical spheres were constructed, and the question about the existence of neighborly cubical polytopes was raised. We start with explaining the choice of parameters in this definition.

Proposition 1 (Characterization of Cubes [6])
Any cubical d-polytope has at least 2^d vertices.
If a cubical d-polytope has exactly 2^d vertices, then it is a combinatorial d-cube.

Corollary 2
If all the k-faces of a d-polytope have 2^k vertices, for all $0 \leq k \leq d-1$, then the polytope is cubical.
If in addition the polytope has 2^d vertices, then it is a combinatorial cube.

It is well-known that the f-vector of a cubical polytope is subject to restrictions that are similar to the Dehn-Sommerville equations for simplicial/simple polytopes.

Proposition 3 (Cubical Dehn-Sommerville Equations [10, 9.4.1])
Let $(f_0, \ldots, f_{d-1})$ be the f-vector of a cubical d-polytope. Then, for $0 \leq k \leq d-2$,

$$\sum_{i=k}^{d-1} (-1)^i 2^{i-k} \binom{i}{k} f_i \quad = \quad (-1)^{d-1} f_k.$$

Lemma 4 (Simple cubical polytopes [15, Exercise 0.1, p. 23])
Every simple cubical d-polytope with $d > 2$ is a d-cube.
(Every 2-polytope is simple and cubical.)

Proposition 5
If a d-polytope P has the r-skeleton of the n-dimensional cube, then all k-faces of P are cubes for $k \leq 2r$.

Proof. Let F be a k-face of P and $k \le 2r$. By induction on k we can assume that F is cubical. If F is simple then F is a cube by Lemma 4. Thus assume that F has a vertex v of degree $k' > k$. Let $a_1, \dots, a_{k+1}$ be $k+1$ distinct vectors such that $v + a_i$ is a neighbor of v in F. As $\dim F = k$, the vectors $a_1, \dots, a_{k+1}$ are linearly dependent, i. e. we can choose $\lfloor \frac{k+1}{2} \rfloor$ vectors among them which do not span a proper face of F. But $\lfloor \frac{k+1}{2} \rfloor \le \lfloor \frac{2r+1}{2} \rfloor = r$. This contradicts the assumption that P has the r-skeleton of a cube. □

We note here that in the simplicial case more is true: If a d-polytope P has the r-skeleton of the n-dimensional *simplex*, then all k-faces of P are simplices for $k \le 2r+1$. In particular, a d-polytope P has the r-skeleton of the n-dimensional simplex for $r \ge \lceil \frac{d}{2} \rceil$, then P is a d-simplex.

Corollary 6
If a d-polytope P has the r-skeleton of the n-dimensional cube for $r \ge \frac{d}{2}$, then P is a d-cube.

But this is not good enough to establish the analogy to the simplicial case. What if P is a $2k$-polytope with the $(k-1)$-skeleton of the $(2k+1)$-cube? By Proposition 5 all $(2k-2)$-faces are cubes, but what about the $(2k-1)$-faces, i. e. the facets? It turns out that the result of Proposition 5 is sharp in the sense that there are $2k$-polytopes which have the $(k-1)$-skeleton of a $(2k+1)$-cube but which are *not* cubical: In Section 5 we present an example of a non-cubical 4-polytope with the graph of the 5-cube.

The proof of the following is based on a theorem of van Kampen and Flores, see [10, 11.1.3 and 11.3].

Proposition 7 (Grünbaum [10, 11.2.1])
Let P be a d-polytope with the r-skeleton of an n-polytope, with $n > d$. Then $r \le \lfloor \frac{d}{2} \rfloor - 1$.

3 Projections.

We discuss the effect of orthogonal projections on polytopes. Everything in this section is well-known; it is included for the sake of completeness.

Let P be a full-dimensional polytope. A vector $\boldsymbol{n}$ is *normal* with respect to a facet F of P if it is orthogonal to F and it points "to the outside," that is, if the linear functional corresponding to $\boldsymbol{n}$ and restricted to P attains its maximum at the points in F. A vector is called *normal* with respect to a face G if it is a positive linear combination of normal vectors of all facets of P containing G. Equivalently, the linear functional corresponding to a normal vector of G attains its maximum at the points in G. Obviously, every facet has a unique normal vector of length 1, while a face of higher codimension does not.

Consider an orthogonal projection π onto some proper affine subspace.

Lemma 8
If the face G has a normal vector orthogonal to the direction of projection, then the image $\pi(G)$ is a face of the polytope $\pi(P)$. Conversely, if $\overline{G}$ is a face of $\pi(P)$, then the full preimage $\pi^{-1}(\overline{G})$ is a face of P with a normal vector orthogonal to the direction of projection.

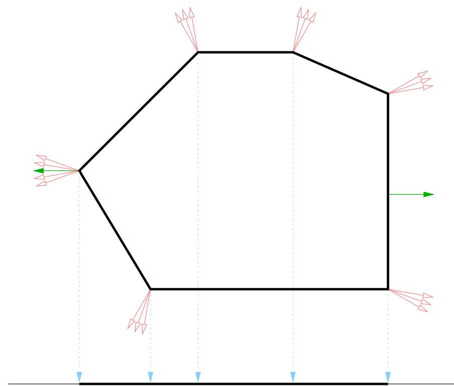

Figure 1: Orthogonal projection of a 2-polytope.

The lemma above characterizes the “shadow boundary,” that is, it describes the faces that are mapped to the boundary of the projection. Note that this may include faces of G that are mapped to faces of lower dimension, cf. Figure 1.

Lemma 9
The restriction of π to a face G is injective if and only if G has a normal vector which not orthogonal to the direction of projection.

Combining the two lemmas above gives the following characterization.

Corollary 10
A face G is mapped onto a face $\pi(G)$ of the same dimension if and only if it has a normal vector which is orthogonal to the direction of projection and another one which is not.

We can also say something about the shape of the projection in this case.

Lemma 11
If and only if a face G is mapped onto a face $\pi(G)$ of the same dimension, the faces G and $\pi(G)$ are affinely isomorphic.

4 The case $n = d + 1$.

4.1 First construction.

Let $d = 2r$ be even, $Q := [-1, +1]^r$, and

$$\widetilde{P} \;:=\; \mathrm{conv}(Q \times 2Q \times \{-1\} \,\cup\, 2Q \times Q \times \{1\}) \;\subseteq\; \mathbb{R}^{d+1}.$$

This is clearly a combinatorial $(d+1)$-cube, with the complete the linear description

$$\begin{aligned} \widetilde{P} \;=\; \{\tbinom{\boldsymbol{x}}{x_{d+1}} \in \mathbb{R}^{d+1} :\;\; & -1 \le x_{d+1} \le 1, \\ & \pm 2x_i \;\le\; 3 - x_{d+1} \quad \text{for } 1 \le i \le r, \\ & \pm 2x_i \;\le\; 3 + x_{d+1} \quad \text{for } r < i \le d \;\; \}. \end{aligned}$$

The projection $\pi : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$ that deletes the last coordinate yields the d-polytope

$$P \;:=\; \pi(\widetilde{P}) \;=\; \mathrm{conv}(Q{\times}2Q \;\cup\; 2Q{\times}Q) \subseteq \mathbb{R}^d.$$

One Fourier-Motzkin elimination step [15, Sect. 1.2] shows that P can also be described in terms of its facets by

$$\begin{aligned} P \;=\; \{\boldsymbol{x} \in \mathbb{R}^d :\quad & \pm x_i \le 2 \quad \text{for } 1 \le i \le 2r, \\ & \pm x_i \pm x_j \le 3 \quad \text{for } 1 \le i \le r < j \le 2r \;\; \}. \end{aligned}$$

This P is a cubical $2r$-polytope with $f_0 = 2^{d+1}$ vertices and $f_{d-1} = 4r + (2r)^2 = d(d+2)$ facets.

It is also easy to see that P has the $(r-1)$-skeleton of a $(d+1)$-cube, either using the criteria of Section 3, or by direct verification from the complete description of the polytope in terms of facets *and* vertices.

This example of a polytope with the $\frac{d}{2}$-skeleton of a cube is amazing because of its simplicity, and also because of its symmetry: It has a vertex-transitive symmetry group, and only two orbits of facets.

4.2 Second construction.

Blind & Blind completed a classification of the (combinatorial types) of cubical d-polytopes with 2^{d+1} vertices in [7]. From this classification, we derive below that for even d, there is exactly one combinatorial type of a d-polytope with the $\frac{d}{2}$-skeleton of the $(d+1)$-cube; for odd d, there are precisely two combinatorial types. The description of the polytopes given by Blind & Blind also implies the following construction given above.

A (cubical) d-polytope P whose boundary complex ∂P is isomorphic to a subcomplex of the d-skeleton of some (higher-dimensional) cube is called *liftable*. We get all distinct combinatorial types of liftable d-polytopes (and $(d-1)$-spheres) with at most 2^{d+1} vertices as follows:

The cube C^{d+1} has $d+1$ pairs of opposite facets $F_i^{\pm}$ $(1 \le i \le d+1)$. For $k, m \ge 0$ and $l \ge 1$ with $k+l+m = d+1$ let $B(k,l,m)$ be the cubical d-ball in the boundary of C^{d+1} formed by $F_i^{\pm}$ $(1 \le i \le k)$ and $F_i^{+}(k+1 \le i \le k+l)$.

The combinatorial types $P(k,l,m)$ of liftable d-polytopes with at most 2^{d+1} vertices are given by the boundary complexes of the cubical balls $B(k,l,m)$. The number of vertices of $P(k,l,m)$ equals 2^{d+1} if and only if $k, m \ge 1$. Note that $P(k,l,m)$ is combinatorially equivalent to $P(m,l,k)$; thus in the following let $k \ge m$. There are $\lfloor d^2/4 \rfloor$ suitable triples (k,l,m) with $k \ge m \ge 1$.

Theorem 12 (Classification for $n = d+1$, Blind & Blind [7, Theorem 3])
For $d \ge 4$, all the combinatorial types of d-polytopes with 2^{d+1} vertices are given by the liftable polytopes $P(k,l,m)$ with $k,l,m \ge 1$, $k+l+m = d+1$ and $k \ge m$, plus in addition the "2-fold non-linearly capped" d-polytope P^d_{NLC}.

A polytope $P(k,l,m)$ is *r-neighborly* if for every r-face H of C^{d+1}, there is some facet of $B(k,l,m)$ containing H, and some facet of the complement $C^{d+1} \backslash B(k,l,m)$ containing H as well. Every r-face of C^{d+1} is contained in exactly $d+1-r$ facets of C^{d+1}. Thus $P(k,l,m)$ is r-neighborly if and only if $k+l \le (d+1-r)-1$ and $l+m \le (d+1-r)-1$, that is $r+1 \le m$ and $r+1 \le k$, where the first condition implies the second because of $k \ge m$.

Now we consider the case $r = \lfloor \frac{d}{2} \rfloor - 1$.

Corollary 13
If d is even, then $P(\frac{d}{2}, 1, \frac{d}{2})$ is the unique $(\frac{d}{2}-1)$-neighborly cubical polytope with 2^{d+1} vertices. If d is odd, then there are precisely two $(\frac{d-1}{2}-1)$-neighborly cubical polytopes with 2^{d+1} vertices, namely $P(\frac{d+1}{2}, 1, \frac{d-1}{2})$ and $P(\frac{d-1}{2}, 1, \frac{d-1}{2})$.

Now we compute the f-vector of $P(k,l,m)$, denoted by $f(P(k,l,m)) = (f_0, \ldots, f_{d-1})$. With the same reasoning as above, an i-face of C^{d+1} is a face of $P(k,l,m)$ if it lies in some facet of $B(k,l,m)$, and also in some facet not in $B(k,l,m)$. We deduce that

$$f_{d+1-i} \;=\; \binom{d+1}{i} 2^i \;-\; \sum_{j=0}^{i} \binom{l}{i-j} \left\{ \binom{k}{j} + \binom{m}{j} \right\} 2^j.$$

4.3 The Cubical Upper Bound Conjecture.

From the analogy to the simplicial case one is tempted to expect that the neighborly cubical polytopes achieve equality for the Cubical Upper Bound Conjecture.

Conjecture 14 (Cubical Upper Bound Conjecture, Kalai [3, Conjecture 4.2]) *Let P be a cubical $(d-1)$-sphere with $f_0(P) = 2^n$ vertices. Then its number of facets is bounded by that of C_d^n, that is, $f_{d-1}(P) \leq f(n,d)$. Moreover,*

$$f_i(P) \leq f_i(C_d^n) \qquad \textit{for } 1 \leq i \leq d-1.$$

Theorem 15 *In the special case $n = d+1$, Conjecture 14 is* true *if restricted to cubical polytopes. However, it is* false *for spheres even for $d = 4$ and $n = 5$.*

Proof.
(1) The proof for polytopes relies on the classification of Theorem 12. Here we can disregard the "2-fold non-linearly capped" d-polytope P_{NLC}^d, since it has the same f-vector as the "2-fold linearly capped" d-polytope, which is $P(d-1,1,1)$, see [7, Figure 1]. Thus our problem is to minimize, for fixed d and i, the function

$$\delta_i(k,l,m) \;=\; \sum_{j=0}^{i} \binom{l}{i-j} \left\{ \binom{k}{j} + \binom{m}{j} \right\} 2^j$$

subject to the restrictions $k, l, m \geq 1$, $k+l+m = d+1$ and $k \geq m$. For this, we note the simple properties and inequalities

$$\begin{aligned} \delta_i(k,l,m) &\geq \delta_i(k-1,l,m+1) \quad \text{for } k > m, \\ \delta_i(k,l,m) &\geq \delta_i(k+1,l-2,m+1) \quad \text{for } l \geq 2, \\ \delta_i(k,2,m) &\geq \delta_i(k,1,m+1) \quad \text{for } k > m, \text{ and} \\ \delta_i(k,2,k) &= \delta_i(k+1,1,k), \end{aligned}$$

from which the claim immediately follows.

For even d, this result also follows from Babson, Billera & Chan [3, Thm. 4.3], who used Adin's "cubical h-vector" [1].

(2) We show that $P(2,1,2)$ does not maximize the f-vector among cubical 3-spheres, as follows. Consider the following chain of three cubical facets $G_1 = F_1^+ \cap F_3^-$, $G_2 = F_1^+ \cap F_5^+$, $G_3 = F_3^+ \cap F_5^+$ in the boundary of $B(2,1,2)$. Observe that

- $G_1 \cap G_2 = F_1^+ \cap F_3^- \cap F_5^+$ and $G_2 \cap G_3 = F_1^+ \cap F_3^+ \cap F_5^+$ are two opposite facets of G_2, and
- there are no edges between vertices in $G_1 \backslash G_2 = F_1^+ \cap F_3^- \cap F_5^-$ and vertices in $G_3 \backslash G_2 = F_1^- \cap F_3^+ \cap F_5^+$. (Any two vertices differ in at least three signs.)

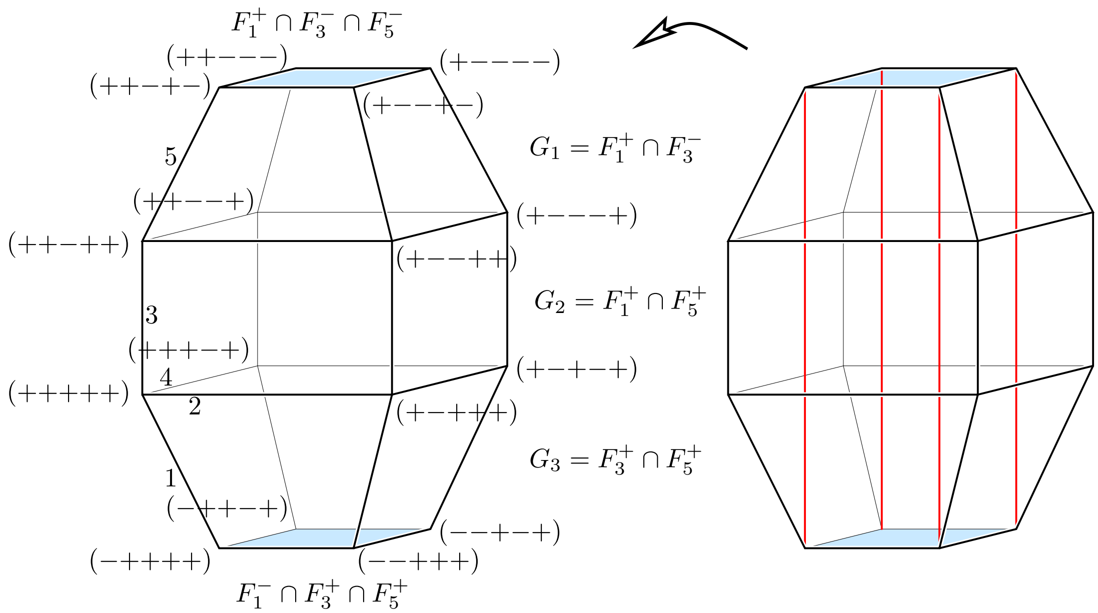

Figure 2: Modifying three facets of $P(2,1,2)$.
(a) The three adjacent facets that are cut out. (b) The ball to be glued in.

Now we modify the boundary complex of $P(2,1,2)$ by a "local surgery," as follows: remove the three facets G_1, G_2, G_3 together with the two 2-faces between them, see Figure 2(a). Into the resulting "hole", glue a cubical ball that consists of four edges (connecting the vertices of $G_1 \backslash G_2$ with the corresponding vertices of $G_3 \backslash G_2$), eight 2-faces and five cubes (four cubes grouped around a central 3-cube whose top facet is in $G_3 \backslash G_2$, and whose bottom facet is in $G_1 \backslash G_2$).

The resulting cubical sphere Φ has the f-vector

$$f(\Phi) \quad = \quad f(P(2,1,2)) + (0, 4, 8-2, 5-3).$$

Thus $P(2,1,2)$ is a neighborly cubical polytope whose f-vector is not maximal among the cubical 3-spheres with 32 vertices. On the other hand, Φ is an example of a *not-polytopal* cubical sphere, since it does not appear in the classification of cubical polytopes with 2^{d+1} vertices of Theorem 12. ∎

5 A non-cubical polytope.

Here we give an example of a non-cubical 4-polytope with the same graph as the 5-cube. For this it is helpful to have yet another coordinate representation of the cubical 4-polytope of the previous section.

Let P be the polytope defined as the convex hull of the following 32 points in $\mathbb{R}^4$. Note that each row corresponds to four distinct points due to arbitrary variation of the signs.

$$
\begin{array}{cccccc}
(& \pm 1 & \pm 1 & 1 & 1 &) \\
(& \pm 1 & \pm 1 & 4 & 1 &) \\
(& \pm 2 & \pm 2 & 3 & 4/5 &) \\
(& \pm 2 & \pm 2 & 2 & 4/5 &) \\
(& \pm 3 & \pm 3 & 2 & 1/2 &) \\
(& \pm 3 & \pm 3 & 3 & 1/2 &) \\
(& \pm 4 & \pm 4 & 0 & 0 &) \\
(& \pm 4 & \pm 4 & 5 & 0 &)
\end{array}
$$

All these points are vertices. Moreover, the last eight vertices span a facet $F = \{\boldsymbol{x} \in P : x_4 = 0\}$, which is a 3-cube. The graph of P is isomorphic to the graph of the 5-cube.

Projecting the polytope P onto the facet F from a point beyond F yields a polytopal complex, the *Schlegel diagram* of P with respect to F, which is essentially equivalent to the boundary complex of P [10, Sect. 5.3] [15, §5.2]. For our example, the whole Schlegel diagram has the same symmetry group of order 16 as a prism over a square. Thus it is sufficient to consider a diagonal section as indicated in Figure 3.

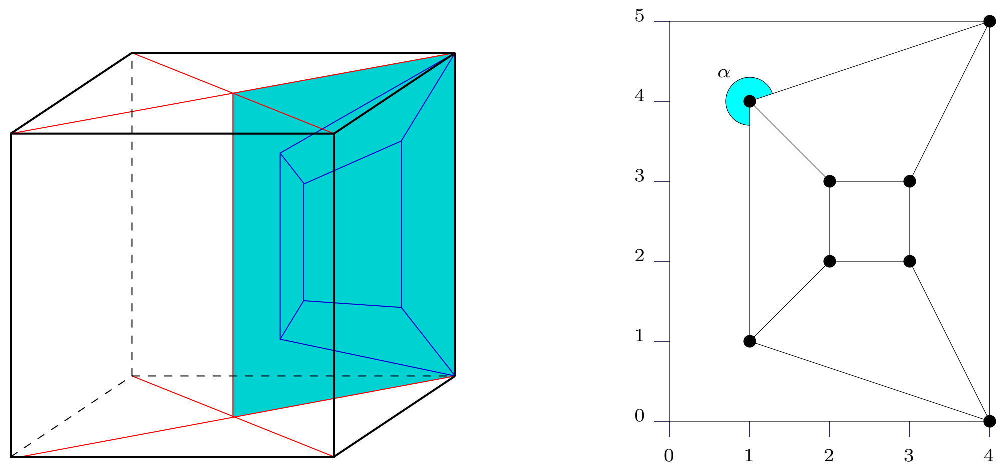

Figure 3: Schlegel diagram and a section.

Now remove the 2-face spanned by the 4 vertices $(\pm 1, \pm 1, 4, 1)$ and merge the two facets containing it. This yields a regular cell complex C whose 1-skeleton is the same as before, that is, it is isomorphic to the graph of the 5-cube. Clearly, $\|C\|$ is homeomorphic to the 3-sphere. But, as realized, C is not polytopal because the angle α in Figure 3 exceeds π.

It is fairly obvious that the modified cell complex C can be realized as a polytopal complex by an appropriate change of the coordinates, such that finally α becomes less than π. It may be less obvious that — for a special choice of coordinates — the transformed diagram with $\alpha < \pi$ can be lifted to a 4-polytope P'. Here is a realization:

(	±1	±1	1	0	)
(	±2	±2	4	0	)
(	±3	±3	3	1	)
(	±3	±3	2	5/4	)
(	±4	±4	2	21/20	)
(	±4	±4	3	9/5	)
(	±56/13	±56/13	0	779/260	)
(	±5	±5	16	0	)

The polytope P' has one facet that has 12 vertices, namely

$$(\pm 1, \pm 1, 1, 0),\ (\pm 2, \pm 2, 4, 0),\ (\pm 5, \pm 5, 16, 0).$$

Therefore P' is a 4-polytope with the graph of the 5-cube which is not cubical. The polytopes P and P' have been constructed with POLYMAKE [8]. Suitable POLYMAKE input files are available on the Internet at

`http://www.math.tu-berlin.de/diskregeom/polymake/examples/NCP/`.

6 Alternating Oriented Matroids and Cyclic Polytopes.

In Section 7 we will construct a class of polytopes which look like "cubical relatives" of the cyclic polytopes. The combinatorial structure of the cyclic polytopes is well-known [10, Sect. 4.6] [15, Example 0.6]. We give a brief account in the framework of oriented matroids, which also captures the "interior combinatorial structure" of the cyclic polytopes.

Let $C_d(n)$ be the cyclic d-polytope on n vertices. It can be realized as the convex hull of n points on the moment curve $t \mapsto (t, t^2, t^3, \ldots, t^d)$ in $\mathbb{R}^d$.

Any point configuration in $\mathbb{R}^d$, and thus any polytope via its vertices, gives rise to an oriented matroid [15, Sect. 6.4] [4]. The positive cocircuits of the oriented matroid bijectively correspond to the facets of the polytope.

For a cyclic polytope $C_d(n)$ the associated oriented matroid is known; it is called the *alternating oriented matroid* $\mathcal{C}(n, d+1)$ of rank $d+1$ on n points [4, Sects. 3.4 and 9.4]. Homogenizing the vertices of $C_d(n)$ it can be represented by the (rows of the) matrix

$$\begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^d \\ 1 & t_2 & t_2^2 & \cdots & t_2^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^d \end{pmatrix}$$

for $t_1 < t_2 < \cdots < t_n$.

There is a notion of duality for oriented matroids which generalizes duality of projective spaces. In the case of the alternating oriented matroids, the dual of $\mathcal{C}(n,d+1)$ is obtained from $\mathcal{C}(n,n-d-1)$ by reorienting every other row in the representation above,

$$\mathcal{C}(n,d+1)^* \quad = \quad {}_{\overline{\{2,4,6,\dots\}}}\mathcal{C}(n,n-d+1),$$

(see [5, pp. 108-109], [14, Sect. 2]), that is, by the rows of the $n \times (n-d+1)$-matrix

$$\begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-d-1} \\ -1 & -t_2 & -t_2^2 & \cdots & -t_2^{n-d-1} \\ 1 & t_3 & t_3^2 & \cdots & t_3^{n-d-1} \\ -1 & -t_4 & -t_4^2 & \cdots & -t_4^{n-d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1} & (-1)^{n+1}t_n & (-1)^{n+1}t_n^2 & \cdots & (-1)^{n+1}t_n^{n-d-1} \end{pmatrix}$$

From the representation of the alternating matroid and its dual given above it is obvious that that a deletion (omitting a row in the primal oriented matroid) or a contraction of the first element (omitting the first row and the first column in the primal oriented matroid, which amounts to omitting the first row in the dual) again gives an alternating matroid (on fewer points and, in the second case, of smaller rank).

The number of facets of $C_d(n)$, that is, the number of positive cocircuits of the alternating matroid $\mathcal{C}(n,d+1)$, is known to be

$$\binom{n-\lceil\frac{d}{2}\rceil}{\lfloor\frac{d}{2}\rfloor} + \binom{n-1-\lceil\frac{d-1}{2}\rceil}{\lfloor\frac{d-1}{2}\rfloor},$$

by Gale's evenness criterion [10, p. 63] [15, Thm. 0.7].

7 The general case.

The following is our main theorem: "Neighborly cubical polytopes exist!" As mentioned in the introduction, the case $d = 2$ and the case $n = d + 1$ were known previously.

Theorem 16 *For any $n \geq d \geq 2r+2$, there exists a combinatorial n-cube $C^n \subseteq \mathbb{R}^n$ and a linear projection map $\pi : \mathbb{R}^n \to \mathbb{R}^d$ such that $C_d^n := \pi(C^n)$ is a cubical d-polytope whose r-skeleton is isomorphic to that of C^n (via π).*

Proof. We first construct a combinatorial n-cube $C^n(\varepsilon) \subseteq \mathbb{R}^n$ that depends on a parameter $\varepsilon > 0$; then we verify that for ε sufficiently small the projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$ to the last d coordinates preserves the r-skeleton; and finally we argue that $\pi(C^n(\varepsilon)) = C_d^n$ is cubical.

(A) For $0 < \varepsilon \leq 1$ define $C^n(\varepsilon) \subseteq \mathbb{R}^n$ as the solution set of

$$\varepsilon|x_k| \quad \leq \quad \frac{2^{\binom{k}{2}}}{\varepsilon^{k-1}} - (-1)^k \sum_{j=1}^{k-1} \binom{k-2}{j-1} x_j \qquad \text{for } 1 \leq k \leq n. \tag{1}$$

This set is a combinatorial n-cube. To see this, we verify by induction on k that all solutions to the first k conditions of (1) satisfy

$$|x_k| \quad < \quad \frac{2^{\binom{k}{2}+1}}{\varepsilon^k}. \tag{2}$$

In fact, the upper bound of (2) increases with k, and for $k = 1$ we have $\varepsilon|x_1| \leq 1$, so $|x_1| < \frac{2}{\varepsilon}$ is surely satisfied. Thus we may use induction, and for $k \geq 2$ estimate

$$\begin{aligned}
\varepsilon|x_k| \quad &\leq \quad \frac{2^{\binom{k}{2}}}{\varepsilon^{k-1}} - (-1)^k \sum_{j=1}^{k-1} \binom{k-2}{j-1} x_j \\
&< \quad \frac{2^{\binom{k}{2}}}{\varepsilon^{k-1}} + \sum_{j=1}^{k-1} \binom{k-2}{j-1} \frac{2^{\binom{k-1}{2}+1}}{\varepsilon^{k-1}} \\
&\leq \quad \frac{2^{\binom{k}{2}}}{\varepsilon^{k-1}} + 2^{k-2}\frac{2^{\binom{k-1}{2}+1}}{\varepsilon^{k-1}} \quad = \quad \frac{2^{\binom{k}{2}+1}}{\varepsilon^{k-1}}.
\end{aligned}$$

In this computation the second term is always smaller in absolute value than the first, that is,

$$\frac{2^{\binom{k}{2}}}{\varepsilon^{k-1}} - \left| \sum_{j=1}^{k-1} \binom{k-2}{j-1} x_j \right| \quad > \quad 0 \qquad \text{for } 1 \leq k \leq n,$$

and from this we see that $C^n(\varepsilon)$ is a combinatorial cube. (It is an iterated deformed product in the sense of [2]).

(B) Now let $\pi : \mathbb{R}^n \to \mathbb{R}^d$ be the projection to the last d coordinates, $(x_1, \ldots, x_n) \longmapsto (x_{n-d+1}, \ldots, x_n)$.

Claim 1. *For sufficiently small $\varepsilon > 0$ and $n \geq d \geq 2r + 2$ the orthogonal projection $\pi : C^n(\varepsilon) \longrightarrow \pi(C^n(\varepsilon)) =: C^n_d(\varepsilon)$ preserves the r-skeleton. That is, π restricts to an isomorphism*

$$\pi : C^n(\varepsilon)^{[r]} \longrightarrow C^n_d(\varepsilon)^{[r]}$$

of polytopal complexes.

To verify this, we need to see that every r-face F of $C^n(\varepsilon)$ is mapped bijectively to an r-face $\pi(F)$ of $C^n(\varepsilon)$. By Corollary 10 this is equivalent to the condition that for every r-face F there is a normal vector which is orthogonal to the direction of projection and another one which is not.

In our specific situation, let F be an r-face of $C^n(\varepsilon)$, and let $\boldsymbol{v} \in C^n(\varepsilon)$ be a vertex of F. Then there are unique signs $\sigma_1, \ldots, \sigma_n \in \{+1, -1\}$ such that $\boldsymbol{v}$ is determined by

$$(-1)^k \sum_{j=1}^{k-1} \binom{k-2}{j-1} v_j + \sigma_k \varepsilon v_k = \frac{2^{\binom{k}{2}}}{\varepsilon^{k-1}} \qquad \text{for } 1 \leq k \leq n,$$

while F is characterized by the additional choice of a set $S \in \binom{[n]}{n-r}$ of $n-r$ indices:

$$F = \{\boldsymbol{x} \in C^n(\varepsilon) : (-1)^k \sum_{j=1}^{k-1} \binom{k-2}{j-1} x_j + \sigma_k \varepsilon x_k = \frac{2^{\binom{k}{2}}}{\varepsilon^{k-1}} \quad \text{for all } k \in S\},$$

where $C^n(\varepsilon)$ itself is given by

$$C^n(\varepsilon) = \{\boldsymbol{x} \in \mathbb{R}^n : (-1)^k \sum_{j=1}^{k-1} \binom{k-2}{j-1} x_j + \varepsilon |x_k| \leq \frac{2^{\binom{k}{2}}}{\varepsilon^{k-1}} \quad \text{for } 1 \leq k \leq n\}.$$

In order to show that all $\pi(\boldsymbol{v})$ are vertices of $\pi(C^n(\varepsilon))$, we must thus check that, for any choice $\Sigma = (\sigma_1, \ldots, \sigma_n) \in \{\pm 1\}^n$ of signs, the rows of the $n \times (n-d)$-matrix

$$A(\Sigma) = \begin{pmatrix} \pm\varepsilon & 0 & & & \\ 1 & \pm\varepsilon & 0 & & \\ -1 & -1 & \pm\varepsilon & \ddots & \\ 1 & 2 & 1 & \ddots & 0 \\ -1 & -3 & -3 & \ddots & \pm\varepsilon \\ 1 & 4 & & & \\ \vdots & & \ddots & & \end{pmatrix}$$

given by

$$a_{kj} = \begin{cases} 0 & \text{for} \quad j > k \\ \sigma_k \varepsilon & \text{for} \quad j = k \\ (-1)^k \binom{k-2}{j-1} & \text{for} \quad j < k \end{cases}$$

have a positive linear dependence.

We also have to show that each r-face has a normal vector which is not orthogonal to the direction of projection. But, this is obvious: for any set of rows of the matrix $A(\Sigma)$ (for an arbitrary vector Σ of signs) set, e. g., the coefficient of the last row to 1 and all the others positive but sufficiently small. This positive linear combination yields a non-zero vector.

At an r-face $F \subseteq C^n(\varepsilon)$ only $n-r$ restrictions are tight, so Claim 1 now reduces to the following.

Claim 2. *For sufficiently small $\varepsilon > 0$ and for every choice $\Sigma \in \{\pm 1\}^n$ of signs, every set of $n-r$ rows of $A(\Sigma)$ has a positive dependence.*

Let $\overline{A}(\Sigma) := (a_{kj})_{2 \leq k \leq n, 1 \leq j \leq n-d} \in \mathbb{R}^{(n-1)\times(n-d)}$ be obtained by deleting the first row of $A(\Sigma)$. An index set $S \subseteq \{2, 3, \ldots, n\}$ will be called *alternating* if it alternates between odd and even numbers; for example $\{2, 3, 6, 9\}$ and $\{3, 4, 5, 8\}$ are alternating, but $\{2, 3, 5, 6\}$ is not. A set of rows of S is *alternating* if the corresponding index set is alternating. Using this concept, we formulate the following Claim 3, which clearly implies Claim 2.

Claim 3.

(i) *If $\varepsilon = 0$, then all maximal minors of $\overline{A}(\Sigma)$ have non-zero determinant.*

(ii) *If $\varepsilon > 0$ is so small that all maximal minors of $\overline{A}(\Sigma)$ have the same sign as for $\varepsilon = 0$, then every alternating subset of $n-d+1$ rows of $\overline{A}(\Sigma)$ positively spans $\mathbb{R}^{n-d}$.*
(iii) *For $n \geq d \geq 2r+1$, every subset of $n-1-r$ rows of $\overline{A}(\Sigma)$ contains an alternating subset of size $n-d+1$.*

To see **(i)**, we note that $\{1, t, \binom{t}{2}, \ldots, \binom{t}{n-d+1}\}$ and $\{1, t, t^2, \ldots, t^{n-d+1}\}$ are two different bases for the vector space of rational polynomials of degree at most $n-d+1$. Thus the matrix $\overline{A} = \overline{A}(\Sigma)$ arises by invertible row operations from the matrix

$$\overline{B} = ((-1)^k (k-1)^{j-1})_{2 \leq k \leq n,\ 1 \leq j \leq n},$$

whose maximal minors are Vandermonde determinants.

For **(ii)**, observe the following: It can be seen from the oriented matroid of the vector configuration whether there is a positive linear relation between a set of vectors, such as the rows of $\overline{A}$. The condition (in order to obtain that it positively spans) is that the configuration must be a positive curcuit, or equivalently, *totally cyclic* [4, Sect. 3.4] [15, Sect. 6.4]. For ε small enough (as indicated), we have the same oriented matroid as for $\varepsilon = 0$, and hence as for $\overline{B}$. The oriented matroid $\mathcal{M}(\overline{B})$ of rank $n-d$ determined by the rows of $\overline{B}$ is the dual of the alternating oriented matroid, which differs from the alternating oriented matroid by a reorientation of $\{3, 5, 7, \ldots\}$, cf. Section 6. In $\mathcal{M}(\overline{B})$, the positive circuits are all subsets of size $n-d+1$. In particular, the ground set is a positive vector, that is, there is a positive linear relation among all the rows of $\overline{B}$.

For part **(iii)**, start with the alternating index set $\{2, 3, \ldots, n\}$ for the rows of $\overline{A}$. Now successively delete any r rows from $\overline{A}$, but whenever a row is deleted, we remove also the next row above or below that has not yet been deleted. Thus in each of the (at most) r deletion steps, we remove two adjacent rows of $\overline{A}$, and hence the index set is kept to be alternating. After all this, we are left with a submatrix $\overline{\overline{A}}$ of A that has at least $n-1-2r$ rows and whose index set is alternating. Since $n-d+1 \geq n-1-2r$, we may take the first $n-d+1$ rows of $\overline{\overline{A}}$.

(C) *The polytopes $C_d^n(\varepsilon)$ are indeed cubical.*

Let F be a facet of $C_d^n(\varepsilon)$. Its preimage $F' = \pi^{-1}(F)$ is a face of $C^n(\varepsilon)$ of dimension at least $d-1$. Suppose $\dim F' \geq d$. From Lemma 8 we infer that some $n-d$ rows of the $n \times (n-d)$-matrix A are (positively) linearly dependent. This is a contradiction to **(i)** of Claim 3. We conclude that $\dim F' = d-1 = \dim F$. The claim now follows from Lemma 11. □

It follows from the discussion in the last paragraph that each facet of $C_d^n(\varepsilon)$ is an affinely isomorphic image of some $(d-1)$-face of $C_d^n(\varepsilon)$.

Proposition 17
The facets of $C_d^n(\varepsilon)$ bijectively correspond to the distinct positive circuits that can be found in the oriented matroids $\mathcal{M}(A(\Sigma))$ associated to the rows of the $n \times (n-d)$-matrices $A(\Sigma)$, for all possible choices $\Sigma \in \{\pm 1\}^n$ of signs.

An equivalent formulation of Proposition 17 is the following: The facets of $C_d^n(\varepsilon)$ bijectively correspond to the set of positive circuits of the $2n \times (n-d)$-matrix

$$\begin{pmatrix} \varepsilon & & & & \\ -\varepsilon & \ddots & & & \\ 1 & \varepsilon & & & \\ 1 & -\varepsilon & \ddots & & \\ -1 & -1 & \varepsilon & & \\ -1 & -1 & -\varepsilon & \ddots & \\ 1 & 2 & 1 & \varepsilon & \\ 1 & 2 & 1 & -\varepsilon & \ddots \\ \vdots & & \ddots & & \ddots \end{pmatrix}$$

which do not contain the first two rows, or the second two rows ... that is, that do not contain both the $(2i-1)$-st and the $2i$-th row, for any i.
This observation allows us to count the facets of $C_d^n(\varepsilon)$.

Corollary 18
For $n \geq d \geq 2$ and for ε sufficiently small, then the number $f(n,d) := f_{d-1}(C_d^n(\varepsilon))$ of facets of $C_d^n(\varepsilon)$ is given by

$$\begin{aligned} f(n,d) &= 2d + 4\sum_{p=0}^{n-d-1}\left(\binom{\lfloor \frac{d}{2}\rfloor + p + 1}{p+2} + \binom{\lfloor \frac{d+1}{2}\rfloor + p}{p+2}\right)2^p \\ &= 2d + \sum_{p=2}^{n-d+1}\left(\binom{\lfloor \frac{d}{2}\rfloor + p - 1}{p} + \binom{\lfloor \frac{d-1}{2}\rfloor + p - 1}{p}\right)2^p \\ &= \begin{cases} 2d + \displaystyle\sum_{p=2}^{n-d+1}\binom{k+p-1}{p}2^{p+1} & \textit{for } d = 2k, \\ 2d + \displaystyle\sum_{p=2}^{n-d+1}\frac{p+2k-2}{p+k-1}\binom{k+p-1}{p}2^p & \textit{for } d = 2k+1. \end{cases} \end{aligned}$$

Proof. A positive circuit is a positive linear combination of $n-d+1$ rows of the matrix $A = A(\Sigma)$ yielding zero.

Consider the set $\mathrm{PC}(p)$ of positive circuits of $\mathcal{M}(A)$ which contain the first p rows of the matrix A but not the $(p+1)$st row, where $0 \le p < n-d$. Clearly, a set C containing the first p rows but not the $(p+1)$st is a circuit of $\mathcal{M}(A)$ if and only if the other $n-d-p$ rows are a circuit C' in the oriented matroid $\mathcal{M}(A_p)$, where the matrix A_p arises from A by removing the first p rows and the first p columns. This follows from the fact that the top minor of A consisting of the first $n-d$ rows is lower triangular. Moreover, each such positive circuit of $\mathcal{M}(A_p)$ can be uniquely extended to an element of $\mathrm{PC}(p)$ for a unique choice of the first p signs.

Now $\mathcal{M}(A_p)$ arises from $\mathcal{M}(A)$ by the contraction of the first p elements, that is, its dual is a p-fold deletion of the first element from the alternating oriented matroid $\mathcal{C}(n-1, d-1)$, which is $\mathcal{C}(n-1-p, d-1)$. We conclude that the elements of $\mathrm{PC}(p)$ bijectively correspond to the positive cocircuits of $\mathcal{C}(n-1-p, d-1)$. Note that we can still choose $2^{n-d+1-p}$ signs.

What is left are the positive circuits containing the first $n-d$ rows. Each of the remaining d rows is contained in precisely 2 positive circuits.

So the total number of facets is

$$2d + \sum_{p=0}^{n-d-1} (\#\text{facets of } C_{d-2}(n-1-p)) \cdot 2^{n-d+1-p}. \qquad \square$$

We evaluate the function $f(n,d)$ for some particularly interesting choices of (n,d):

$$\begin{aligned}
d = 2: &\quad f(n,2) &=&\ 2^n \\
d = 3: &\quad f(n,3) &=&\ 2^n - 2 \\
d = 4: &\quad f(n,4) &=&\ (n-2)2^{n-2} \\
d = 5: &\quad f(n,5) &=&\ (n-4)2^{n-2} + 2 \\
n = d: &\quad f(d,d) &=&\ 2d \\
n = d+1: &\quad f(d+1,d) &=&\ d^2 + d + 2\lfloor \tfrac{d}{2} \rfloor
\end{aligned}$$

For d even, the number $f(n,d)$ is already determined by n and d, together with the fact that we have a cubical d-polytope with the r-skeleton of the n-cube — using the cubical Dehn-Sommerville equations, cf. Proposition 3.

8 Comments.

(1) A cubical d-polytope is *k-stacked* if it has a cubical subdivision without interior $(d-k-1)$-faces [3, Def. 5.3]. Thus every k-stacked cubical polytope is also $(k+1)$-stacked.

Proposition 19 *The neighborly cubical d-polytopes C_d^n are $\lfloor\frac{d+1}{2}\rfloor$-stacked.*

Proof. The projection $\mathbb{R}^{d+1} \to \mathbb{R}^d$ that deletes the last coordinate maps C_{d+1}^n to C_d^n. The "upper faces" of the cubical polytope C_{d+1}^n (those with a normal vector whose last coordinate is positive) thus define a cubical subdivision B_d^n of C_n^d. Since, for $r = \lfloor\frac{d}{2}\rfloor - 1$, all r-faces of C_{d+1}^n get mapped to the boundary of C_d^n, we conclude that C_d^n has no interior r-faces. Thus C_d^n is k-stacked, for $k = d-1-r = d - \lfloor\frac{d}{2}\rfloor = \lfloor\frac{d+1}{2}\rfloor$. □

In the setting of Babson, Billera & Chan [3, Sect. 5] this yields a new extreme ray for the cubical g-cone for even d.

Corollary 20 *For even d and $k := \frac{d}{2}$, the neighborly cubical d-polytopes C_d^n form a sequence of polytopes for which the cubical g-vector is dominated by its k-th component, that is*

$$\lim_{n\to\infty} \frac{g_i^c(C_d^n)}{g_k^c(C_d^n)} = 0 \qquad \text{for all } i \neq k.$$

Thus "the ray $\mathbb{R}^+ e_k$ lies in the closure of the Adin g-cone", for even d and $k = \frac{d}{2}$, in the terminology of [3, Sect. 5]. In particular, for $d = 4$ this implies that the closure of the Adin g-cone, i.e., of

$$\text{cone}\,\{(g_1^c, g_2^c) \quad = \quad \{(f_0 - 16, 4f_3 - 3f_0 + 16) : \text{ cubical 4-polytopes}\}$$

is the complete positive orthant in $\mathbb{R}^2$.

(2) Our proof for Corollary 18 contains an implicit combinatorial description of the polytopes C_d^n for arbitrary parameters. It may be worth-while to make it more explicit.

(3) One can ask (and Raimund Seidel did) whether for *any* polytope P of dimension $n \geq d$ there is a d-polytope Q that has isomorphic r-skeleton, for $r = \lfloor d/2\rfloor - 1$. The case of the cross polytopes, $P = C_n^\triangle$, is particularly interesting.

(4) Is there a construction of (even-dimensional) neighborly cubical polytopes that have all vertices on a sphere? Note that the trigonometric moment curve [10, p. 67] [15, p. 75] yields this in the simplicial case. By explicit construction of Schlegel diagram as the Delaunay subdivisions of a finite point set, Raimund Seidel [13] obtained such a construction for $d = 4$ and $n \leq 7$.

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