The Integral Cohomology Algebras of Ordered Configuration Spaces of Spheres

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Abstract. We compute the cohomology algebras of spaces of ordered point configurations on spheres, $F(S^k, n)$, with integer coefficients. For $k = 2$ we describe a product structure that splits $F(S^2, n)$ into well-studied spaces. For $k > 2$ we analyze the spectral sequence associated to a classical fiber map on the configuration space. In both cases we obtain a complete and explicit description of the integer cohomology algebra of $F(S^k, n)$ in terms of generators, relations and linear bases. There is 2-torsion occuring if and only if k is even. We explain this phenomenon by relating it to the Euler classes of spheres.

Our rather classical methods uncover combinatorial structures at the core of the problem.

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1 Introduction

The space of configurations of n pairwise distinct labelled points in a topological space X,

$$F(X, n) := \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\} \subseteq X^n ,$$

is called the n-th (ordered) *configuration space* of X.
A systematic study of these spaces started with work by Fadell & Neuwirth [FaN] and Fadell [Fa] in the sixties. They introduced sequences of

fibrations for configuration spaces and mainly concentrated on describing their homotopy groups for various instances of X. In 1969 Arnol′d [Ar] derived the integer cohomology algebra of $F(\mathbb{C}, n)$ — the group cohomology of the colored braid group — and thereby initiated still ongoing research on the cohomology algebras of complements of linear subspace arrangements.

Broader interest in the cohomology algebras of configuration spaces came up in the seventies: The cohomology of $F(X, n)$ for a manifold X appeared as a basic ingredient in the E_2-terms of spectral sequences for the Gelfand-Fuks cohomology of the manifold [GF] and for the homology of certain function spaces [An]. Cohen [C1, C2] studied various aspects of the cohomology of configuration spaces of Euclidean spaces in view of its relation to homology operations for iterated loop spaces [C3]. Cohen & Taylor [CT1, CT2] described the cohomology algebras of configuration spaces of spheres with coefficients in a field of characteristic different from 2. Recently, compactifications of configuration spaces of algebraic varieties have been constructed by Fulton and MacPherson [FM]. As an application, they determine the rational homotopy type of configuration spaces of non-singular compact complex algebraic varieties $F(X, n)$ in terms of invariants of X. Compare also work of Kriz [Kr] and Totaro [T], where alternative minimal models for $F(X, n)$ are used.

In contrast to these results on the rational homotopy type of configuration spaces, it seems that so far Arnol′d's computation of the integer cohomology algebra of $F(\mathbb{C}, n)$ remained the only instance where the *integer* cohomology algebra of an ordered configuration space was fully described.

Recently, Raoul Bott asked about the integer cohomology algebra of the ordered configuration space of the 2-sphere. We are able to answer his question by describing a product decomposition for $F(S^2, n)$:

$$F(S^2, n) \;\cong\; \mathrm{PSL}(2, \mathbb{C}) \times M_{0,n},$$

where $M_{0,n}$, the moduli space of n-punctured complex projective lines, is homotopy equivalent to the complement of an affine complex hyperplane arrangement. We deduce that $H^*(F(S^2, n), \mathbb{Z})$ has (only) 2-torsion that can be traced back to $H^2(\mathrm{PSL}(2, \mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}_2$ (Section 2).

For spheres of higher dimension we use spectral sequences to obtain an analogous decomposition on the level of cohomology algebras:

$$\begin{aligned} H^*(F(S^k, n), \mathbb{Z}) &\cong (\mathbb{Z} \oplus \mathbb{Z}) \otimes H^*(\mathcal{M}(\mathcal{A}_{n-2}^{(k)}), \mathbb{Z}) && \text{for odd } k\,, \\ H^*(F(S^k, n), \mathbb{Z}) &\cong (\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}) \otimes H^*(\mathcal{M}(\mathcal{A}_{\Pi_3}), \mathbb{Z}) && \text{for even } k\,, \end{aligned}$$

where $\mathcal{M}(\mathcal{A}_{n-2}^{(k)})$ is the complement of a certain arrangement of real linear subspaces $\mathcal{A}_{n-2}^{(k)}$ and $\mathcal{M}(\mathcal{A}_{\Pi_3})$ is the complement of an arrangement of affine subspaces that is naturally related to the linear arrangement $\mathcal{A}_{n-2}^{(k)}$. For both arrangement complements the integer cohomology algebra is torsion-free and we have explicit descriptions in terms of generators, relations and linear bases. *In the following all (co)homology is taken with $\mathbb{Z}$-coefficients.*

The key for our approach is a family of locally trivial fiber maps on configuration spaces that appears already in the work by Fadell & Neuwirth [FaN] and Fadell [Fa]. The maps are given by "projection to the last r points" of a configuration. For configuration spaces of spheres $F(S^k, n)$ and $1 \leq r < n$ the projection Π_r reads as follows:

$$\begin{array}{rrcl} \Pi_r = \Pi_r(S^k, n): \qquad & F(S^k, n) & \longrightarrow & F(S^k, r) \\ & (x_1, \ldots, x_n) & \longmapsto & (x_{n-r+1}, \ldots, x_n)\ . \end{array}$$

We derive the integer cohomology algebra of $F(S^k, n)$ for $k > 2$ by a complete discussion of the Leray-Serre spectral sequence associated to the fiber map $\Pi_1(S^k, n)$. Our success with this rather classical approach depends on the fact that the fibers of $\Pi_1(S^k, n)$ are complements of linear subspace arrangements. Their cohomology algebras are well-studied objects both from topological and combinatorial viewpoints [GM, BZ, Bj, DP]. The fibers of $\Pi_1(S^k, n)$ are in fact the complements of codimension k versions of the classical braid arrangements, and thus they are particularly prominent examples of arrangement complements. This paves the way for a complete discussion of the associated spectral sequence (Section 3).

A distinction between the configuration spaces of spheres of odd and even dimension emerges from the only possibly non-trivial differential of the spectral sequence. We present two methods to compute this differential (Section 4).

(1) It can be derived from one particular cohomology group of $F(S^k, n)$. To obtain the latter we use an independent, rather elementary approach to the cohomology of configuration spaces, which may be of interest on its own right.

(2) We show that the differential can be interpreted as a map that is induced by "multiplication with the Euler class of S^k." It is well-known that the Euler class depends on the parity of k.

To get the final tableau of the spectral sequence, and to derive the integer cohomology algebra of the configuration space $F(S^k, n)$, we use combinatorially constructed $\mathbb{Z}$-linear bases for the cohomology of the fiber (Section 5).

In the last section of this paper we consider the bundle structures on $F(S^k, n)$ given by the fiber maps $\Pi_r(S^k, n)$, $1 < r < n$. We show that the associated spectral sequences collapse in their second terms unless k is even and r equals 1 or 2. For some parameters we can decide the triviality of the bundle structure, which in general is a difficult question.

For configuration spaces of closed manifolds other than spheres, in principle one can attempt to follow the approach taken in this paper. However, with the cohomology of the manifold (i.e., of the base space of the considered fiber map) getting more complicated, the corresponding spectral sequence will be less sparse, and thus more non-trivial differentials will have to be considered. Even more importantly, if the manifold is not simply connected, then it is not straightforward, and not true in general, that the system of local coefficients on the manifold induced by the fiber map is simple. Already the entries of the second sequence tableau thus will be much harder to compute.

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2 Configuration spaces of the 2-sphere

We first comment on some special cases for small values of n and on the configuration space of the 1-sphere. For $n = 1$, we see from the definition that $F(X,1) = X$ for all spaces X. For $n = 2$, we consider the projection Π_1, sending a configuration in $F(S^k,2)$ to its second point. We obtain a fiber bundle with contractible fiber $\Pi_1^{-1}(x_2) = F(S^k \setminus \{x_2\},1) \cong \mathbb{R}^k$, hence $F(S^k,2) \simeq S^k$. In fact, $F(S^k,2)$ is equivalent to the tangent bundle over S^k.
For the configuration space of the 1-sphere, $F(S^1,n)$, we state an explicit trivialization of the fiber bundle given by Π_1, the projection to the last point of a configuration. Using the group structure on S^1 we define a homeomorphism which shows that $\Pi_1(S^1,n)$ is a trivial fiber map:

$$\begin{array}{rrcl} \varphi_1: & F(S^1 \setminus \{e\}, n-1) \times S^1 & \longrightarrow & F(S^1,n) \\ & ((x_1,\dots,x_{n-1}),\, y) & \longmapsto & (yx_1,\dots,yx_{n-1},y)\,. \end{array}$$

For $r > 1$, the fiber of $\Pi_r(S^1,n)$ is homeomorphic to the space of configurations of $n-r$ points on r disjoint copies of the unit interval. We obtain a homeomorphism

$$\varphi_r: \quad F(\biguplus_r (0,1), n-r) \times F(S^1,r) \quad\longrightarrow\quad F(S^1,n)$$

that trivializes the bundle by "inserting" the points $x_1,\dots,x_{n-r}$ from $\biguplus_r (0,1)$ into the r open segments in which the points of the configuration $(y_1,\dots,y_r)$ in $F(S^1,r)$ separate S^1.

Compared to configuration spaces of higher dimensional spheres we gain the main structural advantage for the 2-dimensional case from the fact that the 2-sphere S^2 is homeomorphic to the complex projective line $\mathbb{C}P^1$. We will freely switch between the resulting two viewpoints on the configuration space in question.
The group of projective automorphisms $\mathrm{PSL}(2,\mathbb{C})$ of $\mathbb{C}P^1$ acts freely on the configuration space $F(\mathbb{C}P^1,n)$ by coordinatewise action, thus exhibiting $F(\mathbb{C}P^1,n)$ as the total space of a principal $\mathrm{PSL}(2,\mathbb{C})$-bundle for $n \geq 3$ [Ge]. We identify the base space — the space of n-tuples of distinct points on the complex projective line modulo projective automorphisms — as the moduli space $M_{0,n}$ of n-punctured complex projective lines. Compactifications of $M_{0,n}$ and their cohomology algebras are the focus of recent research; for a brief account and further references see [FM, p.189].

Theorem 2.1 *The configuration space $F(\mathbb{C}P^1, n)$ of the complex projective line is the total space of a trivial* $\mathrm{PSL}(2,\mathbb{C})$*-bundle over $M_{0,n}$ for $n \geq 3$; hence there is a homeomorphism*

$$F(\mathbb{C}P^1, n) \;\cong\; \mathrm{PSL}(2,\mathbb{C}) \;\times\; M_{0,n}\,.$$

Proof. The automorphism group $\mathrm{PSL}(2,\mathbb{C})$ acts sharply 3-transitive on $\mathbb{C}P^1$. In particular, we obtain a homeomorphism between the configuration space of three distinct points on $\mathbb{C}P^1$ and the automorphism group $\mathrm{PSL}(2,\mathbb{C})$:

$$\phi \;:\; F(\mathbb{C}P^1, 3) \;\longrightarrow\; \mathrm{PSL}(2,\mathbb{C})\,.$$

Here $(x_1, x_2, x_3) \in F(\mathbb{C}P^1, 3)$ is mapped to the *unique* automorphism that transforms x_1 to $\binom{1}{0}$, x_2 to $\binom{0}{1}$, and x_3 to $\binom{1}{1}$, i.e., to the "standard projective basis" of $\mathbb{C}P^1$.
Given a configuration $x = (x_1, \ldots, x_n)$ of n distinct points on $\mathbb{C}P^1$, the group element $\phi(x_1, x_2, x_3)$ transforms x to a configuration on $\mathbb{C}P^1$ that has the standard projective basis in its first three entries. We describe the resulting configuration by the columns of a $(2 \times n)$-matrix:

$$\phi(x_1, x_2, x_3) \circ x \;=\; \begin{pmatrix} 1 & 0 & 1 & z_3 & \ldots & z_{n-1} \\ 0 & 1 & 1 & 1 & \ldots & 1 \end{pmatrix},$$

where $z_i \in \mathbb{C}\backslash\{0,1\}$ for $3 \leq i \leq n-1$, $z_i \neq z_j$ for $3 \leq i < j \leq n-1$, and the columns are understood as vectors in $\mathbb{C}^2\backslash\{0\}$ that represent elements in $\mathbb{C}P^1$. Lifting an element $\bar{x} \in M_{0,n}$ to its "normal form" $\phi(x_1, x_2, x_3) \circ x$ in the total space $F(\mathbb{C}P^1, n)$ defines a section for the $\mathrm{PSL}(2,\mathbb{C})$-bundle. Hence, the principal bundle is trivial [St, Part I, Thm. 8.3]. The resulting product decomposition on $F(\mathbb{C}P^1, n)$ can be described explicitly by the homeomorphism

$$\begin{array}{rccc} \Phi \;: & F(\mathbb{C}P^1, n) & \longrightarrow & \mathrm{PSL}(2,\mathbb{C}) \;\times\; M_{0,n} \\ & (x_1, \ldots, x_n) & \longmapsto & (\,\phi(x_1, x_2, x_3)\,,\, \bar{x}\,)\,. \end{array} \qquad \square$$

Remark 2.2 An analogous argument is not possible for S^4, since there are no *sharply* 3-transitive group actions in the case of a non-commutative field such as $\mathbb{H}$. The structural reason for this can be traced back to a theorem by von Staudt, see [P, Kap. 5.1.4].

In view of a description of the integer cohomology algebra of $F(\mathbb{C}P^1, n)$ we use the intimate relation of the base space $M_{0,n}$ to a complex hyperplane arrangement — the *complex braid arrangement* $\mathcal{A}_{n-2}^{\mathbb{C}}$ *of rank* $n-2$ in $\mathbb{C}^{n-1}$ given by the hyperplanes

$$z_j - z_i \;=\; 0 \qquad \text{for} \;\; 1 \leq i < j \leq n-1\,.$$

This arrangement is a key example in the theory of hyperplane arrangements and initiated much of its development [Ar, OT]. Its complement, $\mathcal{M}(\mathcal{A}_{n-2}^{\mathbb{C}}) := \mathbb{C}^{n-1}\backslash \bigcup \mathcal{A}_{n-2}^{\mathbb{C}}$, coincides with $F(\mathbb{C}, n-1)$, the configuration space of the complex plane.

The base space $M_{0,n}$ is homotopy equivalent to the complement of the affine arrangement ${}^{\mathit{aff}}\!\mathcal{A}^{\mathbb{C}}_{n-2}$, which is obtained from $\mathcal{A}^{\mathbb{C}}_{n-2}$ by restriction to the affine hyperplane $\{z_2 - z_1 = 1\} \cong \mathbb{C}^{n-2}$. A complete description of the integer cohomology algebra of the complement $\mathcal{M}({}^{\mathit{aff}}\!\mathcal{A}^{\mathbb{C}}_{n-2}) := \mathbb{C}^{n-2} \setminus \bigcup {}^{\mathit{aff}}\!\mathcal{A}^{\mathbb{C}}_{n-2}$ is provided by general theory on the topology of complex hyperplane arrangements [OS, BZ, OT]. The description depends only on combinatorial data of the arrangement, i.e., on the semi-lattice of intersections $\mathcal{L}({}^{\mathit{aff}}\!\mathcal{A}^{\mathbb{C}}_{n-2})$ which is customarily ordered by reverse inclusion.

Proposition 2.3 *The base space $M_{0,n}$ is homotopy equivalent to the complement of the affine complex braid arrangement of rank $n-2$, since*

$$M_{0,n} \times \mathbb{C} \cong \mathcal{M}({}^{\mathit{aff}}\!\mathcal{A}^{\mathbb{C}}_{n-2}).$$

Its integer cohomology algebra is torsion-free. It is generated by one-dimensional classes $e_{i,j}$ for $1 \le i < j \le n-1$, $(i,j) \neq (1,2)$, and has a presentation as a quotient of the exterior algebra on these generators:

$$H^*(\mathcal{M}({}^{\mathit{aff}}\!\mathcal{A}^{\mathbb{C}}_{n-2})) \cong \Lambda^* \mathbb{Z}^{\binom{n-1}{2}-1} / I,$$

where I is the ideal generated by elements of the form

$$\begin{array}{rl} e_{i,l} \wedge e_{j,l} - e_{i,j} \wedge e_{j,l} + e_{i,j} \wedge e_{i,l} & \text{for} \quad 1 \le i < j < l \le n-1,\ (i,j) \neq (1,2), \\ e_{1,i} \wedge e_{2,i} & \text{for} \quad 2 < i \le n-1. \end{array}$$

Proof. We consider the homeomorphic image of $M_{0,n}$ under the section defined in the proof of Proposition 2.1:

$$\begin{aligned} M_{0,n} &\cong \left\{ \left(\begin{array}{cccccc} 1 & 0 & 1 & z_3 & \dots & z_{n-1} \\ 0 & 1 & 1 & 1 & \dots & 1 \end{array} \right) \,\middle|\, z_i \in \mathbb{C}\setminus\{0,1\},\, z_i \neq z_j \text{ for } i \neq j \right\} \\ &\cong \{ (z_1, \dots, z_{n-1}) \mid z_i \in \mathbb{C},\, z_i \neq z_j \text{ for } i \neq j,\, z_1 = 0,\, z_2 - z_1 = 1 \}. \end{aligned}$$

From this description we see that $M_{0,n}$ is homeomorphic to the complement of the affine braid arrangement ${}^{\mathit{aff}}\!\mathcal{A}^{\mathbb{C}}_{n-2}$ intersected with the hyperplane $\{z_1 = 0\}$. This intersection operation is equivalent to a projection parallel to the intersection of all the hyperplanes in $\mathcal{A}^{\mathbb{C}}_{n-2}$, $\bigcap \mathcal{A}^{\mathbb{C}}_{n-2} = \{z_1 = \ldots = z_{n-1}\}$. The fibers of this projection map are contractible: they are translates of $\bigcap \mathcal{A}^{\mathbb{C}}_{n-2}$. Hence the projection does not alter the homotopy type, and we conclude that $M_{0,n}$ is homotopy equivalent to $\mathcal{M}({}^{\mathit{aff}}\!\mathcal{A}^{\mathbb{C}}_{n-2})$.
The presentation of the integer cohomology algebra follows from general results on the topology of the complements of complex hyperplane arrangements (compare [OT]). □

We have seen that the fiber $\mathrm{PSL}(2,\mathbb{C})$ is homeomorphic to $F(\mathbb{C}P^1, 3)$, resp. $F(S^2, 3)$. By a result of Fadell [Fa, Thm. 2.4] there is a fiber homotopy equivalence between $F(S^k,3)$ and $V_{k+1,2}$, the Stiefel manifold of orthogonal 2-frames in $\mathbb{R}^{k+1}$. The cohomology of the latter is well-known, see [Bd, Ch. IV, Exp. 13.5].

Combining the product structure on $F(\mathbb{C}P^1,n)$ obtained in Theorem 2.1 with the information on the cohomology algebras of base space and fiber we conclude:

Theorem 2.4 *The cohomology algebra of $F(S^2,n)$ with integer coefficients is given by*

$$\begin{array}{rcl} H^*(F(S^2,n)) & \cong & H^*(F(S^2,3)) \otimes H^*(\mathcal{M}({}^{\mathit{aff}}\!\mathcal{A}^{\mathbb{C}}_{n-2})) \\ & \cong & \big(\mathbb{Z}(0)\oplus\mathbb{Z}_2(2)\oplus\mathbb{Z}(3)\big) \otimes \Lambda^* \bigoplus_{\binom{n-1}{2}-1} \mathbb{Z}(1) \,/\, I\,, \end{array}$$

where $G(i)$ denotes a direct summand G in dimension i, and I is the ideal of relations described in Proposition 2.3.

3 A spectral sequence for $H^*(F(S^k,n))$

Our approach for $k>2$ uses the Leray-Serre spectral sequence associated with the projection Π_1:

$$\begin{array}{rrcl} \Pi_1: & F(S^k,n) & \longrightarrow & S^k \\ & (x_1,\dots,x_n) & \longmapsto & x_n\,. \end{array}$$

For the construction and special features of Leray-Serre spectral sequences we refer to Borel [Bo2, Sect. 2]. Since the base space of the considered fiber bundle is a sphere we could equally work with the Wang sequence [Wh, Ch. VII, Sect. 3], a long exact sequence connecting the cohomology of the total space and of the fiber. However, the derivation of the multiplicative structure of the cohomology algebra gets more transparent with spectral sequence tableaux. Moreover, this approach extends to projections Π_r for $r>1$ (see Section 6).

We meet especially favorable conditions in the second tableau of the Leray-Serre spectral sequence associated to the fiber map $\Pi_1(S^k,n)$: The base space S^k is simply connected for $k\geq 2$, hence the system of local coefficients on S^k induced by Π_1 for $k\geq 2$ is simple. As the fiber over $x_n\in S^k$ we obtain:

$$\begin{array}{rcl} \Pi_1^{-1}(x_n) & = & \{(x_1,\dots,x_{n-1})\in(S^k)^{n-1}\,|\,x_i\neq x_j \text{ for } i\neq j, \\ & & \qquad\qquad x_i\neq x_n \text{ for } i=1,\dots,n{-}1\} \\ & \cong & \{(x_1,\dots,x_{n-1})\in(\mathbb{R}^k)^{n-1}\,|\,x_i\neq x_j \text{ for } i\neq j\}\,. \end{array}$$

This is the complement of the *real k-braid arrangement $\mathcal{A}^{(k)}_{n-2}$ of rank $n{-}2$* which is formed by linear subspaces $U_{i,j}$ in $(\mathbb{R}^k)^{n-1}$, $1\leq i<j\leq n{-}1$,

$$U_{i,j} \;=\; \{(x_1,\dots,x_{n-1})\in(\mathbb{R}^k)^{n-1}\,|\,x_{i_1}=x_{j_1},\dots,x_{i_k}=x_{j_k}\}\,.$$

This arrangement, a direct generalization of the real and complex braid arrangements, is a *k-arrangement* in the sense of Goresky & MacPherson [GM, Part III, p. 239]: the subspaces have codimension k, and the codimensions of their intersections are multiples of k. Such arrangements have combinatorial properties analogous to those of complex hyperplane arrangements, which is reflected by strong similarities in their topological properties: The cohomology algebras of real k-arrangements are torsion-free [GM, Part III, Thm. B]; they are generated in dimension $k-1$ by cohomology classes that naturally correspond to the subspaces of the arrangement [BZ, Sect. 9].

The complement of the real k-braid arrangement $\mathcal{A}_{n-2}^{(k)}$ is an ordered configuration space: the space $F(\mathbb{R}^k, n-1)$ of configurations of $n-1$ pairwise distinct points in $\mathbb{R}^k$. The following thus complements work by Cohen [C1, C2], who discussed the cohomology of $F(\mathbb{R}^k, n-1)$ in connection with homology operations for iterated loop spaces.

Proposition 3.1 *The integer cohomology algebra of $\mathcal{M}(\mathcal{A}_{n-2}^{(k)})$ is generated by $(k-1)$-dimensional cohomology classes $c_{i,j}$, $1 \leq i < j \leq n-1$. It has a presentation as a quotient of the exterior algebra on these generators:*

$$H^*(\mathcal{M}(\mathcal{A}_{n-2}^{(k)})) \cong \Lambda^* \mathbb{Z}^{\binom{n-1}{2}} / I ,$$

where I is the ideal generated by the elements

$$(c_{i,l} \wedge c_{j,l}) + (-1)^{k+1}(c_{i,j} \wedge c_{j,l}) + (c_{i,j} \wedge c_{i,l}) \qquad \text{for } 1 \leq i < j < l \leq n-1 .$$

Remark 3.2 The generating cohomology classes $c_{i,j}$, $1 \leq i < j \leq n-1$, are defined by restricting cohomology generators $\widehat{c}_{i,j}$ for the subspace complements $\mathcal{M}(\{U_{i,j}\}) \simeq S^{k-1}$ to the complement of the arrangement. A canonical choice of the generators $\widehat{c}_{i,j}$ results from fixing the natural "frame of hyperplanes" in the sense of [BZ, Sect. 9].

Proof. Björner & Ziegler [BZ, Sect. 9] derived a presentation for the cohomology algebras of real k-arrangements up to the signs in the relations. For the real k-braid arrangement their presentation specializes up to signs to the one stated above.

Consider the relation for a triple $1 \leq i < j < l \leq n-1$:

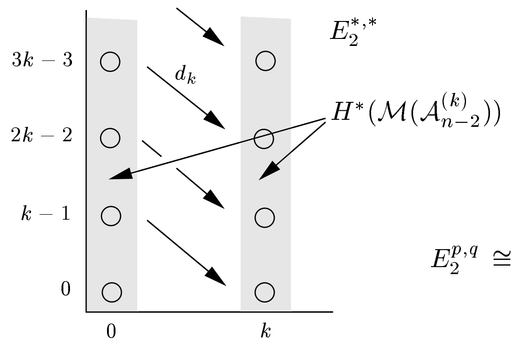
$$\varepsilon_1(c_{i,l} \wedge c_{j,l}) + \varepsilon_2(c_{i,j} \wedge c_{j,l}) + \varepsilon_3(c_{i,j} \wedge c_{i,l}) = 0 , \ \ \varepsilon_r \in \{\pm 1\} \ \text{ for } r = 1,2,3 .$$

Transpositions of (i,j) and (i,l) and of (i,l) and (j,l) in the linear (lexicographic) order of the subspaces in $\mathcal{A}_{n-2}^{(k)}$ lead to similar relations among the cohomology classes $c_{i,l} \wedge c_{j,l}$, $c_{i,j} \wedge c_{j,l}$, and $c_{i,j} \wedge c_{i,l}$:

$$\begin{aligned} \varepsilon_1(c_{i,j} \wedge c_{j,l}) + \varepsilon_2(c_{i,l} \wedge c_{j,l}) + \varepsilon_3(c_{i,l} \wedge c_{i,j}) &= 0 \\ \varepsilon_1(c_{j,l} \wedge c_{i,l}) + \varepsilon_2(c_{i,j} \wedge c_{i,l}) + \varepsilon_3(c_{i,j} \wedge c_{j,l}) &= 0 . \end{aligned}$$

Anti-commutativity of the exterior product yields the signs in the relations. □

We obtain the following tensor product decomposition on the E^2-tableau of the Leray-Serre spectral sequence associated with the fiber map $\Pi_1(S^k, n)$:



$$E_2^{p,q} \;\cong\; H^p(S^k) \otimes H^q(\mathcal{M}(\mathcal{A}_{n-2}^{(k)}))\,, \qquad p, q \geq 0\,.$$

The location of non-zero entries shows that there is only one possibly non-trivial differential on stage k of the sequence.

4 The k-th differential

The tableaux of a cohomological spectral sequence are bigraded algebras. The differentials respect their multiplicative structure. In particular, the differentials are determined by their action on multiplicative generators of the sequence tableaux. Thus, it suffices in our case to describe d_k on the multiplicative generators $c_{i,j}$, $1 \leq i < j \leq n-1$, of $E_k^{0,*} \cong H^*(\mathcal{M}(\mathcal{A}_{n-2}^{(k)}))$ in dimension $k-1$.

Actually, we can restrict our attention even further to the action of d_k on one single generator, say on $c_{1,2}$: The permutation of the first $n-1$ points of a configuration in $F(S^k, n)$ by $\mathfrak{S}_{n-1}$ gives a group action on the considered fiber bundle and hence induces a $\mathfrak{S}_{n-1}$-action on the spectral sequence. The group $\mathfrak{S}_{n-1}$ acts transitively on the generators $c_{i,j}$ of $E_k^{0,k-1}$, whereas it keeps $E_k^{k,0}$ fixed. We conclude that

$$d_k(c_{i,j}) \;=\; d_k(c_{1,2}) \qquad \text{for } 1 \leq i < j \leq n-1\,.$$

In the following we provide two independent ways to evaluate d_k.

4.1 ... via a homology group of the discriminant.

Here the key observation is that knowing $H^k(F(S^k, n))$ is sufficient to determine d_k. To obtain this specific group, we use a "Vassiliev type" argument that allows one to compute, in favorable situations, some cohomology groups of configuration spaces. Using a smooth compactification, in our case given by $F(S^k, n) \subseteq (S^k)^n$, we set

$$F(S^k, n) \;=\; (S^k)^n \setminus \Gamma_n \;=\; (S^k)^n \setminus \bigcup_{1 \leq i < j \leq n} (\Gamma_n)_{i,j}\,,$$

where

$$(\Gamma_n)_{i,j} = \{(x_1,\ldots,x_n) \in (S^k)^n \,|\, x_i = x_j\} \quad \text{for } 1 \le i < j \le n\,.$$

The idea is to use duality theorems in $(S^k)^n$ for transferring homology information about the *discriminant* Γ_n to the cohomology of $F(S^k,n)$. For this, we proceed in three steps.

STEP 1. Determine $H_*(\Gamma_n)$ in dimensions $(n-1)k$ and $(n-1)k-1$.

The spaces $(\Gamma_n)_{i,j}$ are homeomorphic to $(S^k)^{n-1}$; they intersect in spaces homeomorphic to $(S^k)^{n-2}$, hence in dimension $k(n-2)$. By a Mayer-Vietoris argument we obtain the top two homology groups of the discriminant:

$$\begin{aligned} H_{(n-1)k}(\Gamma_n) &\cong \bigoplus_{1\le i<j\le n} H_{(n-1)k}((\Gamma_n)_{i,j}) \;\cong\; \mathbb{Z}^{\binom{n}{2}} \\ H_{(n-1)k-1}(\Gamma_n) &= 0\,. \end{aligned}$$

STEP 2. Determine the relative homology $H_*((S^k)^n,\Gamma_n)$ in dimension $(n-1)k$.

The relevant part of the long exact sequence in homology for the pair $((S^k)^n,\Gamma_n)$ is the following:

$$\to H_{(n-1)k}(\Gamma_n) \xrightarrow{i_*} H_{(n-1)k}((S^k)^n) \to H_{(n-1)k}((S^k)^n,\Gamma_n) \to H_{(n-1)k-1}(\Gamma_n) \to$$

We had computed that the last group is zero, and thus

$$H_{(n-1)k}((S^k)^n,\Gamma_n) \cong \operatorname{coker} i_*\,,$$

where i_* is induced by the inclusion $i : \Gamma_n \hookrightarrow (S^k)^n$. We intend to write i_* as a $(n \times \binom{n}{2})$-matrix over $\mathbb{Z}$ and to read the cokernel from its Smith normal form [Mu, § 11]. For this we choose $\mathbb{Z}$-bases for the homology groups that are involved, and determine i_* in terms of these bases.

According to the Künneth Theorem, $H_{(n-1)k}((S^k)^n)$ has a basis that consists of tensor products of k-dimensional classes ω_j, $j = 1,\ldots,n$, of the form

$$\nu_i \;=\; \omega_1 \otimes \ldots \otimes \widehat{\omega}_i \otimes \ldots \otimes \omega_n\,, \quad i = 1,\ldots,n\,,$$

where ω_j is an orientation class for the j-th factor in $(S^k)^n$, and $\widehat{\omega}_i$ denotes that we omit the i-th class.

Generating homology classes of Γ_n in dimension $(n-1)k$ are given by the $\binom{n}{2}$ generating homology classes for the spaces $(\Gamma_n)_{i,j}$, $1 \le i < j \le n$. These spaces are products of k-spheres,

$$(\Gamma_n)_{i,j} \;\cong\; S_{i,j} \times S_1 \times \ldots \times \widehat{S}_i \times \ldots \times \widehat{S}_j \times \ldots \times S_n\,,$$

with S_l denoting the l-th k-sphere appearing as a factor in $(S^k)^n$, whereas $S_{i,j}$ denotes the k-sphere diagonally embedded in the i-th and j-th k-sphere. A generating homology class for $(\Gamma_n)_{i,j}$ in dimension $(n-1)k$ can be described as

$$\nu_{ij} \;=\; \omega_{ij} \otimes \omega_1 \otimes \ldots \otimes \widehat{\omega}_i \otimes \ldots \otimes \widehat{\omega}_j \otimes \ldots \otimes \omega_n\,, \quad 1 \le i < j \le n\,,$$

where ω_{ij} is a homology generator for $S_{i,j}$ in dimension k.

To understand how i_* maps such generators ν_{ij} we use the following lemma. It tells how to describe the homology generator of the diagonal in $S_i \times S_j$ in terms of homology classes of the product.

Lemma 4.1 *Let ω denote a generating homology class in dimension k for the k-sphere. Under the diagonal map $\Delta : S^k \to S^k \times S^k$, $\Delta(x) = (x,x)$ for $x \in S^k$, the homology class ω is mapped to*

$$\Delta_*(w) = \omega \otimes 1 + 1 \otimes \omega .$$

Proof. By the Künneth Theorem the two summands form a basis of $H_k(S^k \times S^k)$, so $\Delta_*(\omega)$ is a $\mathbb{Z}$-linear combination of those. Moreover, the diagonal map combined with one of the projections pr_i to the respective factor is the identity map on S^k. Hence the result follows from $(\mathrm{pr}_i)_* \circ \Delta_*(\omega) = \omega$ for $i = 1, 2$. □

We conclude that

$$i_*(\nu_{ij}) = \big((\omega_i \otimes 1) + (1 \otimes \omega_j)\big) \otimes \bigotimes_{\substack{l=1 \\ l \neq i,j}}^{n} \omega_l , \quad 1 \leq i < j \leq n .$$

To write this in terms of the generators ν_i for $H_{(n-1)k}((S^k)^n)$ we have to permute the factors of the underlying product space to the order used above in the definition of the classes ν_i. The tensor product of homology classes is anti-commutative [FFG, Ch. II, §16]; i.e., under the transposition map $\tau : X \times X \longrightarrow X \times X$, $(x_1, x_2) \mapsto (x_2, x_1)$, a product of homology classes $\alpha \otimes \beta$, $\alpha, \beta \in H_*(X)$, is mapped to

$$\tau_*(\alpha \otimes \beta) = (-1)^{\deg(\alpha)\deg(\beta)} \beta \otimes \alpha.$$

This is the point where the distinction between odd and even dimensions comes up:

$$i_*(\nu_{ij}) = \begin{cases} (-1)^{i-1}\nu_j + (-1)^{j-2}\nu_i & \text{for odd } k , \\ \nu_j + \nu_i & \text{for even } k \end{cases} \qquad (1 \leq i < j \leq n).$$

Writing i_* as a $(n \times \binom{n}{2})$-matrix $M(n)$ we obtain the (unsigned) incidence matrix of 2-element subsets of an n-set for even k, whereas for odd k a certain sign pattern occurs on the matrix entries. For example,

$$M(3) = \begin{array}{c} \\ {\scriptstyle 1} \\ {\scriptstyle 2} \\ {\scriptstyle 3} \end{array} \begin{array}{c} \begin{array}{ccc} {\scriptstyle 12} & {\scriptstyle 13} & {\scriptstyle 23} \end{array} \\ \begin{pmatrix} 1 & (-1)^k & 0 \\ 1 & 0 & (-1)^k \\ 0 & 1 & (-1)^k \end{pmatrix} \end{array} ,$$

$$M(4) = \begin{array}{c} \\ {\scriptstyle 1} \\ {\scriptstyle 2} \\ {\scriptstyle 3} \\ {\scriptstyle 4} \end{array} \begin{array}{c} \begin{array}{cccccc} {\scriptstyle 12} & {\scriptstyle 13} & {\scriptstyle 14} & {\scriptstyle 23} & {\scriptstyle 24} & {\scriptstyle 34} \end{array} \\ \begin{pmatrix} 1 & (-1)^k & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & (-1)^k & 1 & 0 \\ 0 & 1 & 0 & (-1)^k & 0 & 1 \\ 0 & 0 & 1 & 0 & (-1)^k & 1 \end{pmatrix} \end{array} .$$

We now derive the Smith normal forms of the matrices $M(n)$ by describing elementary row and column operations. Ordering the columns of $M(n)$ – corresponding to the 2-element subsets of $\{1,\ldots,n\}$ – lexicographically, we see that

$$M(n) \;=\; \left(\begin{array}{cccc|ccc} 1 & \cdots\cdots & & 1 & 0 & \cdots\cdots\cdots\cdots & 0 \\ \hline 1 & & & & & & \\ & \ddots & & & & & \\ & & \ddots & & & M(n{-}1) & \\ & & & 1 & & & \end{array}\right) \qquad \text{for even } k, \text{ and}$$

$$M(n) \;=\; \left(\begin{array}{cccc|ccc} 1 & -1 & \cdots & (-1)^n & 0 & \cdots\cdots\cdots\cdots & 0 \\ \hline 1 & & & & & & \\ & \ddots & & & & & \\ & & \ddots & & & -M(n{-}1) & \\ & & & 1 & & & \end{array}\right) \qquad \text{for odd } k.$$

For even k, we subtract the i-th row from the first row for $i = 2,\ldots,n$, and thus create 0-entries in the left part of the first row and entries -2 on top of the submatrix $M(n{-}1)$. Note that the column sum in $M(n{-}1)$ is 2.
Adding multiples of the first $n{-}1$ columns to the rest of the matrix, we transform $M(n{-}1)$ to 0. The remaining entries in the first row can be reduced to one single entry 2, and after switching rows and columns we obtain the following Smith normal form:

$$\mathrm{SNF}(\,M(n)\,) \;=\; \left(\begin{array}{cccc|c} 1 & & & & \\ & \ddots & & & \\ & & 1 & & 0 \\ & & & 2 & \end{array}\right) \qquad \text{for even } k.$$

For odd k, we add the t-th row multiplied with $(-1)^{t-1}$ to the first row for $i = 2,\ldots,n$. This creates 0-entries in the first row. This is obvious for the first $n{-}1$ columns. For an entry on top of a column of the submatrix $-M(n{-}1)$ which contains entries in its i-th and j-th rows, we obtain

$$(-1)^i\cdot(-(-1)^{j-2}) \;+\; (-1)^j\cdot(-(-1)^{i-1}) \;=\; 0\,.$$

As before, we transform the submatrix $-M(n{-}1)$ to 0 by adding multiples of the first $n{-}1$ columns. Thus, after switching rows, we obtain:

$$\mathrm{SNF}(\,M(n)\,) \;=\; \left(\begin{array}{cccc|c} 1 & & & & \\ & \ddots & & & \\ & & 1 & & 0 \\ & & & 0 & \end{array}\right) \qquad \text{for odd } k.$$

We read off the cokernel of i_* as

$$H_{(n-1)k}\,((S^k)^n,\Gamma_n) \;\cong\; \begin{cases} \mathbb{Z} & \text{for odd } k\,, \\ \mathbb{Z}_2 & \text{for even } k\,. \end{cases}$$

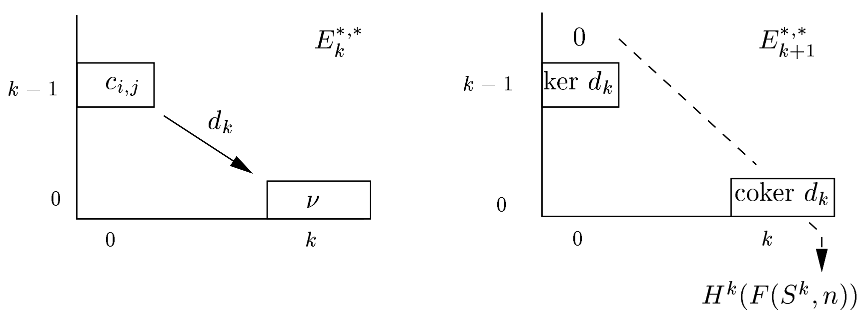
Step 3. Apply Poincaré-Lefschetz duality between relative homology of the pair $((S^k)^n, \Gamma_n)$ and cohomology of $F(S^k, n)$.

Proposition 4.2 *The k-th cohomology group of $F(S^k, n)$, $k > 2$, $n > 2$, is given by*

$$H^k(F(S^k, n)) \cong \begin{cases} \mathbb{Z} & \textit{for odd } k\,, \\ \mathbb{Z}_2 & \textit{for even } k\,. \end{cases}$$

Remark 4.3 In principle, the discriminant approach can be used to determine the cohomology of $F(S^k, n)$ as a graded group. However, to compute $H_*(\Gamma_n)$ is difficult and requires extra tools (interpretation of Γ_n as a homotopy limit of a diagram of spaces, study of a spectral sequence converging to the homology of a homotopy limit [ZŽ, Sect. 3(e)]). Also, the study of the pair sequence gets considerably more involved. Moreover, because of the use of Poincaré-Lefschetz duality the multiplicative structure of $H^*(F(S^k, n))$ seems out of reach for this approach.

The partial result of Proposition 4.2 allows us to determine the differential in the spectral sequence associated to $\Pi_1(S^k, n)$. Taking cohomology of $E_k^{*,*}$ with respect to the differential d_k leads to the *final* sequence tableau $E_{k+1}^{*,*}$:



Since there is only one non-zero entry on the k-th diagonal for $k > 2$, $H^k(F(S^k, n))$ can be read from $E_{k+1}^{*,*}$:

$$H^k(F(S^k, n)) \cong \operatorname{coker} d_k\,.$$

Our result on $H^k(F(S^k, n))$ in Proposition 4.2 implies that

$$d_k(c_{1,2}) = d_k(c_{i,j}) = \begin{cases} 0 & \text{for odd } k \\ 2\nu & \text{for even } k\,, \end{cases}$$

where ν is a generator of $H^k(S^k)$.

4.2 ... VIA AN INTERPRETATION IN TERMS OF THE EULER CLASS.

Our second approach to the differential d_k stays within the setting of fiber bundles. We study an inclusion of fiber bundles and transfer information on the differentials via the induced homomorphism of spectral sequences. We will find that the differential is determined by the Euler class of the base space S^k, which depends on the parity of k.

Consider, for $n \geq 3$, the following space of point configurations on S^k, $k > 2$:

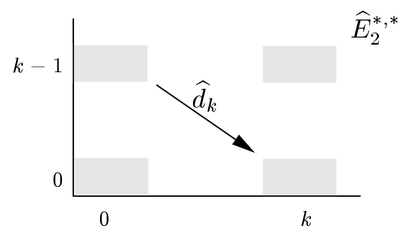
$$\widehat{F} := \{(x_1, \ldots, x_n) \in (S^k)^n \mid x_1 \neq x_2,\, x_j \neq x_n \ \text{ for } \ j = 1, \ldots, n-1\}\,.$$

Projection of a configuration to its last point, $\widehat{\Pi} : \widehat{F} \to S^k$, makes it the total space of a fiber bundle with spherical fiber: the complement of the codimension k subspace $U_{1,2}$ in $(\mathbb{R}^k)^{n-1}$,

$$\begin{aligned} \widehat{\Pi}^{-1}(x_n) &= \{(x_1, \ldots, x_{n-1}) \in (S^k)^{n-1} \mid x_1 \neq x_2,\, x_j \neq x_n \text{ for } 1 \leq j \leq n-1\} \\ &\cong \{(x_1, \ldots, x_{n-1}) \in (\mathbb{R}^k)^{n-1} \mid x_1 \neq x_2\} \\ &= \mathcal{M}(\{U_{1,2}\})\,. \end{aligned}$$

The spectral sequence $\widehat{E}_*$ associated to $\widehat{\Pi}$ has an $\widehat{E}_2$-tableau of the form

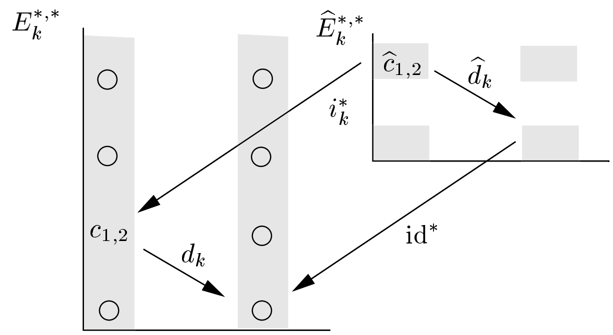
$$\widehat{E}_2^{p,q} \cong H^p(S^k) \otimes H^q(\mathcal{M}(\{U_{1,2}\}))\,, \qquad p, q \geq 0\,.$$



¿From the location of non-zero entries in $\widehat{E}_2^{*,*}$ we easily see that there is only one possibly non-trivial differential $\widehat{d}_k$ on stage k of the sequence.
The inclusion of $F(S^k, n)$ into $\widehat{F}$ is a map of fiber bundles.

$$\begin{tikzcd} & \mathcal{M}(\{U_{1,2}\}) \arrow[rr] & & \widehat{F} \arrow[dd] \\ \mathcal{M}(\mathcal{A}^{(k)}_{n-2}) \arrow[ur] \arrow[rr] & & F(S^k, n) \arrow[ur] \arrow[dd] & \\ & & & S^k \\ & & S^k \arrow[ur] & \end{tikzcd}$$

The homomorphism of spectral sequences induced by the inclusion of the fiber bundles factors on the $\widehat{E}_k$-tableau into the induced map between the cohomology of the fibers and the identity on the cohomology of the base space [Bo1, Exp. VIII, Thm. 4]. The map i^* between the cohomology of the fibers maps the generator $\widehat{c}_{1,2}$ of $H^{k-1}(\mathcal{M}(\{U_{1,2}\}))$ to $c_{1,2}$ in $H^{k-1}(\mathcal{M}(\mathcal{A}_{n-2}^{(k)}))$ (compare Remark 3.2). Hence, we are left to determine the action of the k-th differential on $\widehat{E}_k^{0,k-1}$:



$$d_k(c_{1,2}) \,=\, d_k(i^*(\widehat{c}_{1,2})) \,=\, \widehat{d}_k(\widehat{c}_{1,2})\,.$$

Proposition 4.4 *The fiber bundle $\widehat{F}$ over S^k is fiber homotopy equivalent to $V_{k+1,2}$, the Stiefel manifold of orthogonal 2-frames in $\mathbb{R}^{k+1}$, considered as fiber bundle over S^k.*

Proof. $\widehat{F}$ is fiber homotopy equivalent to $F(S^k,3)$, both spaces considered as fiber bundles over S^k. The fiber homotopy equivalence is realized by the projection of configurations in $\widehat{F}$ to their first, second and last points. In turn, $F(S^k,3)$ is fiber homotopy equivalent to the Stiefel manifold $V_{k+1,2}$ [Fa, Thm. 2.4]. □

For a simply connected, k-dimensional, orientable manifold M the only possibly non-trivial differential in the spectral sequence associated to the unit tangent bundle can be described as a cup product multiplication with the Euler class of the manifold:

$$d_k(x\otimes\mu) \,=\, d_k(\mu)\smile x \,=\, \chi_M\smile x\,,$$

where μ is a generator of $H^{k-1}(S^{k-1})$, $x\in H^*(M)$, and χ_M denotes the Euler class of the manifold (compare [MS, Thm. 12.2]).
The Stiefel manifold $V_{k+1,2}$ coincides with the unit tangent bundle on S^k. Given an orientation on S^k and a generator ν of $H^k(S^k)$ that evaluates to 1 on the orientation class, the Euler class of S^k is given by

$$\chi_{S_k} \,=\, \begin{cases} 0 & \text{for odd } k\,,\\ 2\nu & \text{for even } k\,.\end{cases}$$

We conclude that in the spectral sequence for $\widehat{F}$ the differential $\widehat{d}_k$ maps the generator $\widehat{c}_{1,2}$ of $H^{k-1}(\mathcal{M}(\{U_{1,2}\}))$ to the Euler class χ of the base space, once an orientation for the base S^k and with it the Euler class have been chosen appropriately. In particular, $\widehat{d}_k$ is the zero-map for odd k. For our initial fiber bundle we thus derive

$$d_k(c_{i,j}) \;=\; d_k(c_{1,2}) \;=\; \begin{cases} 0 & \text{for odd } k\,, \\ 2\nu & \text{for even } k\,, \end{cases}$$

where 2ν is the Euler class of the k-sphere under appropriate orientation.

5 Recovering $H^*(F(S^k, n))$ from the spectral sequence

For configuration spaces of odd-dimensional spheres we now have enough information to derive a complete description of the integer cohomology algebra. In the previous section we showed that the k-th differential is trivial on multiplicative generators of the sequence tableau $E_k^{*,*}$, therefore it is trivial on all of $E_k^{*,*}$. The spectral sequence collapses in its second term; a favorable location of non-zero tableau entries allows us to get both the linear and the multiplicative structure of $H^*(F(S^k, n))$ directly from the second tableau:

Theorem 5.1 *For a sphere S^k of odd dimension $k \geq 3$, and $n \geq 3$, the integer cohomology algebra of $F(S^k, n)$ is given by*

$$\begin{aligned} H^*(F(S^k,n)) \;&\cong\; H^*(S^k) \otimes H^*(\mathcal{M}(\mathcal{A}_{n-2}^{(k)})) \\ &\cong\; (\,\mathbb{Z}(0) \oplus \mathbb{Z}(k)\,) \otimes \Lambda^* \bigoplus_{\binom{n-1}{2}} \mathbb{Z}(k-1)\,/\,I\,, \end{aligned}$$

where I is the ideal described in Proposition 3.1. *In particular, the cohomology is free.*

For the case of even-dimensional spheres the considerations in the previous section show that the k-th differential is non-zero. We have to describe the kernel and cokernel of that differential and with it the final sequence tableau $E_{k+1}^{*,*}$ in a manageable form.
The cohomology algebra of the fiber, hence of the left-most column of the second, resp. k-th tableau, is given by Proposition 3.1. A linear basis for this algebra is given by the products of $(k-1)$-dimensional classes $c_{i,j}$ associated with the faces of the broken circuit complex $\mathrm{BC}(\mathcal{L})$ of the intersection lattice $\mathcal{L} = \mathcal{L}(\mathcal{A}_{n-2}^{(k)})$ [BZ, Sect. 9]:

$$\mathcal{B}_{BC} \;=\; \{c_{\alpha_1} \wedge \ldots \wedge c_{\alpha_t} \,|\, \{\alpha_1, \ldots, \alpha_t\} \in \mathrm{BC}(\mathcal{L})\}\,.$$

Here is a different basis which enables us to describe the kernel of d_k both as a direct summand and as a subalgebra of $H^*(\mathcal{M}(\mathcal{A}_{n-2}^{(k)}))$:

Proposition 5.2 *The following set is a $\mathbb{Z}$-linear basis for $H^*(\mathcal{M}(\mathcal{A}_{n-2}^{(k)}))$:*

$$\begin{aligned} \mathcal{B}' \;=\;& \{c_{1,2} \wedge (c_{\alpha_1} - c_{1,2}) \wedge \ldots \wedge (c_{\alpha_t} - c_{1,2}) \,|\, \{\alpha_1, \ldots, \alpha_t\} \in \mathrm{BC}(\mathcal{L}),\, \alpha_i \neq (1,2)\} \\ &\cup \{(c_{\alpha_1} - c_{1,2}) \wedge \ldots \wedge (c_{\alpha_t} - c_{1,2}) \,|\, \{\alpha_1, \ldots, \alpha_t\} \in \mathrm{BC}(\mathcal{L}),\, \alpha_i \neq (1,2)\}\,. \end{aligned}$$

Proof. Each element in $\mathcal{B}_{\mathrm{BC}}$ can be written as a linear combination of elements in $\mathcal{B}'$. This is true for each element having $c_{1,2}$ as a factor because those are themselves elements in $\mathcal{B}'$. For $c_{\alpha_1} \wedge \ldots \wedge c_{\alpha_t}$, $\{\alpha_1, \ldots, \alpha_t\} \in \mathrm{BC}(\mathcal{L})$, $\alpha_i \neq (1,2)$,

$$(c_{\alpha_1} - c_{1,2}) \wedge \ldots \wedge (c_{\alpha_t} - c_{1,2}) \,=\, c_{\alpha_1} \wedge \ldots \wedge c_{\alpha_t} + \beta \,,$$

where β is a linear combination of products containing $c_{1,2}$, hence of elements in $\mathcal{B}'$. Thus $c_{\alpha_1} \wedge \ldots \wedge c_{\alpha_t}$ can be written as a linear combination of those. □

Let $\mathcal{T}^\bullet$ denote the submodule of $H^*(\mathcal{M}(\mathcal{A}_{n-2}^{(k)}))$ generated by those elements of $\mathcal{B}'$ that contain $c_{1,2}$ as a factor, whereas $\mathcal{T}^\circ$ denotes the submodule generated by all other elements of $\mathcal{B}'$:

$$H^*(\mathcal{M}(\mathcal{A}_{n-2}^{(k)})) \,\cong\, \mathcal{T}^\circ \oplus \mathcal{T}^\bullet \,.$$

Obviously, multiplication within $\mathcal{T}^\bullet$ is trivial, whereas for $\mathcal{T}^\circ$ we can state the following:

Proposition 5.3 *The submodule $\mathcal{T}^\circ$ is a subalgebra of $H^*(\mathcal{M}(\mathcal{A}_{n-2}^{(k)}))$ generated by the elements $\bar{c}_{i,j} := (c_{i,j} - c_{1,2})$ in dimension $k-1$, $1 \leq i < j \leq n-1$, $(i,j) \neq (1,2)$. It has a presentation as a quotient of the exterior algebra on these generators:*

$$\mathcal{T}^\circ \,\cong\, \Lambda^*\mathbb{Z}^{\binom{n-1}{2}-1} \,/\, J \,,$$

where J is the ideal generated by elements of the form

$$\begin{array}{rr} (\bar{c}_{i,l} \wedge \bar{c}_{j,l}) \,+\, (-1)^{k+1}(\bar{c}_{i,j} \wedge \bar{c}_{j,l}) \,+\, (\bar{c}_{i,j} \wedge \bar{c}_{i,l}) \,, & 1 \leq i < j < l \leq n-1 \,, \\ & (i,j) \neq (1,2) \,, \\ (\bar{c}_{1,i} \wedge \bar{c}_{2,i}) \,, & 2 < i \leq n-1 \,. \end{array}$$

Proof. It is clear that $\mathcal{T}^\circ$ has a presentation as a quotient of the exterior algebra on the generators $\bar{c}_{i,j} = (c_{i,j} - c_{1,2})$, $1 \leq i < j \leq n-1$, $(i,j) \neq (1,2)$. Moreover, it is easy to check that the proposed relations hold in $H^*(\mathcal{M}(\mathcal{A}_{n-2}^{(k)}))$. To see that they generate the ideal for a presentation of $\mathcal{T}^\circ$ note that they allow one to write each product in the generators $\bar{c}_{i,j}$ as a linear combination of elements from the linear basis for $\mathcal{T}^\circ$: Assume that for a product of generators

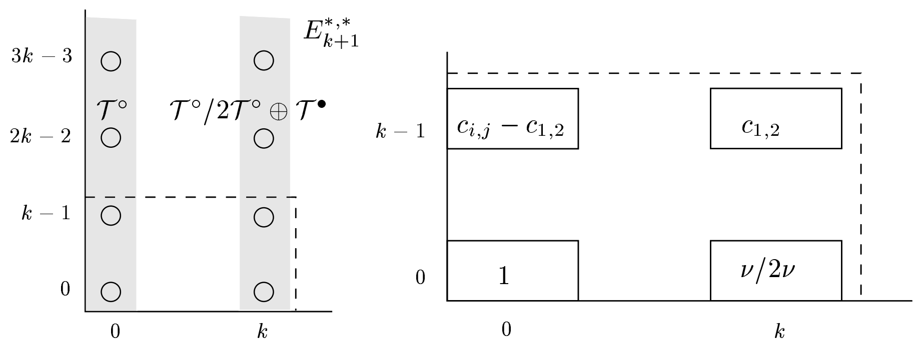
$$\bar{c}_{\alpha_1} \wedge \ldots \wedge \bar{c}_{\alpha_t}$$

all products with lexicographically smaller index set can be written as a linear combination of basis elements from $\mathcal{T}^\circ$. If this product is not itself a basis element then $\{\alpha_1, \ldots, \alpha_t\}$ contains a broken circuit of $\mathcal{L}(\mathcal{A}_{n-2}^{(k)})$. In case $(1,2)$ extends it to a circuit the product is zero by a relation of the second type. Otherwise, a relation of the first type allows to write it as a linear combination of products with lexicographically smaller index set, and hence as a linear combination of basis elements. □

Our results on d_k now read as follows:

$$\begin{aligned} d_k(c_{1,2}) &= 2\nu \\ d_k(c_{i,j} - c_{1,2}) &= 0 \qquad \text{for } 1 \le i < j \le n-1\,, \end{aligned}$$

where ν is a generator of $H^k(S^k)$. Evaluating d_k by the Leibniz rule on the basis elements of $\mathcal{B}'$ we exhibit $\mathcal{T}^\circ$ as the kernel of d_k, whereas $\operatorname{im} d_k = 2\mathcal{T}^\circ$, and hence coker $d_k \cong \mathcal{T}^\circ/2\mathcal{T}^\circ \oplus \mathcal{T}^\bullet$. We thus obtain the final sequence tableau $E_{k+1}^{*,*}$ with entries $E_{k+1}^{0,*} = \mathcal{T}^\circ$ and $E_{k+1}^{k,*} = \mathcal{T}^\circ/2\mathcal{T}^\circ \oplus \mathcal{T}^\bullet$.
From the sequence tableau $E_{k+1}^{*,*}$ we can read the cohomology algebra of $F(S^k,n)$: Free generators for $\mathcal{T}^\circ = E_{k+1}^{0,*}$ are located in $E_{k+1}^{0,0}$ and $E_{k+1}^{0,k-1}$. Together with the free generator in $E_{k+1}^{k,k-1}$ and the generator of order two in $E_{k+1}^{k,0}$ they generate $\mathcal{T}^\circ/2\mathcal{T}^\circ \oplus \mathcal{T}^\bullet = E_{k+1}^{k,*}$.



Linearly, the cohomology of $F(S^k,n)$ is isomorphic to a tensor product of two free generators in dimension 0 and $2k-1$ and a generator of order 2 in dimension $k-1$ with the algebra $\mathcal{T}^\circ$:

$$H^*(F(S^k,n)) \cong (\mathbb{Z}(0) \oplus \mathbb{Z}_2(k) \oplus \mathbb{Z}(2k-1)) \otimes \mathcal{T}^\circ\,.$$

This isomorphism is an algebra isomorphism: This is obvious for multiplication among elements represented by entries in the left-most column $E_{k+1}^{0,*}$. Also, multiplication between entries of $E_{k+1}^{0,*}$ *and* $E_{k+1}^{k,*}$ is correctly described in the proposed tensor product. Moreover, the trivial multiplication among entries in $E_{k+1}^{k,*}$ has its correspondence in the tensor algebra since multiplication within the left-hand factor is trivial. We conclude:

Theorem 5.4 *For a sphere S^k of even dimension, $k \ge 4$, the integer cohomology algebra of $F(S^k,n)$, $n \ge 3$, is given by*

$$\begin{aligned} H^*(F(S^k,n)) &\cong (\mathbb{Z}(0) \oplus \mathbb{Z}_2(k) \oplus \mathbb{Z}(2k-1)) \otimes \mathcal{T}^\circ \\ &\cong (\mathbb{Z}(0) \oplus \mathbb{Z}_2(k) \oplus \mathbb{Z}(2k-1)) \otimes \Lambda^* \bigoplus_{\binom{n-1}{2}-1} \mathbb{Z}(k-1)\,/\,J\,, \end{aligned}$$

where J is the ideal described in Proposition 5.3.

In the next section we will give a topological interpretation for this product decomposition of the cohomology algebra (see Remark 6.1).

6 A family of fiber bundles

The bundle structure on $F(S^k,n)$ given by the projection Π_1 was the key to determine the integer cohomology algebra of $F(S^k,n)$. This projection Π_1 is one instance from a family of fiber maps $\Pi_r = \Pi_r(S^k,n)$, $1 \le r < n$, that are given by projection of a configuration in $F(S^k,n)$ to its last r points. In this section we will have a closer look at these fiber maps, at their spectral sequences, and at the question whether the induced bundle structures are trivial.

For the fiber map $\Pi_r(S^k,n)$, $1 \le r < n$, we obtain the following space as the fiber over a point configuration $q = (q_1,\ldots,q_r)$ on S^k:

$$\begin{aligned}\Pi_r^{-1}(q) \quad = \quad \{(x_1,\ldots,x_{n-r}) \in (S^k)^{n-r} \,|\, x_i \ne x_j \text{ for } i \ne j,\ x_i \ne q_t \\ \text{for } i = 1,\ldots,n-r,\, t = 1,\ldots,r\}\,.\end{aligned}$$

This space is again a configuration space:

$$\Pi_r^{-1}(q) \;=\; F(\,S^k \setminus \{q_1,\ldots,q_r\},\, n-r\,)\,.$$

Configurations on S^k that avoid $r \ge 1$ (fixed) points $q_1,\ldots,q_r$ are equivalent to configurations in $\mathbb{R}^k$ that avoid $r-1$ points $q_1,\ldots,q_{r-1}$. Thus the fiber of Π_r is homeomorphic to the complement of the arrangement $\mathcal{A}_{\Pi_r(S^k,n)}$ of (affine) subspaces in $\mathbb{R}^{k(n-r)}$ given by

$$\begin{aligned}U_{i,j} \quad &= \quad \{(x_1,\ldots,x_{n-r}) \in (\mathbb{R}^k)^{n-r} \,|\, x_i \,=\, x_j\,\}, \qquad 1 \le i < j \le n-r,\\ U_i^t \quad &= \quad \{(x_1,\ldots,x_{n-r}) \in (\mathbb{R}^k)^{n-r} \,|\, x_i \,=\, t\cdot(1,0,\ldots,0)^T\,\},\\ &\qquad\qquad 1 \le i \le n-r,\ 0 \le t \le r-2\,.\end{aligned}$$

For $r = 1$, the arrangement $\mathcal{A}_{\Pi_1(S^k,n)}$ coincides with the k-braid arrangement $\mathcal{A}_{n-2}^{(k)}$ — a fact that we used extensively in the previous sections. For $r > 2$, $\mathcal{A}_{\Pi_r(S^k,n)}$ contains *affine* subspaces, the subspaces U_i^t for $0 < t \le r-2$. In the complex case, for $k = 2$, these arrangements were extensively studied by Welker [We].

6.1 The spectral sequences

We proved in the previous sections that the spectral sequence $E_*(\Pi_1)$ associated to the fiber map $\Pi_1(S^k,n)$ collapses in E_2 for odd k, and in E_{k+1} for even k. We obtain a similar picture for the spectral sequence $E_*(\Pi_2)$ associated to the fiber map $\Pi_2(S^k,n)$: The base space $F(S^k,2)$ is homotopy equivalent to S^k. Hence, it is simply connected for $k \ge 2$, and the system of local coefficients on S^k induced by Π_2 is simple. The fiber $\mathcal{M}(\mathcal{A}_{\Pi_2(S^k,n)})$ is homotopy equivalent to the complement of the k-braid arrangement $\mathcal{A}_{n-2}^{(k)}$. In fact, the homotopy equivalence is realized by projection of $\mathcal{M}(\mathcal{A}_{n-2}^{(k)})$ along $\bigcap \mathcal{A}_{n-2}^{(k)}$ on the linear subspace

$$U_{n-1}^{0} \;=\; \{\, (x_1, \ldots, x_{n-1}) \in (\mathbb{R}^k)^{n-1} \,|\, x_{n-1} = 0 \,\}\,.$$

Thus, the E_2-tableaux of the spectral sequences induced by Π_1 and Π_2 coincide. For dimensional reasons, the collapsing results on $E_*(\Pi_1)$ translate to analogous collapsing results on $E_*(\Pi_2)$.

The picture changes for the spectral sequences $E_*(\Pi_3)$ associated to $\Pi_3(S^k, n)$. In fact, we have all arguments at hand to discuss them briefly: The base space $F(S^k, 3)$ of the fiber map $\Pi_3(S^k, n)$ is homotopy equivalent to the Stiefel manifold $V_{k+1,2}$ of orthogonal 2-frames in $\mathbb{R}^{k+1}$ [Fa, Thm. 2.4], hence it is simply connected for $k \geq 2$. We conclude that the system of local coefficients on $F(S^k, 3)$ induced by Π_3 is simple. We have seen above that the fiber of Π_3 is homeomorphic to the complement of the (affine) subspace arrangement $\mathcal{A}_{\Pi_3(S^k,n)}$. Comparison to the complement of the k-braid arrangement $\mathcal{A}_{n-2}^{(k)}$ yields a homotopy equivalence,

$$\mathcal{M}(\mathcal{A}_{\Pi_3(S^k,n)}) \;\simeq\; \mathcal{M}(\mathcal{A}_{n-2}^{(k)}{}_{\lceil U})\,,$$

where $\mathcal{A}_{n-2}^{(k)}{}_{\lceil U}$ denotes the restriction of the k-braid arrangement to the affine subspace

$$U \;=\; \{(x_1, \ldots, x_{n-1}) \in (\mathbb{R}^k)^{n-1} \,|\, x_{n-2} - x_{n-1} = (1, 0, \ldots, 0)^T \,\}\,.$$

The homotopy equivalence is realized by projection of $\mathcal{M}(\mathcal{A}_{n-2}^{(k)}{}_{\lceil U})$ along the intersection $\bigcap \mathcal{A}_{n-2}^{(k)}$ to the linear subspace

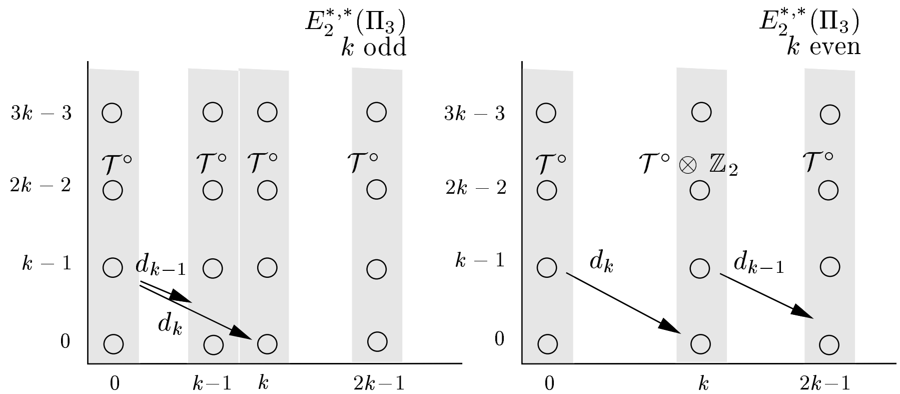
$$U_{n-1}^{0} \;=\; \{(x_1, \ldots, x_{n-1}) \in (\mathbb{R}^k)^{n-1} \,|\, x_{n-1} = 0 \,\}\,.$$

The affine arrangement $\mathcal{A}_{n-2}^{(k)}{}_{\lceil U}$ is associated to the k-braid arrangement in the same way as we associated before an affine complex hyperplane arrangement to the complex braid arrangement (compare Section 2). This analogy allows one to state a presentation for its cohomology algebra in terms of generators and relations. In fact, one obtains an algebra presentation that coincides with the one that we stated for $\mathcal{T}^\circ$ in Proposition 5.3:

$$H^*(\mathcal{M}(\mathcal{A}_{n-2}^{(k)}{}_{\lceil U})) \;\cong\; \mathcal{T}^\circ\,.$$

In particular, $H^*(\mathcal{M}(\mathcal{A}_{n-2}^{(k)}{}_{\lceil U}))$ is torsion-free and it is generated in dimension $k-1$ by cohomology classes that are in one-to-one correspondence with the inclusion maximal subspaces of the arrangement.

For both odd and even k the E_2-tableaux of the spectral sequences associated to $\Pi_3(S^k, n)$ carry the structure of tensor products. We content ourselves with discussing the spectral sequences for $k \geq 3$; for $k = 2$, we already showed in Section 2 that the bundle structure induced by Π_3 is trivial.



$$E_2^{p,q}(\Pi_3) \cong H^p(V_{k+1,2}) \otimes H^q(\mathcal{M}(\mathcal{A}^{(k)}_{n-2\lceil U})), \quad p,q \geq 0\,.$$

It is easy to see that $E_*(\Pi_3)$ collapses in its second term for *both* odd and even k: The location of non-zero entries in the respective tableaux suffices to see the triviality of differentials d_r with $r \neq k$. The k-th cohomology group of $F(S^k,n)$ can be read already from the k-th diagonal in $E_{k+1}(\Pi_3)$. Our results on $H^k(F(S^k,n))$ (Proposition 4.2) allow to deduce triviality of the differential d_k as we did in Section 4.1.

Remark 6.1 There is a topological explanation for the product decomposition of the integer cohomology algebra of $F(S^k,n)$ for even k that we obtained in Theorem 5.4: The factors are the cohomology algebras of base space and fiber for the fiber bundle structure on $F(S^k,n)$ given by Π_3. We showed above that the associated spectral sequence $E_*(\Pi_3)$ collapses in its second term, which explains the product structure in cohomology.
The collapsing result on $E_*(\Pi_3)$ extends to the spectral sequences associated to the fiber maps Π_r for $r > 3$, and we can summarize as follows:

Proposition 6.2 *The spectral sequence $E_*(\Pi_r)$ of the fiber map $\Pi_r(S^k,n)$ on the configuration space $F(S^k,n)$ collapses in its second term unless k is even and r equals* 1 *or* 2. *For those parameters the spectral sequence collapses in E_{k+1}.*

Proof. We are left to show the triviality of the spectral sequence $E_*(\Pi_r)$ for $r > 3$. This we will derive from the triviality of $E_*(\Pi_3)$, thereby involving several applications of the following Lemma.

Lemma 6.3 [Bo2, Ch. II, Thm. 14.1] *Let $F \overset{i}{\hookrightarrow} E \overset{\Pi}{\to} B$ be a fiber bundle with path-connected base B and assume that the cohomology of the base or the cohomology of the fiber is torsion-free. Then the following assertions are equivalent:*

(1) *The system of local coefficients on B induced by* Π *is simple* and *the associated spectral sequence with integer coefficients collapses in its second term.*

(2) *The induced map $i^* : H^*(E) \to H^*(F)$ is surjective.*

Consider the map of fiber bundles between $\Pi_r(S^k, n)$ and $\Pi_3(S^k, n)$ given by $(\mathrm{id}, \Pi_3(S^k, r))$. For simplicity of notation we denote with Q_t a fixed set of pairwise distinct points $\{q_1, \ldots, q_t\}$ in S^k and thus write $F(S^k \backslash Q_t, n-t)$ for the respective fibers. The fibers are complements of affine k-arrangements, thus their cohomology algebras are torsion-free.

$$\begin{tikzcd}
 & F(S^k \setminus Q_3, n-3) \arrow[rr, "i_{\Pi_3}"] & & F(S^k, n) \arrow[dd, "\Pi_3"] \\
F(S^k \setminus Q_r, n-r) \arrow[ur, "i"] \arrow[rr, "i_{\Pi_r}"] & & F(S^k, n) \arrow[ur, "\mathrm{id}"] \arrow[dd, "\Pi_r"] & \\
 & & & F(S^k, 3) \\
 & & F(S^k, r) \arrow[ur, "\Pi_3"'] &
\end{tikzcd}$$

The configuration space $F(S^k, 3)$ is simply connected for $k \geq 2$, due to the homotopy equivalence with the Stiefel manifold $V_{k+1,2}$. With the collapsing result on $E_*(\Pi_3)$ we deduce that $i^*_{\Pi_3}$ is surjective by the equivalence stated above. We are left to show that the inclusion i between the fibers induces a surjective homomorphism in cohomology. Then $i^*_{\Pi_r} = i^* \circ i^*_{\Pi_3}$ is surjective, and another application of Lemma 6.3 yields the collapsing result on $E^*(\Pi_r)$.
To see that i^* is surjective we interpret i as a concatenation of inclusions in a sequence of fiber maps. Namely, we consider the sequence of fiber maps obtained by successively projecting $F(S^k \setminus Q_1, n-1)$ to its last coordinate. We picture the part of this sequence which is relevant to our investigation:

$$\begin{tikzcd}
F(S^k \backslash Q_r, n-r) \arrow[r, "j_{r-1}"] & \ldots \arrow[r, "j_4"] & F(S^k \backslash Q_4, n-4) \arrow[r, "j_3"] \arrow[d, "p_4"] & F(S^k \backslash Q_3, n-3) \arrow[d, "p_3"] \\
 & & S^k \setminus Q_4 & S^k \setminus Q_3
\end{tikzcd}$$

The base spaces of the fiber bundles given by p_t, $1 \leq t \leq n-2$, are simply connected for $k > 2$, thus the systems of local coefficients are simple. The same holds for $k = 2$, which is a result of Cohen [C2, Lemma 6.3]. The fibers are complements of affine k-arrangements, thus their cohomology groups are non-trivial only in dimensions that are multiples of $k-1$ [GM, Part III, Thm. B]. For dimensional reasons, the associated spectral sequences $E_*(p_t)$ collapse in their second terms and we conclude by Lemma 6.3 that the j_t^* are surjective for $1 \leq t \leq n-2$. Thus, $i^* = j_{r-1}^* \circ \ldots \circ j_3^*$ is a surjective map, which concludes our proof. $\square$

6.2 Triviality of the fiber bundles

The fiber bundle structure induced by Π_3 on $F(S^2, n)$ for $n \geq 3$ is trivial (Theorem 2.1). One is led to ask: For which parameters do the fiber maps Π_r induce a trivial fiber bundle structure on $F(S^k, n)$?

We observed in Section 2 that the bundle structure on $F(S^k, 2)$ given by Π_1 is equivalent to the tangent bundle over S^k. Thus, $\Pi_1(S^k, 2)$ is a trivial fiber map if and only if S^k is parallelizable (see Hirzebruch [H]). This indicates that the triviality question for the fiber maps Π_r is difficult in general.
Our results on the cohomology algebra of $F(S^k, n)$ for even k, $k \geq 2$, exclude a trivial bundle structure on $F(S^k, n)$ induced by Π_1: There is 2-torsion in $H^*(F(S^k, n))$ while the cohomology algebra of the cartesian product of base space and fiber is torsion-free. However, the cohomology algebra of $F(S^k, n)$ for odd k coincides with the cohomology algebra of the cartesian product of base space and fiber. Such product decomposition might as well hold beyond the level of cohomology.
Recall from previous arguments that $F(S^k, 3)$ is fiber homotopy equivalent to the Stiefel manifold $V_{k+1,2}$ of orthogonal 2-frames in $\mathbb{R}^{k+1}$, both considered as fiber bundles over S^k. Fiber bundles are trivial if and only if their associated principal bundles are trivial [St, Part I, Cor. 8.4]. Hence, $V_{k+1,2}$ is a trivial fiber bundle if and only if $O(k+1)$, considered as a fiber bundle over S^k, admits a section — which again is the case iff $k = 1, 3$ or 7. Moreover, $V_{k+1,2}$ is fiber homotopy equivalent to a trivial bundle if and only if it is trivial itself, hence iff $k = 1, 3$ or 7 [Ja, Thm. 1.11]. We conclude that $F(S^k, 3)$ is a *non-trivial* fiber bundle over S^k for $k \neq 1, 3$ or 7.
For the 1-sphere we have shown triviality of $F(S^1, n)$ as a fiber bundle over S^1 in Section 2. Analogously, we obtain a trivialization of the fiber bundle structure on $F(S^3, n)$ given by Π_1, using the group structure of S^3. The 7-sphere does not carry the structure of a topological group [Bd, VI, Cor. 15.21]. However, one can establish an explicit equivalence of fiber bundles between $F(S^7, 3)$ and $V_{8,2} \times \mathbb{R}^7 \times \mathbb{R}$, both considered as fiber bundles over S^7 in the natural way. As mentioned above, $V_{8,2}$ is a trivial fiber bundle over S^7, and we can thus conclude triviality of $F(S^7, 3)$ over S^7.
Thus it remains to decide whether the bundle structure on $F(S^k, n)$ induced by Π_1 is trivial for $n > 3$ and odd $k \geq 5$.

We have seen in Section 2 that Π_3 induces a trivial bundle structure on $F(S^2, n)$. Our collapsing results on the spectral sequences $E_*(\Pi_3(S^k, n))$ for both odd and even k would be consistent with triviality of the fiber bundle structure given by Π_3. However, except for $k = 2$ this leaves us with an open question.

Remark 6.4 After completion of this paper, we learned about recent work by Fadell & Husseini [FaH] which addresses the question of configuration space bundles being (fiber-homotopically) trivial. The paper is mostly concerned with configuration spaces of Euclidean spaces; a complete discussion for configuration spaces of spheres is announced, but the results are stated only for spheres of odd dimension.

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