Oriented Matroids Today

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Abstract

This dynamic survey offers an “entry point” for current research in oriented matroids. For this, it provides updates on the 1993 monograph “Oriented Matroids” by Björner, Las Vergnas, Sturmfels, White & Ziegler [85], in three parts:

1. a sketch of a few “Frontiers of Research” in oriented matroid theory,
2. an update of corrections, comments and progress as compared to [85], and
3. an extensive, complete and up-to-date bibliography of oriented matroids, comprising and extending the bibliography of [85].

1 Introduction(s).

Oriented matroids are both important and interesting objects of study in Combinatorial Geometry, and indispensable tools of increasing importance and applicability for many other parts of Mathematics. The main parts of the theory and some applications were, in 1993, compiled in the quite comprehensive monograph by Björner, Las Vergnas, Sturmfels, White & Ziegler [85]. For other (shorter) introductions and surveys, see Bachem & Kern [35], Bokowski & Sturmfels [146], Bokowski [116], Goodman & Pollack [333], Ziegler [712, Chapters 6 and 7], and, most recently, Richter-Gebert & Ziegler [565].

This dynamic survey provides three parts:

1. a sketch of a few “Frontiers of Research” in oriented matroid theory,
2. an update of corrections, comments and progress as compared to [85], and
3. an extensive, complete and up-to-date bibliography of oriented matroids, comprising and extending the bibliography of [85].

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2 What is an Oriented Matroid?

Let $V = (v_1, v_2, \ldots, v_n)$ be a finite, spanning, sequence of vectors in $\mathbb{R}^r$, that is, a finite vector configuration. With this vector configuration, one can associate the following sets of data, each of them encoding the combinatorial structure of $V$.

- The chirotope of $V$ is the map
  \[ \chi_V : \{1, 2, \ldots, n\}^r \rightarrow \{+, -, 0\} \]
  \[ (i_1, i_2, \ldots, i_r) \mapsto \text{sign}(\det(v_{i_1}, v_{i_2}, \ldots, v_{i_r})) \]
  that records for each $r$-tuple of the vectors whether it forms a positively oriented basis of $\mathbb{R}^r$, a basis with negative orientation, or not a basis.

- The set of covectors of $V$ is
  \[ V^*(V) := \{ (\text{sign}(a^t v_1), \ldots, \text{sign}(a^t v_n)) \in \{+, -, 0\}^n : a \in \mathbb{R}^n \} \]
  that is, the set of all partitions of $V$ (into three parts) induced by hyperplanes through the origin.

- The collection of cocircuits of $V$ is the set
  \[ C^*(V) := \{ (\text{sign}(a^t v_1), \ldots, \text{sign}(a^t v_n)) \in \{+, -, 0\}^n : a \in \mathbb{R}^n \text{ is orthogonal to a hyperplane spanned by vectors in } V \} \]
  of all partitions by “special” hyperplanes that are spanned by vectors of the configuration $V$.

- The set of vectors of $V$ is
  \[ V(V) := \{ (\text{sign}(\lambda_1), \ldots, \text{sign}(\lambda_n)) \in \{+, -, 0\}^n : \lambda_1 v_1 + \ldots + \lambda_n v_n = 0 \text{ is a linear dependence between vectors in } V \} \]

- The set of circuits is
  \[ C(V) := \{ (\text{sign}(\lambda_1), \ldots, \text{sign}(\lambda_n)) \in \{+, -, 0\}^n : \lambda_1 v_1 + \ldots + \lambda_n v_n = 0 \text{ is a minimal linear dependence between vectors in } V \} \]

A simple, but basic, result now states that all of these sets of data are equivalent, except for a global sign change that identifies $\chi$ with $-\chi$. Thus whenever one of the data

\[ \{\chi_{V}, -\chi_{V}\}, \quad V^*(V), \quad C^*(V), \quad V(V), \quad \text{or } C(V) \]

is given, one can from this uniquely reconstruct all the others.

Furthermore, one has axiom systems (see [85, Chap. 3]) for chirotopes, covectors, cocircuits, vectors and circuits that are easily seen to be satisfied by the corresponding collections above. Thus there are combinatorial structures, called oriented matroids, that can equivalently be given by any of these five different sets of data, and defined/characterized in terms of any of the five corresponding axiom systems. (The proofs for the equivalences between these data sets resp. axiom systems are not simple.)

Vector configurations as discussed above give rise to oriented matroids of rank $r$ on $n$ elements (or: on a ground set of size $n$). Usually the ground set is identified with $E = \{1, 2, \ldots, n\}$.

Equivalent to vector configurations, one has the model of (real, linear, essential, oriented) hyperplane arrangements: finite collections $A := (H_1, H_2, \ldots, H_n)$ of hyperplanes (linear subspaces of codimension one) in $\mathbb{R}^r$, with the extra requirement that $H_1 \cap \ldots \cap H_n = \{0\}$, and with a choice of a positive halfspace $H_i^+$ for each of the hyperplanes. In fact, every vector configuration gives rise to such an arrangement.
via $H_i^+ := \{ x \in \mathbb{R}^r : v_i^T x \geq 0 \}$, and from an oriented hyperplane arrangement we recover a vector configuration by taking the positive unit normals.

More specialized, one has the model of directed graphs: if $D = (V, A)$ is a finite directed graph (with vertex set $V = \{0, 1, 2, \ldots, r\}$ and arc set $A = \{a_1, \ldots, a_n\} \subseteq V^2$), then one has the obvious “directed circuits” in the digraph that give rise to circuits in the sense of sign vectors in $C(V) \subseteq \{+,-,0\}^n$, while directed cuts give rise to cocircuits, and minimal directed cuts give rise to cocircuits. Thus one obtains the oriented matroid of a digraph, which can also, equivalently, be constructed by associating with each arc $(i, j)$ the vector $e_i - e_j \in \mathbb{R}^r$, where we take $e_i$ to be the $i$-th coordinate vector in $\mathbb{R}^r$ for $i \geq 1$, and $e_0 := 0$.

Although the axiom systems of oriented matroids describe the data arising from vector configurations very well, it is not true that every oriented matroid corresponds to a real vector configuration. In other words, there are oriented matroids that are not realizable. This points to basic theorems and problems in Oriented Matroid Theory:

- The Topological Representation Theorem (see [85, Chap. 5]) shows that while real vector configurations can equivalently be represented by oriented arrangements of hyperplanes, general oriented matroids can be represented by oriented arrangements of pseudo-hyperplanes.

- There is no finite set of axioms that would characterize the oriented matroids that are representable by vector configurations. In fact, even for $r = 3$ there are oriented matroids on $n$ elements that are minimally non-realizable for arbitrarily large $n$.

- The realization problem is a difficult algorithmic task: for a given oriented matroid, to decide whether it is realizable, and possibly find a realization. This statement is a by-product of the constructions for the Universality Theorem for oriented matroids, see below.

3 Some Frontiers of Research.

Currently there is substantial research done on a variety of aspects and questions; among them are several deep problems of oriented matroid theory that were thought to be both hard and fundamental, and are now gradually turning out to be just that.

Here I give short sketches and a few pointers to the (recent) literature, for just a few selected topics. (By construction, the selection is very much biased. I plan to expand and update regularly. Your help and comments are essential for that.)

3.1 Realization spaces.

Mnëv’s Universality Theorem of 1988 [503] states that every primary semialgebraic set defined over $\mathbb{Z}$ is “stably equivalent” to the realization space of some oriented matroid of rank 3. In other words, the semialgebraic sets of the form

$$\mathcal{R}(X) := \{ Y \in \mathbb{R}^{3 \times n} : \text{sign}(\det(Y_{i,j,k})) = \text{sign}(\det(Y_{i,j,k})) \text{ for all } 1 \leq i < j < k \leq n \},$$

for real matrices $X \in \mathbb{R}^{3 \times n}$, can be arbitrarily complicated, both in their topological and their arithmetic properties. Mnëv’s even stronger Universal Partition Theorem [504] announced in 1991 says that essentially every semialgebraic family appears in the stratification given by the determinant function on the $(3 \times 3)$-minors of $(3 \times n)$-matrices.

These results are fundamental and far-reaching. For example, via oriented matroid (Gale) duality they imply universality theorems for $d$-polytopes with $d+4$ vertices.
For some time, no complete proofs were available. This has only recently changed with the complete proof of the Universal Partition Theorem by Günzel [301] and by Richter-Gebert [557]. Richter-Gebert [556, Sect. 2.5] has also — finally! — provided a suitable notion of “stable equivalence” of semialgebraic sets that is weak enough to make the Universality Theorems true, and strong enough to imply both homotopy equivalence and arithmetic equivalence (i.e., it preserves the existence of $K$-rational points in the semialgebraic set for every subfield $K$ of $\mathbb{R}$).

Further, surprising recent progress is now available with Richter-Gebert’s [556, 564] Universality Theorem (and Universal Partition Theorem) for 4-dimensional polytopes, and related to this his Non-Steinitz theorem for 3-spheres. (See [302] for a second proof.)

For still another, very recent, interesting universality result, concerning the configuration spaces of planar polygons, see Kapovich & Millson [411]. Kapovich & Millson trace the history of their result back to a universality theorem by Kempe [419] from 1875!

Here are two major challenges that remain in this area:

- To construct and understand the smallest oriented matroids with non-trivial realization spaces. The smallest known examples are Suvorov’s [640] oriented matroid of rank 3 on 14 points with a disconnected realization space (see also [85, p. 365]), and Richter-Gebert’s [558] new example $\Omega^3_{14}$ with the same parameters, which additionally has rational realizations, and a non-realizable symmetry.
- To provide Universality Theorems for simplicial 4-dimensional polytopes. (The Bokowski-Ewald-Kleinschmidt polytope [122] is still the only simplicial example known with a non-trivial realization space; see also Bokowski & Guedes de Oliveira [124].)

### 3.2 Extension spaces, combinatorial Grassmannians, and matroid bundles

*Thanks to Laura Anderson for help on this section!*

The consideration of spaces of oriented matroids brings several very different lines of thinking into a common topological framework. Given a set $S$ of oriented matroids, we obtain a partial order on $S$ by weak maps, and from this we obtain a topological space by taking the order complex (the simplicial complex given by chains in the partial order; see Björner [82]). This simplicial complex can be viewed as a combinatorial analog to a vector bundle. Just as a vector bundle represents a continuous parametrization of a set of vector spaces, this topological space can be viewed as a “continuous” parametrization of elements of $S$. Such spaces have arisen in several contexts:

- If $S$ is the set of non-trivial single-element extensions of a fixed oriented matroid $M$, the resulting space is the extension space $E(M)$ of $M$.
- If $S$ is the set of all rank $r$ oriented matroids on a fixed set of $n$ elements, this space is the MacPhersonian $\text{MacP}(r, n)$.
- If $S$ is the set of all rank $r$ strong map images of a fixed oriented matroid $M$, this space is the combinatorial Grassmannian $G(r, M)$. (In fact, this example essentially encompasses the previous two: The extension space $E(M)$ of a oriented matroid $M$ is a double cover of $G(r(M) - 1, M)$, while if $M$ is the unique rank $n$ oriented matroid on a fixed set of $n$ elements, then $G(r, M) = \text{MacP}(r, n)$.)

Extension spaces are closely related to zonotopal tilings (via the Bohn-Dress Theorem) and to oriented matroid programs: see Sturmfels & Ziegler [639]. The MacPhersonian and combinatorial Grassmannian arise in MacPherson’s theory of combinatorial differential manifolds and matroid bundles [476] [16] in which oriented matroids serve as combinatorial analogs to real vector spaces.
Among the basic conjectures in the field are:

- For a rank $n$ oriented matroid $M^n$, the topology of $G(k, M^n)$ should be similar to that of the real Grassmannian $G(k, \mathbb{R}^n)$. In particular, it is known that a realization of an oriented matroid determines a canonical homotopy class of maps $c : G(k, \mathbb{R}^n) \to G(k, M^n)$ [18], and it is hoped that $c$ is a homotopy equivalence, or at least leads to an isomorphism in cohomology with various coefficients.

- The extension space $\mathcal{E}(M^n)$ should have the homotopy type of an $(n - 1)$-sphere if $M^n$ is realizable. (This is essentially a special case of the above.)

There are substantial grounds for pessimism on both conjectures. For instance, there are examples of non-realizable $M^n$ such that $G(n - 1, M^n)$ and $\mathcal{E}(M^n)$ are not even connected (Mnëv & Richter-Gebert [505]). In addition, Mnëv's Universality Theorem implies that for realizable $M^n$ the inverse images under $c$ of points in $G(r, M^n)$ can have arbitrarily complicated topology. However, substantial progress has been made on the topology of $\mathcal{E}(M)$ for realizable $M$ [639] [505] and on the topology of $G(k, M)$ under various conditions: for small values of $k$ (Babson [29]), for the first few homotopy groups of the MacPhersonian (Anderson [15]), and for mod 2 cohomology (Anderson & Davis [18]). Three related survey articles are Mnëv & Ziegler [506], Anderson [16], and Reiner [546].

The analogy between oriented matroids and real vector bundles leads to an intriguing and useful interplay between topology and combinatorics. On the one hand, appropriate combinatorial adaptations of classical topological methods for real vector bundles prove that for realizable $M^n$ the map $c : G(k, \mathbb{R}^n) \to G(k, M^n)$ induces split surjections in mod 2 cohomology [18]. On the other hand, combinatorial methods can be applied to topology as well. Any real vector bundle over a triangulated base space can be "combinatorialized" into a matroid bundle [476] [18], giving a combinatorial approach to the study of bundles. The most notable success in this direction has been Gelfand & MacPherson's [308] combinatorial formula for the rational Pontrjagin classes of a differential manifold.

The topological problems discussed in this section have close connections to classical problems of oriented matroid theory, such as the following: Las Vergnas' conjectures that every oriented matroid has at least one mutation (simplicial top) and that the set of uniform oriented matroids of rank $r$ on a given finite set is connected under performing mutations. In fact, if these conjectures are false, then the "top level" of the MacPhersonian, given by all oriented matroids without circuits of size smaller than $r$ and at most one circuit of size $r$, cannot be connected. As for the Las Vergnas conjecture, Bokowski [114] and Richter-Gebert [552] have the strongest positive resp. negative partial results; more work is necessary.

Further work also remains in the understanding of weak and strong maps — currently the only comprehensive source is [85, Section 7.7]. One still has to derive structural information from the failure of Las Vergnas' strong map factorization conjecture (disproved by Richter-Gebert in [552]) and derive criteria for situations where factorization is possible.

### 3.3 Affine and infinite oriented matroids.

The Bohne-Dress Theorem, announced by Andreas Dress at the 1989 "Combinatorics and Geometry" Conference in Stockholm, provides a bijection between the zonotopal tilings of a fixed $d$-dimensional zonotope $Z$ and the single-element liftings of the realizable oriented matroid associated with $Z$. This theorem turned out to be, at the same time,

- fundamental (see e. g. the connection to extension spaces of oriented matroids [639]),
- "intuitively obvious" (just draw pictures!), and
- surprisingly hard to prove; see Bohne [105] and Richter-Gebert & Ziegler [563].
Just recently, however, a new and substantially different proof of the Bohne-Dress theorem has become available, by Huber, Rambau & Santos [387]. In particular, there are bijections

\[
\begin{align*}
\{ \text{zonotopal tilings of} \} & \quad \longleftrightarrow \quad \{ \text{subdivisions of the} \} & \quad \longleftrightarrow \quad \{ \text{extensions of the dual} \\
\{ \text{the zonotope } Z(A) \} & \quad \text{Lawrence polytope } \Lambda(A) \} & \quad \text{oriented matroid } \mathcal{M}^*(A) \}
\end{align*}
\]

The first bijection follows from the “Cayley trick”, see Huber, Rambau & Santos [387]. The second, more difficult one was already before established by Santos [590, 387].

A separate, simpler proof for the case of rank 3 — pseudoline arrangements are in bijection with zonotopal tilings of a centrally symmetric 2n-gon — is contained in the work by Felsner & Weil [275].

The Bohne-Dress theorem provides a connection to several other areas of study. On the one hand, the classification and enumeration of rhombic tilings of a hexagon relates to the theory of plane partitions and symmetric functions; see e.g. Ehnitzky [264], Edelman & Reiner [249].

On the other hand, there is a definite need for a better understanding of zonotopal tilings of the entire plane (or of \(\mathbb{R}^d\)). Two different approaches have been started by Bohne [106, Kapitel 5] and by Crapo & Senechal [210], but no complete picture has emerged, yet. This is of interest, for example, in view of the mathematical problems posed by understanding quasiperiodic tilings and quasicrystals; see Senechal [598, 599].

### 3.4 Realization algorithms.

The realizability problem — given a “small” oriented matroid, find a realization or prove that none exists — is a key problem not only in oriented matroid theory, but also for various applications, such as the classification of “small” simplicial spheres into polytopal and non-polytopal ones (see Bokowski & Sturmfels [144, 146], Altshuler, Bokowski & Steinberg [12], Bokowski & Shemer [139]). The universality theorems mentioned above tell us that the problem is hard: in fact, in terms of Complexity Theory it is just as hard as the “Existential Theory of the Reals,” the problem of solving general systems of algebraic equations and inequalities over the reals. While it is not known whether the problem over \(\mathbb{Q}\) is at all algorithmically solvable (see Sturmfels [630]), there are algorithms available that (at least theoretically) solve the problem over the reals. For the general problem Basu, Pollack & Roy [5] currently have the best result:

Let \(\mathcal{P} = \{P_1, \ldots, P_s\}\) be a set of polynomials in \(k < s\) variables each of degree at most \(d\) and each with coefficients in a subfield \(K \subseteq \mathbb{R}\).

There is an algorithm which finds a solution in each connected component of the solution set, for each sign condition on \(P_1, \ldots, P_s\), in at most \(\binom{s(k)}{k} s \mathcal{O}^* = (s/k)^k s \mathcal{O}^*\) arithmetic operations in \(K\).

However, until now this is mostly of theoretical value. What can be done for specific, explicit, small examples? Given an oriented matroid of rank 3, it seems that

- the most efficient algorithm (in practice) currently available to find a realization (if one exists) is the iterative “rubber band” algorithm described in Pock [532].
- the most efficient algorithm (in practice) currently available to show that it is not realizable (if it isn’t) is the “binomial final polynomials” algorithm of Bokowski & Richter-Gebert [130] which uses solutions of an auxiliary linear program to construct final polynomials. (An explicit example of a non-realizable oriented matroid \(\Omega_{14}\) without a biquadratic final polynomial was just recently constructed by Richter-Gebert [558].)

Neither of these two parts is guaranteed to work: but still the combination of both parts was good enough for a (still unpublished) complete classification of all 312,356 (unlabeled reorientation classes
of uniform oriented matroids of rank 3 on 10 points into realizable and non-realizable ones (Bokowski, Laffaille & Richter-Gebert [128]).

A very closely related topic is that of Automatic Theorem Proving in (plane) geometry. In fact, the question of validity of a certain incidence theorem can be viewed as the realizability problem for (oriented or unoriented) matroids of the configuration. Richter-Gebert’s Thesis [550] and Wu’s book [692] here present two recent (distinct) views of the topic, both with many of its ramifications.

Here we are far from having reached the full scope of current possibilities. For an (impressive) demonstration I refer to the spectacular new Interactive Geometry Software system Cinderella by Ulrich Kortenkamp and Jürgen Richter-Gebert [561], whose prover includes the idea of “binomial proofs” [207] as well as new randomized methods. An amazing piece of work!

4 Some Additions and Corrections.

In this section, I collect some notes, additions, corrections and updates to the 1993 book by Björner, Las Vergnas, Sturmfels, White & Ziegler [83]. The list is far from complete (even in view of the points that I know about), and with your help I plan to expand it in the future.

Page 150, Section 3.9 “Historical Sketch”

Jaritz [401, 402] gives a new axiomatic of oriented matroids in terms of “order functions” whose axioms and concepts she traces back to Sperner [611] (1949!), Karzel [416] etc. At the same time, Kalhoff [407] reduces embedding questions about pseudoline arrangements, as solved by Goodman, Pollack, Wenger & Zamfirescu [338, 341], back to 1967 results of Priëß-Crampe [537].

All this gets us closer to confirming the suspicion that probably Hilbert knew about oriented matroids.

Page 220, Exercise 4.28*.

Part (a) of this was already proved by Zaslavsky [694, Sect. 9]. However, part (b) remains open and should be an interesting challenge.

Page 227, Definition 5.1.3.

For condition (A2), if \( S_A \cap S_e = S^{-1} = \emptyset \) is the empty sphere in a zero sphere \( S_A \cong S^0 \), then the sides of this empty sphere are the two points of \( S_A \).

Page 244, Exercise 5.2(c).

Hochstättler [381] has shown that quite general arrangements of wild spheres also yield oriented matroids.

Page 270, Proposition 6.5.1.

Felsner [273] has constructed a new and especially effective encoding scheme for wiring diagrams, which implies improved upper bound for the number of wiring diagrams and hence of simple pseudoline arrangements, namely

\[ \log_2 s_n < 0.6988 n^2. \]
Richter-Gebert [560] has proved (in 1996, and written up in 1998) that orientability is NP-complete [560]. (It’s a beautiful paper!)

Page 279, Exercises 6.21(a) (*)

The answer is “yes”: this problem was solved in 1997, with an explicit construction, by Forge & Ramírez Alfonsin [281].

Page 334, Exercises 7.15(b) (*) and 7.17.

An explicit example of an oriented matroid that has a simple adjoint, but not a double adjoint was constructed by Hochstätter & Kromberg [383, 436].

Also, they observed [382, 436] that some assertions in Exercise 7.17 are not correct: Jürgen Richter-Gebert’s [550, p. 117] 8-point torus is realizable over an ordered skew field, but not over \( \mathbb{R} \). Therefore the oriented matroid given by such a skew realization has an infinite sequence of adjoints, but it is not realizable in \( \mathbb{R}^d \).

Page 337, Exercises 7.44*.

No one seems to remember the example: so consider this to be an open problem. (The non-existence of such an example is also discussed, as a Conjecture of Brylawski, in McNullty [493].)

Page 385, McMullen’s problem on projective transformations.

Forge & Schuchert [282] have found a configuration of 10 points in general position in affine 4-space that no projective transformation can put into convex position. This solves McMullen’s problem for \( d = 4 \) resp. \( r = 5 \) with \( f(4) = g(5) = 9 \).

This is consistent with the conjecture that the inequalities \( 2d + 1 \leq g(d + 1) \leq f(d) \) [sic!:] hold with equality also for \( d > 4 \).

Page 396.

Proposition 9.4.2 is true only for \( n \geq r + 2 \). For \( n = r + 1 \) the matroid is one single circuit, the inseparability graph is a complete graph, etc.

Page 405 (top).

It is not true that the sphere \( S = M_{963}^0 \) is neighborly: the edges 13 and 24 are missing (in the labeling used in [85]). Thus Schmer’s Theorem 9.4.13 cannot be applied here. A proof that the sphere admits at most one matroid polytope, \( \mathbb{A} \mathbb{B}(9) \), was given by Bokowski [109] in 1978 (see also Altshuler, Bokowski & Steinberg [12] and Antonin [20]). It is described in detail in Bokowski & Schuchert [137]. (The oriented matroid \( R(8) \) discussed in [85, Sect. 1.5] arises as a contraction of the oriented matroid \( \mathbb{A} \mathbb{B}(9) \)).

Page 413, Exercise 9.12(*).

Bokowski & Schuchert [137] showed that the smallest example (both in terms of its rank \( r = 5 \) and in terms of its number of vertices \( n = 9 \)), is given by Altshuler’s sphere \( M_{963}^0 \).
In Definition 10.1.8, delete “infeasible oriented matroid program” resp. “unbounded oriented matroid program.”

After this, the cocircuit $Y$ should be $Y = (00+++|+-)$, the circuit $X$ should be $X = (0+00|+-)$, and the circuit $X_0$ should be $X_0 = (000++|+-)$.

**Page 426, Proof of Corollary 10.1.10.**

“Orthogonality of circuits and cocircuits”
5 The Bibliography.

The purpose of the following is to keep the bibliography of the book [85] up-to-date electronically. For this, the following contains all the references of this book (including those which are not directly concerned with oriented matroids). Into this I have inserted all the corrections, missing references, additions and updates that I am currently aware of. Any corrections, new papers concerned with oriented matroids, and other updates that you tell me about will be entered asap. I am eager to hear about your corrections, updates and comments!

Related bibliographies on the web are:

- Bibliography of matroids, by Sandra Kingan, at http://members.aol.com/matroids/biblio.htm

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