Combinatorial Stratification of Complex Arrangements

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Abstract.

We present a method for discretizing complex hyperplane arrangements by encoding their topology into a finite partially ordered set of "sign vectors". This is used in the following ways:

(1) A general method is given for constructing regular cell complexes having the homotopy type of the complement of the arrangement.

(2) For the case of complexified arrangements this specializes to the construction of Salvetti [S]. We study the combinatorial structure of complexified arrangements and the Salvetti complex in some detail.

(3) This general method simultaneously produces cell decompositions of the singularity link.This link is shown to have the homotopy type of a wedge of spheres for arrangements in $\mathbb{C}^d$, $d \geq 4$.

(4) The homology of the link and the cohomology of the complement is computed in terms of explicit bases, which are matched by Alexander duality. This gives a new, more elementary and more generally valid proof for the Brieskorn-Orlik-Solomon theorem and some related results.

(5) Our setup leads to a more general notion of "2-pseudoarrangements", which can be thought of as topologically deformed complex arrangements (retaining only the essential topological and combinatorial structure). We show that all of the above remains true in this generality, except for the sign patterns of the Orlik-Solomon relations.

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CONTENTS.

1. Introduction.

Every real hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ in $\mathbb{R}^d$ with $\bigcap_{a=1}^n H_a = \{0\}$ induces a regular cell decomposition of the unit sphere S^{d-1}. The choice of a defining equation ℓ_a for every hyperplane $H_a \in \mathcal{A}$ leads to a representation of the face poset of this cell complex by a set $\mathcal{L} \subseteq \{+, -, 0\}^n$ of sign vectors, for which the partial ordering of cells by inclusion is encoded into a very simple combinatorial relation.

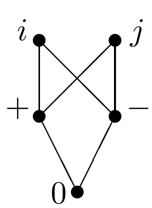
In this paper, we present an analogous combinatorial description for complex arrangements. For this we associate a *complex sign* $\mathbf{s}^{(1)}(z)$ with every complex number z, as follows:

$$\mathbf{s}^{(1)}(x+iy) = \begin{cases} i & \text{if } y > 0, \\ j & \text{if } y < 0, \\ + & \text{if } y = 0 \text{ and } x > 0, \\ - & \text{if } y = 0 \text{ and } x < 0, \\ 0 & \text{if } y = x = 0. \end{cases}$$

Now let $\mathcal{B} = \{H_1, \ldots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^d$, and fix linear forms ℓ_a with $H_a = \ker(\ell_a)$. The position of any point $\mathbf{z}$ in space with respect to the hyperplanes in $\mathcal{B}$ is encoded in the complex sign vector

$$\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z}) := (\mathbf{s}^{(1)}(\ell_1(\mathbf{z})), \ \ldots \ , \mathbf{s}^{(1)}(\ell_n(\mathbf{z}))) \in \{i, j, +, -, 0\}^n.$$

The points in $\mathbb{C}^d$ that have the same sign vector form a relative-open convex cone. Furthermore, the intersections of these cones with the unit sphere S^{2d-1} in $\mathbb{C}^d$ form a regular cell decomposition of S^{2d-1}, whose face poset is given by the set $\mathcal{K}^{(1)}_{\mathcal{B}} = \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbb{C}^d)$ of all sign vectors, ordered componentwise according to the paradigm:



Note that this diagram descibes the (augmented) face poset of the decomposition of $S^1 \subseteq \mathbb{C}$ induced by the sign function $\mathbf{s}^{(1)}$. See Figure 2.3 for a larger example.

The poset $\mathcal{K}^{(1)}$ naturally splits into two parts. The first one, $\mathcal{K}^{(1)}_{link}$, consisting of all sign vectors that have a 0-component, is the (augmented) face poset a regular CW decomposition of the singularity link $V_{\mathcal{B}} \cap S^{2d-1}$, encoding the local structure near the origin of the singular complex variety $V_{\mathcal{B}} = \bigcup_{a=1}^{n} H_a$. The complementary poset $\mathcal{K}^{(1)}_{comp} = \mathcal{K}^{(1)} \backslash \mathcal{K}^{(1)}_{link}$ yields a regular CW complex having the homotopy type of the connected complex manifold $C_{\mathcal{B}} = \mathbb{C}^d \backslash V_{\mathcal{B}}$.

In the special case of a complexified arrangement $\mathcal{B} = \mathcal{A}^{\mathbb{C}}$ (i.e., when $\mathcal{B}$ is given by real forms) the $\mathcal{K}^{(1)}_{comp}$ cell complex coincides with the one earlier described by Salvetti [S]. Salvetti's work implies that if two real arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ have isomorphic face posets, then $C_{\mathcal{A}_1^{\mathbb{C}}}$ and $C_{\mathcal{A}_2^{\mathbb{C}}}$ are homotopy equivalent. It follows from our analysis that $C_{\mathcal{A}_1^{\mathbb{C}}}$ and $C_{\mathcal{A}_2^{\mathbb{C}}}$ are then actually homeomorphic. The same is true for $V_{\mathcal{A}_1^{\mathbb{C}}}$ and $V_{\mathcal{A}_2^{\mathbb{C}}}$.

We use the combinatorial description of these cell complexes to determine the homology of the link $V_{\mathcal{B}} \cap S^{2d-1}$ via a simple Mayer-Vietoris argument and induction on n. This leads via Alexander duality to a new and quite elementary proof of the description of the structure of the cohomology algebra $\mathrm{H}^*(C_{\mathcal{B}}; \mathbb{Z})$ given by Brieskorn [Br] and Orlik & Solomon [OS] together with the combinatorial basis for it provided (via [OS]) by Björner [B1] and Jambu & Leborgne [JL]. We describe explicit bases for the homology of the link and for the cohomology algebra of the complement, which are exactly matched by Alexander duality. The spherical classes giving a basis for homology, together with a classical result of Whitehead, imply that the singularity link $V_{\mathcal{B}} \cap S^{2d-1}$ has the homotopy type of a wedge of spheres if $d \geq 4$.

The posets $\mathcal{K}^{(1)}$ of complex sign vectors can also be interpreted as defining an equivalence relation on $G_d(\mathbb{C}^n)$, the Grassmann variety of d-dimensional subspaces of $\mathbb{C}^n$. This induces a decomposition of $G_d(\mathbb{C}^n)$ which refines the "matroid stratification" of complex Grassmannians studied by Gel'fand, Goresky, MacPherson and Serganova [GGMS].

Our analysis of arrangements, its method to describe links and complements, and the computations of homology and cohomology are elementary and valid in greater generality than for complex arrangements. We outline a framework of 2-*pseudoarrangements* that provides sufficient generality to cover also some cases where the tools of algebraic geometry [Br] and of differential topology [F1] [GM] are not available. These objects are essentially the topologically deformed generalizations of the even codimension subspace arrangements considered by Goresky & MacPherson [GM]. Everything that is done for complex arrangements in this paper goes through for 2-pseudoarrangements, including cell complexes and (co)homology computations for links and complements, except for the precise sign patterns of the Orlik-Solomon relations in the cohomology algebra. (See Section 9 for even greater generality.)

2-pseudoarrangements give rise to combinatorial objects that we call 2-matroids. Examples include complex arrangements, arrangements of subspaces of real codimension 2 with even-dimensional intersections, and *complexified oriented matroids.* In this paper we show how the process of complexification can be described combinatorially, so that it applies to any oriented matroid (whether realizable or not). This extends the results of Gel'fand & Rybnikov [GR], who constructed the "Salvetti" part of the complexification of an oriented matroid. We also show that every complexified oriented matroid is represented by a 2-pseudoarrangement, so our cohomology computation applies.

The decomposition of $\mathbb{C}^d$ given by the fibers of the complex sign map $\mathbf{s}^{(1)}_{\mathcal{B}} : \mathbb{C}^d \longrightarrow \mathcal{K}^{(1)}_{\mathcal{B}}$ is in this paper called the $\mathbf{s}^{(1)}$*-stratification of* $\mathcal{B}$ (it depends on the choice of linear forms ℓ_a for $\mathcal{B}$). Other complex sign maps $\mathbf{s}$ are possible and give rise to a corresponding $\mathbf{s}$-stratification. In particular, the $\mathbf{s}^{(2)}$-stratification obtained by separate consideration of the real and the imaginary parts of $\ell_a(\mathbf{z})$ is useful, since it refines the $\mathbf{s}^{(1)}$-stratification and carries the structure of a real hyperplane arrangement (and an oriented matroid). However, the $\mathbf{s}^{(1)}$-stratification is the most economical one (being the coarsest) and also seems to be more intrinsic than the others. Most of the results of this paper have straightforward generalizations to arbitrary *combinatorial stratifications* of complex arrangements, and are treated in this generality. In fact, the simple underlying ideas, particularly that of the $\mathbf{s}^{(1)}$-stratification, can easily be adapted to arbitrary subspace arrangements.

We gratefully acknowledge the inspiration we received from papers by I.M. Gel'fand and G.L. Rybnikov [GR] on the use of $\{i, j, +, -\}$ sign vectors and by M. Falk [F1,F2] on the geometric duality between homology and cohomology classes in the link and the complement of an arrangement.

In parallel work, P. Orlik [O2] has developed ideas concerning stratifications of the $\mathbf{s}^{(2)}$ type for subspace arrangements.

2. Sign functions, cell complexes and stratifications.

In this section, we construct cell decompositions of the unit sphere S^{2d-1} in $\mathbb{C}^d$ derived from complex arrangements, and establish their basic properties. For this we formalize the concept of a *combinatorial stratification* and its face poset. We then show how such stratifications can be induced from the one-dimensional case. The special case of *induced stratifications* permits an encoding of the combinatorial structure in a family of sign vectors with the structure of a poset (partially ordered set).

We will make frequent use of the fact that finite regular *CW* complexes are completely encoded by their face posets, so that the combinatorics of the face poset gives us complete control of the topology. To fix notation and terminology we list a few key points concerning this encoding; for further details see Section 4.7 of [BLSWZ].

- All (simplicial and *CW*) complexes we consider are finite, and all *CW* complexes are regular. We will often refer to finite regular *CW* complexes simply as *cell complexes.*
- We denote cell complexes by Γ, whereas the letter Δ is reserved for simplicial complexes.
- The *face poset* P_Γ of a cell complex Γ is the poset of its closed cells, ordered by inclusion. The minimal elements are the vertices of Γ, and P_Γ has a combinatorial rank function given by $r(\sigma) = \dim(\sigma)$. The *augmented face poset* $0 \uplus P_\Gamma$ has a least element 0 (corresponding to the empty cell) adjoined. Deletion of 0 from an augmented face poset will be written without set brackets as $P\backslash 0$. The covering relation, written $X \lessdot Y$, means that $X < Y$ and no element is strictly between X and Y.
- Γ is determined by P_Γ up to homeomorphism, since the order complex $\Delta(P_\Gamma)$ (simplicial complex of chains) triangulates Γ.
- Construct $\widehat{P} := P \uplus \{0, \hat{1}\}$ by adjoining a minimal and a maximal element, 0 and $\hat{1}$ to $P = P_\Gamma$. The bounded poset $\widehat{P}$ is a lattice if and only if Γ has the *intersection property*, that is, if the intersection of two closed cells always is a closed cell or empty.
- A *CW poset* is the face poset of a cell complex. A *CW* poset P determines its cell complex $\Gamma(P)$ uniquely up to cellular homeomorphism.
- A cell complex homeomorphic to a ball or a sphere is *PL* (“*piecewise linear*”) if some triangulation of it has a piecewise linear homeomorphism with a simplex (respectively the boundary of a simplex). This is a combinatorial property, which only depends on the face poset.
- The boundary of every convex polytope is a *PL* sphere.
- If Γ has a subdivision that is *PL*, then Γ is *PL* as well.
- Let Γ be a *PL* sphere, whose face poset is $(P, \leq)$. Define the *opposite poset* $P^{op} := (P, \leq^{op})$ on the same set by

$$x \leq^{op} y \iff y \leq x.$$

  Then P^{op} is again the face poset of a regular *CW* sphere, the *opposite sphere* Γ^{op}, see [BLSWZ, Proposition 4.7.26]. In fact, Γ^{op} is the *dual block complex* [Mu, §64] of

Γ. The requirement that Γ is *PL* guarantees that the dual block complex is a cell complex. The barycentric subdivisions of Γ and Γ^{op} coincide with the chain complex $\Delta(P)$, so Γ^{op} is also *PL*.

The strata that we use in the following are convex cones: non-empty subsets of real vector spaces that are closed under taking linear combinations with positive coefficients. Furthermore, they are *relative-open*, that is, they are open subsets of their linear hulls. Relative-open convex cones can also be characterized as the solution sets of systems of homogeneous equations and strict homogeneous inequalities.

Definition 2.1. *A combinatorial stratification $\mathcal{K}$ of a complex arrangement $\mathcal{B}$ in $\mathbb{C}^d$ is a partition of $\mathbb{R}^{2d} \cong \mathbb{C}^d$ into finitely many subsets ("strata") that have the following properties:*
(i) *the strata are relative-open convex cones,*
(ii) *the intersections of the strata with the unit sphere S^{2d-1} in $\mathbb{C}^d$ are the open cells of a regular CW-decomposition $\Gamma_{\mathcal{K}}$ of S^{2d-1},*
(iii) *every hyperplane $H \in \mathcal{B}$ is a union of strata, that is, every $H \cap S^{2d-1}$ is a subcomplex of $\Gamma_{\mathcal{K}}$.*
The face poset of a combinatorial stratification $\mathcal{K}$ is defined to be the augmented face poset of the cell complex $\Gamma_{\mathcal{K}}$.

It follows easily from (i) that the strata are polyhedral, that is, they are defined by *finite* sets of equations and inequalities. However, note that the faces of such a cone are not necessarily strata, because they might be subdivided. The face poset of the stratifcation is isomorphic to the set of closures of strata, ordered by containment.

We now begin the construction of a special class of *induced* combinatorial stratifications, whose structure is derived from the case of dimension 1.

Definition 2.2. *An admissible sign function is a surjective map*

$$\mathbf{s} : \mathbb{C} \longrightarrow \Sigma,$$

where Σ is a finite set, which we call a set of complex signs, such that the following conditions hold:
(i) *for all $\sigma \in \Sigma$, the preimage $\mathbf{s}^{-1}(\sigma) = \{z \in \mathbb{C} : \mathbf{s}(z) = \sigma\}$ is a relative-open convex cone,*
(ii) *the set Σ contains a distinguished element 0, such that $\{0\} = \mathbf{s}^{-1}(0)$.*
There is canonical partial order on the set Σ of signs of an admissible sign function, with $\sigma \leq \tau$ if and only if $\mathbf{s}^{-1}(\sigma)$ is contained in the closure of $\mathbf{s}^{-1}(\tau)$. The unique minimal element of Σ is 0.

Equivalently, an admissible sign function is a surjective map from $\mathbb{C}$ to a finite set of signs such that the preimages of the signs form a combinatorial stratification of the 1-dimensional arrangement with $\{0\}$ as its only hyperplane. We can, in fact, identify the

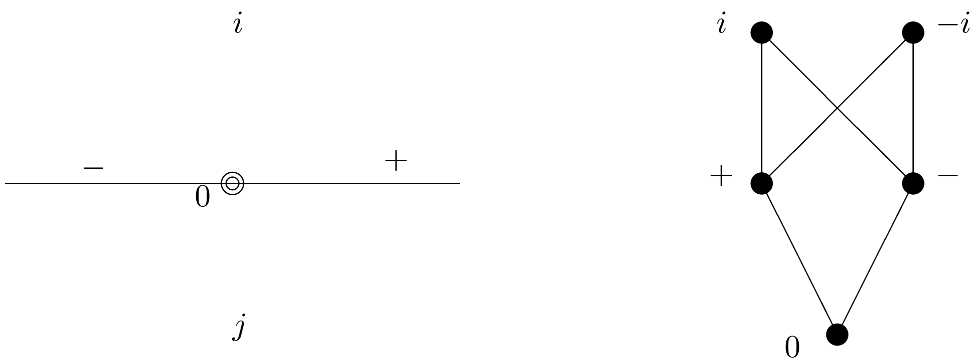


Figure 2.1: $\mathbf{s}^{(1)}$, and its poset of signs.

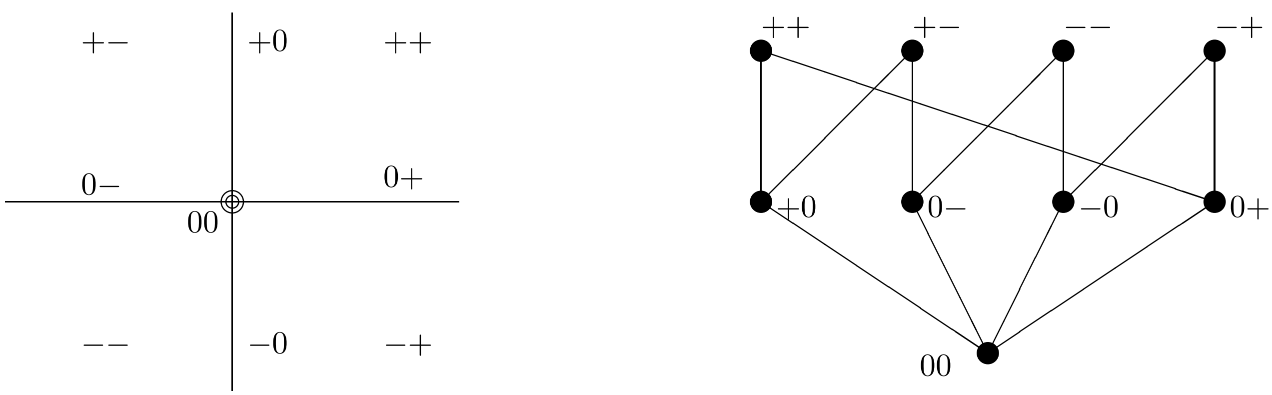


Figure 2.2: $\mathbf{s}^{(2)}$, and its poset of signs.

signs in Σ with the strata of the induced stratification. The augmented face poset of the induced (regular) cell decomposition of S^1 is $(\Sigma, \leq)$.

Examples 2.3. Denote the usual sign function on $\mathbb{R}$ by $\mathbf{s}^{\mathbb{R}}\colon \mathbb{R} \longrightarrow \{+,-,0\}$.

(1) On $\mathbb{C}$, we say that a complex number with positive imaginary part has sign "i", and a complex number with negative imaginary part has sign "j", where "$j = -i$". This extends $\mathbf{s}^{\mathbb{R}}$ to a *complex sign* function

$$\begin{aligned} \mathbf{s}^{(1)}: \quad \mathbb{C} &\longrightarrow \{i, j, +, -, 0\} \\ x+iy &\longmapsto \begin{cases} \mathbf{s}^{\mathbb{R}}(y)i & \text{for } \mathbf{s}^{\mathbb{R}}(y) \neq 0, \\ \mathbf{s}^{\mathbb{R}}(x) & \text{otherwise.} \end{cases} \end{aligned}$$

(2) A second possibility is to consider real and imaginary part separately, to get the sign

function

$$\begin{aligned} \mathbf{s}^{(2)}: \quad \mathbb{C} &\longrightarrow \{+,-,0\}^2 \\ x+iy &\mapsto (\mathbf{s}^{\mathbb{R}}(y), \mathbf{s}^{\mathbb{R}}(x)). \end{aligned}$$

We obtain partial orderings of the sets of signs $\{i,j,+,-,0\}$ and $\{+,-,0\}^2$ that correspond to inclusion of the closures of the strata. See Figures 2.1 and 2.2.

(3) More generally, the regular division of the complex plane into congruent regions by $k \geq 2$ rays from the origin (including the positive real axis) with successive angles $2\pi/k$ gives an admissible sign function. Figures 2.1 and 2.2 show the $k=2$ and $k=4$ cases.

The induced stratifications of complex arrangements are now obtained by associating a sign vector to every point of $\mathbb{C}^d$.

Definition 2.4. *Let $\mathcal{B} = \{H_1, \ldots, H_n\}$ be a complex arrangement in $\mathbb{C}^d$, given by functionals $\ell_a : \mathbb{C}^d \longrightarrow \mathbb{C}$ $(1 \leq a \leq n)$, and let $\mathbf{s} : \mathbb{C} \longrightarrow \Sigma$ be an admissible sign function on $\mathbb{C}$. Then the function*

$$\begin{aligned} \mathbf{s}_{\mathcal{B}}: \quad \mathbb{C}^d &\longrightarrow \Sigma^n \\ \mathbf{z} &\longmapsto (\mathbf{s}(\ell_1(\mathbf{z})), \ldots, \mathbf{s}(\ell_n(\mathbf{z}))) \end{aligned}$$

associates a complex sign vector with every point in $\mathbb{C}^d$.

The strata of the $\mathbf{s}$-stratification of $\mathcal{B}$ are the maximal subsets of $\mathbb{C}^d$ on which the map $\mathbf{s}_{\mathcal{B}}$ is constant.

The set $\mathbf{s}_{\mathcal{B}}(\mathbb{C}^d) \subseteq \Sigma^n$ of complex sign vectors inherits a partial order as a subset of Σ^n, which as a direct product is ordered component-wise.

This construction is analogous to the situation for real arrangements, where the usual sign function $\mathbf{s}^{\mathbb{R}}$: $\mathbb{R} \longrightarrow \{+,-,0\}$ defines the only combinatorial stratification of the real one-dimensional arrangement whose hyperplane is the origin, so that the construction in $\mathbb{R}^n$ of induced stratification and face poset is canonical.

In the complex case there are several choices involved. For a fixed sign function $\mathbf{s}$, the combinatorial structure of the $\mathbf{s}$-stratification depends on the choice of linear forms ℓ_a.

We remark that in Definition 2.4, one could alternatively allow a different choice of admissible sign function for each hyperplane $H_a = \ker(\ell_a)$. This might be of interest for unitary reflection groups, where it seems natural to associate the regular k_a sign function (Example 2.3(3)) with a hyperplane H_a whose reflection has order k_a.

Theorem 2.5. *Let $\mathcal{B}$ be an essential arrangement, and $\mathbf{s}$ an admissible sign function.*

Then the $\mathbf{s}$-stratification of $\mathcal{B}$ is combinatorial (Definition 2.1).

Its face poset is isomorphic to the poset of sign vectors $\mathbf{s}_{\mathcal{B}}(\mathbb{C}^d)$ (Definition 2.4).

Proof. Consider first the arrangement $\mathcal{B}^{[n]} = \{H_1^{[n]}, \ldots, H_n^{[n]}\}$ of coordinate hyperplanes in $\mathbb{C}^n$, with $H_a^{[n]} = \{\mathbf{z} \in \mathbb{C}^n : z_a = 0\}$. For this arrangement

$$\mathbf{s}^n : \mathbb{C}^n \longrightarrow \Sigma^n, \qquad (z_1, \ldots, z_n) \longmapsto (\mathbf{s}(z_1), \ldots, \mathbf{s}(z_n))$$

Figure 2.3. The $\mathbf{s}^{(1)}$-stratification of the complexified arrangement in $\mathbb{C}^2$ defined by $Q = wz(w - z)$.

clearly defines a combinatorial stratification.

Now the injective map

$$\mathbf{l} : \mathbb{C}^d \longrightarrow \mathbb{C}^n, \qquad \mathbf{z} \longmapsto (\ell_1(\mathbf{z}), \ldots, \ell_n(\mathbf{z}))$$

embeds $\mathbb{C}^d$ as a d-subspace $T_{\mathcal{B}} := \mathbf{l}(\mathbb{C}^d)$ of $\mathbb{C}^n$, so that $\mathbf{l}(H_a) = T_{\mathcal{B}} \cap H_a^{[n]}$ for $1 \leq a \leq n$. This induces a combinatorial stratification of the arrangement $\{T_{\mathcal{B}} \cap H_1^{[n]}, \ldots, T_{\mathcal{B}} \cap H_n^{[n]}\}$ in $T_{\mathcal{B}}$, which is (via $\mathbf{l}$) isomorphic to $\mathcal{B}$.

Furthermore, the face poset of the $\mathbf{s}$-stratification of $\mathcal{B}^{[n]}$ is given by Σ^n, ordered componentwise. This property is preserved when restricting to $T_{\mathcal{B}}$. □

We have shown that for a stratification $\mathcal{K}$ induced by an admissible sign function $\mathbf{s}$, the face poset of $\mathcal{K}$ is isomorphic to the poset $\mathbf{s}_{\mathcal{B}}(\mathbb{C}^d)$ of sign vectors. For the rest of this paper we will freely make the identification $\mathcal{K} = \mathbf{s}_{\mathcal{B}}(\mathbb{C}^d)$. In particular, for $i = 1, 2$ we will let $\mathcal{K}^{(i)}$ denote both the $\mathbf{s}^{(i)}$-stratification of $\mathcal{B}$ and its poset of complex sign vectors.

Theorem 2.6. *If $\mathcal{K}$ is a combinatorial stratification of an essential complex arrangement in $\mathbb{C}^d$, then the induced cell decomposition $\Gamma_{\mathcal{K}}$ of the unit sphere S^{2d-1} is PL.*

Proof. Every stratum of $\mathcal{K}$ is a relative-open polyhedral cone. Therefore, we can for every stratum construct a real hyperplane arrangement in which the stratum appears as a face. The union of all these arrangements is a real hyperplane arrangement $\mathcal{A}(\mathcal{K})$ whose cell complex $\Gamma_{\mathcal{A}(\mathcal{K})}$ is a refinement of $\Gamma_{\mathcal{K}}$.

But $\Gamma_{\mathcal{A}(\mathcal{K})}$ is polytopal: it is the boundary complex of $\mathcal{Z}_{\mathcal{A}}^*$, the polar of the zonotope of $\mathcal{A}(\mathcal{K})$, see [BLSWZ, Theorem 2.2.2]. Thus $\Gamma_{\mathcal{A}(\mathcal{K})}$ is *PL*, and so is therefore $\Gamma_{\mathcal{K}}$. □

A different proof of Theorems 2.5 and 2.6 for the $\mathbf{s}^{(1)}$-stratification is given in Section 8 (the "linear" case of the proof of Theorem 8.10).

For specific stratifications, like those induced by many admissible sign functions, stronger statements than that of Theorem 2.6 are possible. For this, we call a stratification of $\mathbb{C}^d$ *polytopal* if there is a polytope P in $\mathbb{R}^{2d} \cong \mathbb{C}^d$ (with 0 in its interior) such that the cones of the stratification are the cones $cone(F)$ over the faces F of P. An admissible sign function is *polytopal* if its stratification of $\mathbb{C}$ is polytopal, that is, if every cone in it is pointed. For example, $\mathbf{s}^{(2)}$ is polytopal, but $\mathbf{s}^{(1)}$ is not.

Theorem 2.7.

Let $\mathbf{s}$ be an admissible sign function that is polytopal, and let $\mathcal{B}$ be an essential arrangement in $\mathbb{C}^d$. Then the $\mathbf{s}$-stratification of $\mathcal{B}$ is polytopal.

Proof. We use the same setup as for the proof of Theorem 2.5. If $\mathbf{s}$ is polytopal (with 2-polytope P), then the $\mathbf{s}$-stratification of the boolean arrangement $\mathcal{B}^{[n]}$ of coordinate hyperplanes is polytopal as well: its polytope $P^{\langle n \rangle}$ is the convex hull of the n copies of P in the 1-dimensional coordinate subspaces of $\mathbb{C}^n$. Restriction to the subspace $T_{\mathcal{B}} = \mathbf{l}(\mathbb{C}^d)$ yields that the $\mathbf{s}$-stratification of $\mathcal{B}$ is polytopal as well (with polytope $P^{\langle n \rangle} \cap T_{\mathcal{B}}$). □

Corollary 2.8. *Let $\mathcal{B}$ be an essential complex arrangement, and let $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$ be (the face posets of) the $\mathbf{s}^{(1)}$- respectively the $\mathbf{s}^{(2)}$-stratification.*

(i) *$\hat{\mathcal{K}}^{(2)}$ is the face lattice of a convex polytope.*

(ii) *The cell complex $\Gamma_{\mathcal{K}^{(2)}}$ is a subdivision of the cell complex $\Gamma_{\mathcal{K}^{(1)}}$.*

Proof.

(i) The sign function $\mathbf{s}^{(2)}$ is polytopal, so this follows from Theorem 2.7.
More explicitly, the $\mathbf{s}^{(2)}$-stratification corresponds to the real hyperplane arrangement that is given by the hyperplanes

$$H_a^{\mathrm{Re}} = \{\mathbf{z} \in \mathbb{C}^d : \mathrm{Im}(\ell_a(\mathbf{z})) = 0\},$$
$$H_a^{\mathrm{Im}} = \{\mathbf{z} \in \mathbb{C}^d : \mathrm{Re}(\ell_a(\mathbf{z})) = 0\},$$

in $\mathbb{R}^{2d} = \mathbb{C}^d$. But real hyperplane arrangements are polytopal, as was mentioned in the proof of Theorem 2.6.

(ii) is induced from the corresponding property of the sign functions $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$: the $\mathbf{s}^{(2)}$-stratification of $\mathbb{C}$ is a subdivision of the $\mathbf{s}^{(1)}$-stratification. □

Note that the $\mathbf{s}^{(1)}$-stratifications are not polytopal, and they do not have the intersection property. This is intimately connected to the combinatorial properties of this coarse stratification, which will be further studied in Section 4. We refer to Figure 2.3, which depicts the $\mathbf{s}^{(1)}$-stratification of the complexification of the real arrangement given by $\{x, y, y-x\}$ in $\mathbb{R}^2$. Possibly the barycentric subdivision $\mathrm{sd}(\Gamma_{\mathcal{K}})$ is polytopal for every combinatorial stratification, but this is not clear even for the $\mathbf{s}^{(1)}$-stratification.

It would be interesting to know whether the $\mathbf{s}^{(1)}$-stratifications of complex arrangements are shellable cell complexes. At the moment, this is an open problem even for the complexification of a real arrangement $\mathcal{A}$ (where $\mathcal{K}^{(1)} = \mathcal{L} \circ i\mathcal{L}$, and $\mathcal{L} \subseteq \{+,-,0\}^n$ is the face poset of $\mathcal{A}$, see Section 5).

Gel'fand, Goresky, MacPherson and Serganova [GGMS] have studied the decomposition of the Grassmann manifold $G_d(\mathbb{C}^n)$ of all d-dimensional subspaces of $\mathbb{C}^n$ into "matroid strata". A similar matroid decomposition exists for real Grassmannians $G_d(\mathbb{R}^n)$, and it is subdivided into "oriented matroid strata". See [BLSWZ, Section 2.4] for a discussion of this topic with further details and references. Here we point out that the matroid stratification of the complex Grassmannian $G_d(\mathbb{C}^n)$ is subdivided into "$\mathcal{K}^{(1)}$-strata" in the following sense.

For each d-dimensional subspace $V \subseteq \mathbb{C}^n$, let $\mathcal{K}^{(1)}(V) := \{\mathbf{s}^{(1)}(\mathbf{z}) : \mathbf{z} \in V\}$, where $\mathbf{s}^{(1)}(\mathbf{z})$ is the sign vector with component $\mathbf{s}^{(1)}(z_a)$ in position a, for $1 \le a \le n$. In the generic case where V is not contained in a coordinate hyperplane, $\mathcal{K}^{(1)}(V)$ is the $\mathcal{K}^{(1)}$ face poset of a certain arrangement of n hyperplanes in $V \cong \mathbb{C}^d$, namely the intersections of V with the coordinate hyperplanes given by the forms $z_a = 0$, for $1 \le a \le n$. The $\mathcal{K}^{(1)}$-*stratification* of $G_d(\mathbb{C}^n)$ is given by the equivalence relation of having the same $\mathcal{K}^{(1)}$-face poset. We have not attempted to study the topology of the $\mathcal{K}^{(1)}$-strata, but hope to return to this question.

3. Cell Complexes for the Link and the Complement.

Regular cell complexes are completely determined (up to homeomorphism) by their face posets. In this section, we use this elementary fact and a simple retraction argument to obtain combinatorial descriptions of cell complexes for the complements and links of complex arrangements. The following general fact will be used.

Proposition 3.1. *Let P be the face poset of a PL regular cell decomposition Γ of the k-sphere. Let P_0 be the order ideal corresponding to a subcomplex $\Gamma_0 \subseteq \Gamma$. Then $(P \backslash P_0)^{op}$ is the face poset of a regular CW complex Γ_{comp} which is homotopy equivalent to $|\Gamma| \backslash |\Gamma_0|$.*

Proof. The fact that Γ is PL guarantees the existence of Γ^{op}, the "opposite" regular cell decomposition of the k-sphere with face poset P^{op}, as was explained at the beginning of Section 2. Clearly, $(P \backslash P_0)^{op}$ is the face poset of a subcomplex Γ_{comp} of Γ^{op}.

Finally, $|\Delta(P \backslash P_0)| \cong |\Gamma_{comp}|$ is homotopy equivalent to $|\Delta(P)| \backslash |\Delta(P_0)| \cong |\Gamma| \backslash |\Gamma_0|$ as a consequence of the next lemma. □

Retraction Lemma 3.2. ([BLSWZ, Lemma 4.7.27], [Mu, Lemma 70.1])
Let P be a finite poset, and let $P = P' \uplus P''$ be a partition into two parts. Then $|\Delta(P')|$ is a strong deformation retract of $|\Delta(P)| \backslash |\Delta(P'')|$.

Since the face poset of a combinatorial stratification is a PL sphere (Theorem 2.6), Proposition 3.1 is directly applicable. Before stating the conclusion formally we will make another definition.

Definition 3.3. *Let $\mathcal{K}$ be the face poset of a combinatorial stratification of the complex arrangement $\mathcal{B}$ in $\mathbb{C}^d$.*

The link poset is the subposet $\mathcal{K}_{link} \subseteq \mathcal{K}$ of those strata that are contained in $V_{\mathcal{B}} = H_1 \cup \ldots \cup H_n$.

The complement poset is $\mathcal{K}_{comp} := \mathcal{K} \backslash \mathcal{K}_{link}$, corresponding to those strata that are contained in $C_{\mathcal{B}} = \mathbb{C}^d \backslash V_{\mathcal{B}}$.

Note that by construction $\mathcal{K}_{link}$ is the augmented face poset of a pure, $(2d-3)$-dimensional complex. Thus if $r : \mathcal{K} \longrightarrow \{0, 1, \ldots, 2d\}$ denotes the poset rank function on $\mathcal{K}$, then $\mathcal{K}_{link}$ is an order ideal whose maximal elements all have rank $2d-2$.

Similarly, $\mathcal{K}_{comp}$ is a filter in $\mathcal{K}$. In general the minimal elements of $\mathcal{K}_{comp}$ do not all have the same rank in $\mathcal{K}$.

Lemma 3.4. *Let $\mathcal{B}$ be an essential complex arrangement,* $\mathbf{s}$ *an admissible sign function, and let $\mathcal{K} \subseteq \Sigma^n$ be the face poset of the* $\mathbf{s}$*-stratification of $\mathcal{B}$. Then*

$$\mathcal{K}_{link} = \{X \in \mathcal{K} : X_a = 0 \text{ for some } a \in [n]\}$$

and

$$\mathcal{K}_{comp} = \mathcal{K} \backslash \mathcal{K}_{link} = \mathcal{K} \cap (\Sigma \backslash 0)^n.$$

Thus, in the induced case we have an entirely combinatorial encoding of the data. See Figure 2.3 for an illustration.

Theorem 3.5. *Let $\mathcal{K}$ be the face poset of a combinatorial stratification of a complex arrangement $\mathcal{B}$, and let $\Gamma := \Gamma_{\mathcal{K}}$ be the corresponding CW-sphere. Then,*

(i) *$\mathcal{K}_{link} \backslash 0$ is the face poset of a subcomplex of Γ that is homeomorphic to the link $D_{\mathcal{B}} := V_{\mathcal{B}} \cap S^{2d-1}$,*

(ii) *$(\mathcal{K}_{comp})^{op}$ is the face poset of a regular CW complex Γ_{comp} that is homotopy equivalent to the complement $C_{\mathcal{B}}$.*

Proof. This follows from Proposition 3.1, together with the definition of a combinatorial stratification and Theorem 2.6. We also use that $C_{\mathcal{B}} \cong (C_{\mathcal{B}} \cap S^{2d-1}) \times \mathbb{R}$, which shows that $C_{\mathcal{B}}$ retracts to its intersection with S^{2d-1}. □

It is clear from the preceding that the space $|\Delta(\mathcal{K}\backslash 0)| \backslash |\Delta(\mathcal{K}_{link}\backslash 0)| \cong S^{2d-1} \backslash D_{\mathcal{B}}$ is homeomorphic to $C_{\mathcal{B}} \cap S^{2d-1}$. Also, $V_{\mathcal{B}} \cong \mathrm{cone}^{\circ}(D_{\mathcal{B}})$, where $\mathrm{cone}^{\circ}(T)$ denotes the open cone over a space T, i.e., $\mathrm{cone}(T)$ minus its base T. Hence, we can draw the following conclusion.

Proposition 3.6. *Let $\mathcal{K}$ be the face poset of a combinatorial stratification of a complex arrangement $\mathcal{B}$ in $\mathbb{C}^d$. Then:*

(i) *the complement $C_{\mathcal{B}}$ is homeomorphic to*

$$(|\Delta(\mathcal{K}\backslash 0)| \setminus |\Delta(\mathcal{K}_{link}\backslash 0)|) \times \mathbb{R}.$$

(ii) *the variety $V_{\mathcal{B}}$ is homeomorphic to*

$$\mathrm{cone}^{\circ}\left(|\Delta(\mathcal{K}\backslash 0)|\right).$$

Corollary 3.7. *The face poset $\mathcal{K}$ determines the complement and the variety of a complex arrangement up to homeomorphism.*

We do not know whether $\mathcal{K}$ determines the complement up to diffeomorphism. For complex arrangements whose matroid has a connected realization space (over $\mathbb{C}$) this follows from a result by Randell [R].

4. Combinatorics of the $\mathbf{s}^{(1)}$-Stratification.

Let $\mathcal{B}$ be an essential complex arrangement in $\mathbb{C}^d$, and let $\mathcal{K}^{(1)} = \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbb{C}^d) \subseteq \{i, j, +, -, 0\}^n$ be the face poset of its $\mathbf{s}^{(1)}$-stratification. The combinatorics of the face poset $\mathcal{K}^{(1)}$ is very interesting. In general, the structure is more complicated than might be suggested by the case of complexified arrangements (see Section 5).

In Theorem 4.3, we will list some basic combinatorial properties satisfied by $\mathcal{K}^{(1)}$. For that, we need some operations on complex sign vectors, as follows. The canonical partial order on the set of complex signs (Figure 2.1) is presumed throughout.

Definition 4.1 *Let $Z, W \in \{i, j, +, -, 0\}^n$ be complex sign vectors.*

(i) *The real set of Z is*

$$\mathrm{Re}(Z) := \{a \in [n] : Z_a \in \{+, -, 0\}\},$$

the zero set of Z is

$$\mathrm{Ze}(Z) := \{a \in [n] : Z_a = 0\}.$$

Hence, $\mathrm{Ze}(Z) \subseteq \mathrm{Re}(Z) \subseteq [n]$.

The support of Z is the pair of sets $\mathrm{supp}(Z) = (\mathrm{Ze}(Z), \mathrm{Re}(Z))$.

(ii) *A sign vector X will be called real if it lies in $\{+, -, 0\}^n$, that is, if $\mathrm{Re}(X) = [n]$. A sign vector Z is imaginary if it lies in $\{i, j, 0\}^n$, that is, if $\mathrm{Re}(Z) = \mathrm{Ze}(Z)$. In this case we can write it as $Z = iY$, for a real sign vector Y.*

(iii) *The composition of sign vectors Z and W is the sign vector $Z \circ W \in \{i, j, +, -, 0\}^n$ defined component-wise by*

$$(Z \circ W)_a := Z_a \circ W_a = \begin{cases} W_a & \text{if } W_a > Z_a, \\ Z_a & \text{otherwise.} \end{cases}$$

(iv) *The separation set of Z and W is*

$$\begin{aligned} S(Z, W) &:= \{a \in [n] : Z_a = -W_a \neq 0\} \\ &= \{a \in [n] : (Z \circ W)_a \neq (W \circ Z)_a\}. \end{aligned}$$

Later in the paper the following elementary properties of the composition operation will be useful. They follow easily from the partial order on the signs in $\{i, j, +, -, 0\}$, because composition is defined componentwise.

Lemma 4.2.

(i) *Composition is associative: for arbitrary sign vectors $Z, Z', Z'' \in \{i, j, +, -, 0\}^n$ we have*

$$Z \circ (Z' \circ Z'') = (Z \circ Z') \circ Z''.$$

(ii) *Real and imaginary sign vectors commute: for real sign vectors $X, Y \in \{+, -, 0\}^n$, we have*

$$X \circ iY = iY \circ X.$$

More generally, arbitrary sign vectors Z and W commute (that is, $W \circ Z = Z \circ W$) if and only if $S(Z,W)=\emptyset$.

(iii) *$W \circ Z = Z$ if and only if $W \le Z$.*
$W \circ Z = W$ if and only if $\operatorname{supp}(Z) \subseteq \operatorname{supp}(W)$.

The following theorem lists a set of basic combinatorial properties possessed by the $\mathcal{K}^{(1)}$-poset of a complex arrangement.

Theorem 4.3. *Let $\mathcal{K}^{(1)} \subseteq \{i,j,+,-,0\}^n$ be the face poset of an $\mathbf{s}^{(1)}$-stratified complex arrangement. Then:*

(0) $0 \in \mathcal{K}^{(1)}$.

(1) *$Z \in \mathcal{K}^{(1)}$ implies that $-Z \in \mathcal{K}^{(1)}$.*

(1′) *$Z \in \mathcal{K}^{(1)}$, $Z_a \neq 0$ and $\sigma \in \{i,j,+,-\}$ implies that there exists a vector $W \in \mathcal{K}^{(1)}$ such that $\mathrm{Ze}(W) = \mathrm{Ze}(Z)$ and $W_a = \sigma$.*

(2) *$Z, W \in \mathcal{K}^{(1)}$ implies that $Z \circ W \in \mathcal{K}^{(1)}$.*

(3) *$Z, W \in \mathcal{K}^{(1)}$ with $a \in S(Z,W)$ implies that there exists a vector $U \in \mathcal{K}^{(1)}$ such that $U_a < Z_a, W_a$ and $U_b = (Z \circ W)_b = (W \circ Z)_b$ for all $b \notin S(Z,W)$.*

(3′) *$Z, W \in \mathcal{K}^{(1)}$ with $W \not\le Z$ and $Z \circ W = Z$ implies that there exists a vector $Z' \in \mathcal{K}^{(1)}$ such that $Z' < Z$ and $Z'_b = Z_b$ for all $b \notin S(Z,W)$.*

Proof. The conditions (0), (1) and (1′) are obvious via rescaling. For (2), use that for $\mathbf{z}, \mathbf{w} \in \mathbb{C}^d$, we get

$$\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z} + \epsilon \mathbf{w}) \quad = \quad \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z}) \circ \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{w}),$$

when $\epsilon > 0$ is small enough.

For (3), let $\mathbf{z}, \mathbf{w} \in \mathbb{C}^d$ be points such that $Z = \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z})$ and $W = \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{w})$. Then $z := \ell_a(\mathbf{z})$ and $w := \ell_a(\mathbf{w})$ are complex numbers with opposite nonzero $\mathbf{s}^{(1)}$-sign, so they are "separated" (on their connecting line) by a complex number with a smaller $\mathbf{s}^{(1)}$-sign: there exists a λ, with $0 < \lambda < 1$, such that $\mathbf{s}^{(1)}(\lambda z + (1-\lambda) w) < \mathbf{s}^{(1)}(z), \mathbf{s}^{(1)}(w)$. Now put

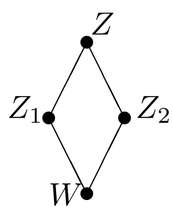
$$U := \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{u}) \quad \text{for} \quad \mathbf{u} := \lambda \mathbf{z} + (1-\lambda)\mathbf{w}.$$

Then the first condition of (3) on U is clear, and the second one follows from the convexity of the strata of $\mathbf{s}^{(1)}$.

For (3′), choose again $\mathbf{z}, \mathbf{w} \in \mathbb{C}^d$ with $Z = \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z})$ and $W = \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{w})$. The condition $Z \circ W = Z$ implies that $\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z} + \epsilon(\mathbf{w} - \mathbf{z})) = \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z})$ for small $\epsilon > 0$. Let $\epsilon' > 0$ be minimal such that $\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z} + \epsilon(\mathbf{w} - \mathbf{z})) \neq Z$, and set $Z' := \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z} + \epsilon'(\mathbf{w} - \mathbf{z}))$. Since strata are relative-open, this is well-defined; and from the definition of the ordering of the signs we get $Z' < Z$. Now from $W \not\le Z$ we derive $\epsilon' < 1$. Thus the assumption $Z \circ W = Z$, together with the convexity of the strata, implies that $W_b \le Z_b$, and hence $Z'_b = Z_b$, for all $b \notin S(Z,W)$. □

The properties of $\mathcal{K}^{(1)}$ listed in Theorem 4.3 are analogues of the "covector axioms" for oriented matroids of Edmonds & Mandel [EM] [BLSWZ, Sections 3.7, 4.1]. In fact, they can be used to define "2-matroids" that abstract the combinatorial structure of $\mathbf{s}^{(1)}$-stratified complex arrangements – see Section 9.4. A different, more combinatorial proof for them will be given in Proposition 8.12.

Proposition 4.4.
(i) *$\mathcal{K}^{(1)}$ is a ranked poset of rank $2d$. Its unique minimal element is* 0.
(ii) *For $Z \in \mathcal{K}^{(1)}$, the rank $\rho(Z)$ is the dimension of the (linear span of) the stratum $\{\mathbf{z} \in \mathbb{C}^d : \mathbf{s}^{(1)}(\mathbf{z}) = Z\}$ of Z.*
(iii) *The maximal elements of $\mathcal{K}^{(1)}$ are the sign vectors in $\mathcal{K}^{(1)} \cap \{i, j\}^n$.*
(iv) *Every interval $[W, Z]$ in $\mathcal{K}^{(1)} \cup \{\hat{1}\}$ of length $\rho Z - \rho W = 2$ has the form*



Proof. This follows from the fact that $\mathcal{K}^{(1)}$ is the augmented face poset of a $(2d-1)$-sphere. For part (iii) we use that every point $\mathbf{z} \in \mathbb{C}^d$ in general position has $\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z}) \in \{i, j\}^n$, together with Theorem 4.3(2). $\square$

There are two matroids associated to (the $\mathcal{K}^{(1)}$-stratification of) a complex arrangement $\mathcal{B}$:

- The matroid $M = M(\mathcal{B})$ is the matroid of $\mathbb{C}$-dependencies of the linear forms ℓ_a. Its lattice of flats is the intersection lattice L of $\mathcal{B}$. It can be reconstructed from $\mathcal{K}^{(1)}$ as

$$L = \{\mathrm{Ze}(W) : W \in \mathcal{K}^{(1)}\},$$

ordered by inclusion. Here $r(M) = d$, because $\mathcal{B}$ is assumed to be essential.

- The second matroid is the matroid $M^{\mathbb{R}}$ of linear dependencies with real coefficients of the complex forms ℓ_a. Its intersection lattice is given by $\mathcal{K}^{(1)}$ as

$$L^{\mathbb{R}} = \{\mathrm{Re}(W) : W \in \mathcal{K}^{(1)}\},$$

ordered by inclusion. This is the matroid of the real arrangement

$$\mathcal{B}^{\mathbb{R}} = \{H_1^{\mathrm{Re}}, \ldots, H_n^{\mathrm{Re}}\},$$

where

$$H_a^{\mathrm{Re}} = \{\mathbf{z} \in \mathbb{C}^d : \mathrm{Im}(\ell_a) = 0\}.$$

From this we see that $d = r(M) \le r(M^{\mathbb{R}}) \le 2d$. There is in fact an oriented matroid $\mathcal{M}^{\mathbb{R}}$ associated to $\mathcal{B}^{\mathbb{R}}$, whose underlying matroid is $M^{\mathbb{R}}$ – its covectors can be read off from $\mathcal{K}^{(1)}$ as the "imaginary parts" of the vectors in $\mathcal{K}^{(1)}$.

Both intersection lattices L and $L^{\mathbb{R}}$ are combined in the $\mathbf{s}^{(1)}$-*intersection lattice* $L^{(1)}$: the lattice of all intersections of subspaces in the set

$$\{H_1, \ldots, H_n, H_1^{\mathrm{Re}}, \ldots, H_n^{\mathrm{Re}}\},$$

(ordered by reverse inclusion). This is a semimodular (hence graded) lattice of length $2d$. Its rank function is given by the real codimension of the corresponding subspace. The hyperplanes H_a^{Re} are its atoms (rank 1), whereas the subspaces H_a appear as its join-irreducible elements of rank 2. Both L and $L^{\mathbb{R}}$ are contained in $L^{(1)}$ as sublattices.

The support map allows us to derive $L^{(1)}$ from $\mathcal{K}^{(1)}$, since $L^{(1)}$ is canonically isomorphic to the *support lattice* of $\mathcal{K}^{(1)}$, that is, the set $\mathrm{supp}(\mathcal{K}^{(1)}) := \{\mathrm{supp}(Z) : Z \in \mathcal{K}^{(1)}\}$, ordered componentwise by inclusion. The isomorphism is given by

$$(\mathrm{Ze}(Z), \mathrm{Re}(Z)) \quad \longmapsto \quad \bigcap_{a \in \mathrm{Ze}(Z)} H_a \ \cap \bigcap_{a \in \mathrm{Re}(Z)} H_a^{\mathrm{Re}},$$

its inverse by

$$V \ \longmapsto \ \mathrm{supp}(\mathbf{s}_{\mathcal{B}}^{(1)}(\mathbf{v})) \qquad \text{for some generic } \mathbf{v} \in V.$$

The following result describes a further aspect of the connection between the two matroids M and $M^{\mathbb{R}}$ on the same ground set $[n]$, yielding a canonical injection $L \hookrightarrow L^{\mathbb{R}}$. For this recall (see [W]) that $N \longrightarrow M$ is a *strong map* between two matroids on the same ground set if every flat of M is also a flat of N.

Proposition 4.5.

$$M^{\mathbb{R}} \quad \longrightarrow \quad M$$

is a strong map of matroids.

Proof. Let $\mathrm{Ze}(Z)$ be a flat of M, and choose $\mathbf{z} \in \mathbb{C}^d$ with $\mathbf{s}^{(1)}(\mathbf{z}) = Z$. Now consider the finite set of complex numbers

$$e^{i\epsilon}\ell_1(\mathbf{z}), \ldots, e^{i\epsilon}\ell_n(\mathbf{z}).$$

For small enough $\epsilon > 0$, none of them is real and nonzero. Therefore

$$\mathrm{Re}(\mathbf{s}^{(1)}(e^{i\epsilon}\mathbf{z})) = \mathrm{Ze}(\mathbf{s}^{(1)}(e^{i\epsilon}\mathbf{z})) = \mathrm{Ze}(\mathbf{s}^{(1)}(\mathbf{z})) = \mathrm{Ze}(Z),$$

and we have $Z' := \mathbf{s}^{(1)}(e^{i\epsilon}\mathbf{z}) \in \mathcal{K}^{(1)}$ with $\mathrm{Re}(Z') = \mathrm{Ze}(Z)$. Hence, $\mathrm{Ze}(Z)$ is also a flat of $M^{\mathbb{R}}$. $\square$

We close this section with a rather technical lemma that will be useful later.

Lemma 4.6. *Let $\mathcal{K}^{(1)}$ be the face poset of a complex arrangement $\mathcal{B}$, in its $\mathbf{s}^{(1)}$-stratification. Let $A = \{a_1, \ldots, a_k\}_<$ be an ordered subset of $[n]$, and call a chain*

$$Z = Z^0 < Z^1 < \ldots < Z^k$$

an A-chain if

$$Z^s(a_t) = \begin{cases} i & \text{for } t \leq s, \\ + & \text{for } t > s, \end{cases}$$

for all $0 \leq s \leq k$ *and* $1 \leq t \leq k$.

(i) *If A is dependent in the matroid $M^{\mathbb{R}}$ of $\mathcal{B}$, then there is no A-chain in $\mathcal{K}^{(1)}$.*

(ii) *If A is an independent set in $M^{\mathbb{R}}$, and if $Z \in \mathcal{K}^{(1)}$ has rank $\rho(Z) = 2d-k$ and satisfies $Z_{a_t} = +$ for $1 \leq t \leq k$, then there is a unique A-chain in $\mathcal{K}^{(1)}$ that starts at Z.*

Proof.

(i) Considering a suitable subchain, we may assume that A is a circuit of $M^{\mathbb{R}}$. But this means that there is a relation

$$\ell_{a_1} = \sum_{j=2}^{k} \alpha_j \ell_{a_j} \qquad (\alpha_j \in \mathbb{R}),$$

so that if $\ell_{a_j}(\mathbf{z}) \in \mathbb{R}$ for $2 \leq j \leq k$, then $\ell_{a_1}(\mathbf{z}) \in \mathbb{R}$. Hence Z^1 cannot exist as required.

(ii) By induction on k, it suffices to see that Z^1 as required exists and is unique with $\rho(Z^1) = 2d - k + 1$. So Z^1 has to be a full-dimensional stratum in the subspace $V := H^{\mathrm{Re}}_{a_2} \cap \ldots \cap H^{\mathrm{Re}}_{a_k}$ of dimension $\dim(V) = 2d - k + 1$. In V, $H^{\mathrm{Re}}_{a_1} \cap V$ is a hyperplane, which is spanned by the stratum Z. Thus there are exactly two strata Z^1 and W^1 in V that cover Z (on the two sides of $H^{\mathrm{Re}}_{a_1} \cap V$ in V), and $Z^1_{a_1} = i$ and $W^1_{a_1} = j$, or conversely. □

A more formal, axiomatic approach is possible that uses only the properties of Theorem 4.3 to derive all the other results of this section. This will partially be carried out in Section 8, see also [Z2].

5. Complexified Arrangements.

We will now turn our attention to the case of a complexified arrangement, which has special combinatorial structure. We will show that in this case the cell complex $\Gamma^{(1)}_{comp}$ of Theorem 3.5 specializes to the complex earlier described by Salvetti [S]. The $\mathbf{s}^{(1)}$-stratification, with the embedding of the Salvetti complex as a subcomplex, is determined by the face lattice of the real arrangement $\mathcal{A}$. Therefore the topology of the complement and the variety of a complexified arrangement (up to homeomorphism) can be studied in entirely combinatorial terms, namely in terms of the oriented matroid $\mathcal{L}_{\mathcal{A}}$ of $\mathcal{A}$. In Section 8 we will show that this analysis can be extended to *any* oriented matroid (whether realizable or not), by "formal complexification".

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement in $\mathbb{R}^d$, given by real linear forms $\ell_a \in (\mathbb{R}^d)^*$. The *complexified arrangement* $\mathcal{A}^{\mathbb{C}} = \{H_1^{\mathbb{C}}, \ldots, H_n^{\mathbb{C}}\}$ in $\mathbb{C}^d$ is given by the same forms: $H_a^{\mathbb{C}} = \ker_{\mathbb{C}^d}(\ell_a)$. The face poset $\mathcal{L}_{\mathcal{A}} = \mathbf{s}_{\mathcal{A}}(\mathbb{R}^d) \subseteq \{+,-,0\}^n$ of $\mathcal{A}$ is the collection of position vectors arising from $\mathbf{s}_{\mathcal{A}} : \mathbf{x} \mapsto (\mathbf{s}^{\mathbb{R}}(\ell_1(\mathbf{x})), \ldots, \mathbf{s}^{\mathbb{R}}(\ell_n(\mathbf{x})))$, cf. Definition 2.4 and Example 2.3. It is isomorphic to the augmented face poset of the regular cell decomposition of S^{d-1} induced by $\mathcal{A}$. See [BLSWZ, Chapter 4] for a detailed treatment of such posets $\mathcal{L}_{\mathcal{A}}$.

Theorem 5.1. *Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a real arrangement with face poset $\mathcal{L} \subseteq \{+,-,0\}^n$, and let $\mathcal{A}^{\mathbb{C}}$ be its complexification, stratified (for the same choice of real forms ℓ_a) with face posets $\mathcal{K}^{(1)} \subseteq \{i,j,+,-,0\}^n$ and $\mathcal{K}^{(2)} \subseteq \{+,-,0\}^{2n}$. Then:*

(i) $\mathcal{K}^{(1)} = \mathcal{L}\circ i\mathcal{L} = \{X\circ iY : X, Y \in \mathcal{L}\}$. *That is, $\mathcal{K}^{(1)}$ can be constructed from $\mathcal{L}$ using composition of sign vectors.*

(ii) *Let* $\mathrm{Int}(\mathcal{L}) = \{[Y,X] : Y \leq X\}$ *denote the set of intervals in $\mathcal{L}$. Then the mapping $[Y,X] \mapsto X\circ iY$ defines a bijection* $\mathrm{Int}(\mathcal{L}) \leftrightarrow \mathcal{K}^{(1)}$.

(iii) *The rank function of $\mathcal{K}^{(1)}$ is given by $\rho^{(1)}(X\circ iY) = \rho(X) + \rho(Y)$ for $Y \leq X$, where ρ denotes poset rank in $\mathcal{L}$.*

(iv) $\mathcal{K}^{(2)} \cong \mathcal{L}\times\mathcal{L} = \{(Y,X) : X, Y \in \mathcal{L}\}$. *That is, $\mathcal{K}^{(2)}$ is isomorphic to the poset product of $\mathcal{L}$ with itself.*

(v) *The rank function of $\mathcal{K}^{(2)}$ is given by $\rho^{(2)}(Y,X) = \rho(X) + \rho(Y)$.*

(vi) *The map $\phi : \mathcal{K}^{(2)} \longrightarrow \mathcal{K}^{(1)}$, $(Y,X) \longmapsto X\circ iY$ is an order-preserving surjection.*

Proof. For $\mathbf{x} + i\mathbf{y} \in \mathbb{C}^d$, with $\mathbf{s}_{\mathcal{A}}(\mathbf{x}) = X$ and $\mathbf{s}_{\mathcal{A}}(\mathbf{y}) = Y$, we compute (cf. Example 2.3):

$$\begin{aligned}
\mathbf{s}^{(1)}_{\mathcal{A}^{\mathbb{C}}}(\mathbf{x}+i\mathbf{y}) &= (\mathbf{s}^{(1)}(\ell_1(\mathbf{x}+i\mathbf{y})), \ldots, \mathbf{s}^{(1)}(\ell_n(\mathbf{x}+i\mathbf{y}))) \\
&= (\mathbf{s}^{(1)}(\ell_1(\mathbf{x})+i\ell_1(\mathbf{y})), \ldots, \mathbf{s}^{(1)}(\ell_n(\mathbf{x})+i\ell_n(\mathbf{y}))) \\
&= (\mathbf{s}^{\mathbb{R}}(\ell_1(\mathbf{x}))\circ i\mathbf{s}^{\mathbb{R}}(\ell_1(\mathbf{y})), \ldots, \mathbf{s}^{\mathbb{R}}(\ell_n(\mathbf{x}))\circ i\mathbf{s}^{\mathbb{R}}(\ell_n(\mathbf{y})) \\
&= \mathbf{s}_{\mathcal{A}}(\mathbf{x})\circ i\mathbf{s}_{\mathcal{A}}(\mathbf{y}) = X\circ iY.
\end{aligned}$$

This computation shows that

$$\mathcal{K}^{(1)} \;=\; \mathbf{s}^{(1)}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{C}^d) \;=\; \mathcal{L}\circ i\mathcal{L}.$$

Furthermore, we observe that $X\circ iY = (Y\circ X)\circ iY$ with $Y \leq Y\circ X$. Here $X\circ iY$ completely determines Y and $Y\circ X$, because they can be read off componentwise. Therefore we have a bijection between $\mathcal{K}^{(1)}$ and the set of intervals in $\mathcal{L}$, as claimed.

An analogous computation to the above establishes $\mathbf{s}^{(2)}_{\mathcal{A}^{\mathbb{C}}}(\mathbf{x}+i\mathbf{y}) = (Y,X)$, and thus $\mathcal{K}^{(2)} = \mathcal{L}\times\mathcal{L}$. The rank function of $\mathcal{K}^{(2)}$ can be deduced from this.

To see that the map $\mathcal{K}^{(2)} \longrightarrow \mathcal{K}^{(1)}$ is order-preserving, suppose that $(Y',X')_a \leq (Y,X)_a$, i.e., $Y'_a \leq Y_a$ and $X'_a \leq X_a$.
Case 1: If $Y'_a = Y_a = 0$, then $X'_a\circ iY'_a = X'_a \leq X_a = X_a\circ iY_a$.
Case 2: If $Y'_a = Y_a \neq 0$, then $X'_a\circ iY'_a = iY'_a = iY_a = X_a\circ iY_a$.
Case 3: If $Y'_a = 0, Y_a \neq 0$, then $X'_a\circ iY'_a = X'_a < iY_a = X_a\circ iY_a$.

It remains to prove part (iii). For this, first observe that

$$\phi(\mathbf{s}^{(2)}(\mathbf{z})) \;=\; \mathbf{s}^{(1)}(\mathbf{z}),$$

for all $\mathbf{z} \in \mathbb{C}^d$. Geometrically this means that the $\mathbf{s}^{(1)}$-stratum of $X\circ iY$ is subdivided into the $\mathbf{s}^{(2)}$-strata of all the pairs (Y',X') such that $X'\circ iY' = X\circ iY$. Since poset rank in $\mathcal{K}^{(1)}$ and in $\mathcal{K}^{(2)}$ is equal to the dimension of the corresponding stratum, this implies in view of part (v) that

$$\rho^{(1)}(X\circ iY) \;=\; \max\{\rho(X') + \rho(Y') : X'\circ iY' = X\circ iY\}.$$

Now, suppose that $Y \leq X$ in $\mathcal{L}$. Then $X'\circ iY' = X\circ iY$ implies that $Y' = Y$ and $Y'\circ X' = Y\circ X = X$, and hence

$$\rho(X') + \rho(Y') \;\leq\; \rho(Y'\circ X') + \rho(Y') \;=\; \rho(X) + \rho(Y).$$

Thus the maximum is achieved for the pair $Y \leq X$, which proves part (iii). ☐

Theorem 5.1 shows that for every complexified arrangement $\mathcal{A}^{\mathbb{C}}$ the face posets $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$ are completely determined by, and can be combinatorially computed from, the face poset $\mathcal{L}$ of the real arrangement $\mathcal{A}$. For instance, the $\mathcal{K}^{(1)}$ shown in Figure 2.3 is easily constructed from the face poset of 3 lines in $\mathbb{R}^2$, as shown in Figure 5.1.

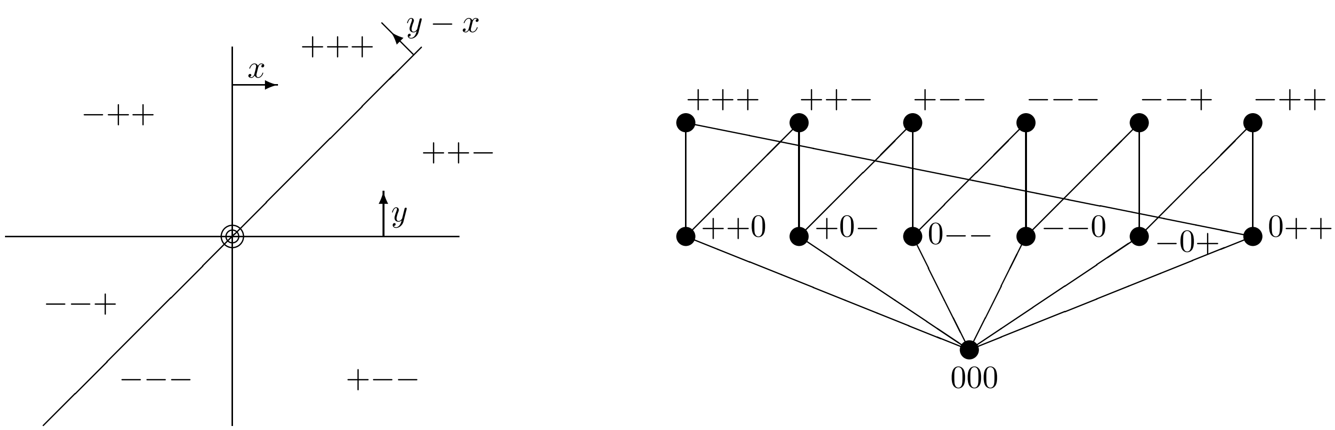


Figure 5.1. Face poset $\mathcal{L}$ of an arrangement of three lines in $\mathbb{R}^2$.

The bijection $\mathrm{Int}(\mathcal{L}) \longleftrightarrow \mathcal{K}^{(1)}$ makes it possible to translate the partial order of $\mathcal{K}^{(1)}$ into a corresponding partial ordering of $\mathrm{Int}(\mathcal{L})$, which can be independently described.

Proposition 5.2. *For intervals* $[Y,X],[Y',X'] \in \mathrm{Int}(\mathcal{L})$, *define an order relation by*

$$[Y,X] \leq [Y',X'] \iff \begin{cases} Y \leq Y' & \textit{and} \\ Y'\circ X \leq X'. \end{cases}$$

With this ordering $\mathrm{Int}(\mathcal{L}) \cong \mathcal{K}^{(1)}$ *is a poset isomorphism.*

Proof. The translation via part (ii) of Theorem 5.1 is straightforward. $\square$

The poset $\mathcal{K}^{(1)} \cong \mathrm{Int}(\mathcal{L})$ produces cell complexes for the link $V_{\mathcal{A}^{\mathbb{C}}} \cap S^{2d-1}$ and the complement $C_{\mathcal{A}^{\mathbb{C}}}$, as shown in Section 3. Since everything can be described in terms of $\mathcal{L}$ we can draw the following conclusion from Corollary 3.7.

Theorem 5.3. *The face poset* $\mathcal{L}$ *(that is, the oriented matroid) of a real arrangement* $\mathcal{A}$ *determines the complement and the variety of the complexified arrangement* $\mathcal{A}^{\mathbb{C}}$ *up to homeomorphism.*

We know from Theorem 3.5 that $(\mathcal{K}^{(1)}_{comp})^{op}$ is the face poset of a regular CW complex $\Gamma^{(1)}_{comp}$ that is homotopy equivalent to $C_{\mathcal{A}^{\mathbb{C}}}$. Since $\mathcal{K}^{(1)}_{comp} = \{Z \in \mathcal{K}^{(1)} : Z_a \neq 0 \text{ for all } a\}$, by Lemma 3.4, we can easily translate the combinatorial description of $\Gamma^{(1)}_{comp}$ into the language of intervals in $\mathcal{L}$. Clearly, if $Y \leq X$ then $\mathrm{Ze}(X\circ iY) = \emptyset$ if and only if $\mathrm{Ze}(X) = \emptyset$, which means that X is a maximal element of $\mathcal{L}$. Geometrically such maximal elements correspond to the *regions* of $\mathcal{A}$, i.e., the connected components of $\mathbb{R}^d \backslash (\bigcup \mathcal{A})$. We have shown that the cells of $\Gamma^{(1)}_{comp}$ are in bijection with the special intervals $[Y,T]$, where T is maximal, and their order relation is a special case of Proposition 5.2. Hence, we have arrived at the following description of $\Gamma^{(1)}_{comp}$. (Remember that the opposite of the ordering of $\mathcal{K}^{(1)}$ is used.)

Proposition 5.4. *The regular cell complex* $\Gamma^{(1)}_{comp}$ *has one* $(d-\rho(Y))$*-dimensional cell* $\sigma_{[Y,T]}$ *for each upper interval* $[Y,T]$ *in* $\mathcal{L}$ *(i.e., such that* $Y \leq T$ *and* T *is maximal), and the inclusion of closed cells is determined by*

$$\sigma_{[Y',T']} \leq \sigma_{[Y,T]} \iff \begin{cases} Y \leq Y' & \textit{and} \\ Y'\circ T = T'. \end{cases}$$

It is now apparent that this is precisely the cell complex for $C_{\mathcal{A}^{\mathbb{C}}}$ that was constructed by Salvetti [S]. In fact, this description of Salvetti's complex in terms of upper intervals $[Y,T]$ in $\mathcal{L}$ was given by Ziegler [Z1], and the equivalent description in terms of complex sign vectors $T\circ iY$ was given by Gel'fand & Rybnikov [GR]. From Theorem 3.5 we can deduce Salvetti's main result.

Theorem 5.5. (Salvetti [S])
Let $\Gamma_{Sal} := \Gamma^{(1)}_{comp}$ *denote the cell complex described in Proposition 5.4. Then* Γ_{Sal} *has the same homotopy type as* $C_{\mathcal{A}^{\mathbb{C}}}$*.*

In the following we will describe some special properties of the Salvetti complex of a real arrangement $\mathcal{A}$. As pointed out by Gel'fand & Rybnikov [GR] the construction of Γ_{Sal} is entirely combinatorial in terms of $\mathcal{L}$, so everything that will be said is also true for the Salvetti complex Γ_{Sal} of an oriented matroid $\mathcal{L}$.

Proposition 5.4 shows that the vertices of Γ_{Sal} correspond to the intervals $[T, T]$, and the maximal cells, all of dimension d, correspond to the intervals $[0, T]$ in $\mathcal{L}$. Hence, both vertices and maximal cells of Γ_{Sal} are in bijection with the set of regions of $\mathcal{A}$.

The fact that $\Gamma_{Sal} = \Gamma^{(1)}_{comp}$ is d-dimensional seems to be a quite special property of complexified arrangements.

Example 5.6. Consider the complex arrangement $\mathcal{B}$ in $\mathbb{C}^2$ defined by $\{z, w, (1+i)w + (2-i)z\}$. One computes $\mathbf{s}^{(1)}_{\mathcal{B}}(1,1) = (+++)$, $\mathbf{s}^{(1)}_{\mathcal{B}}(i,1) = (i{+}{+})$, $\mathbf{s}^{(1)}_{\mathcal{B}}(i, i+3) = (ii{+})$ and $\mathbf{s}^{(1)}_{\mathcal{B}}(i,i) = (iii)$. This yields a chain

$$(+++) \;\lessdot\; (i{+}{+}) \;\lessdot\; (ii{+}) \;\lessdot\; (iii)$$

of length 3 in $\mathcal{K}^{(1)}_{comp}$, so $\Gamma^{(1)}_{comp}$ has dimension 3.

There is of course an analogous cell complex $\Gamma^{(2)}_{comp}$ for the $\mathbf{s}^{(2)}$-stratification of $\mathcal{A}^{\mathbb{C}}$. Its barycentric subdivision $\mathrm{sd}(\Gamma^{(2)}_{comp})$ appears in the work of Orlik [O2]. The order-preserving map $\phi : \mathcal{K}^{(2)} \longrightarrow \mathcal{K}^{(1)}$ defined in Theorem 5.1(vi) restricts to $\mathcal{K}^{(2)}_{comp} \longrightarrow \mathcal{K}^{(1)}_{comp}$, and therefore provides a simplicial map

$$\phi_\Delta : \ \mathrm{sd}(\Gamma^{(2)}_{comp}) \ \longrightarrow \ \mathrm{sd}(\Gamma_{Sal}).$$

This simplicial map was shown to be a homotopy equivalence by Arvola [Ar]. We remark that the dimension of $\Gamma^{(2)}_{comp}$ is usually higher than d. In fact, whenever the matroid of $\mathcal{A}$ has two disjoint hyperplanes (e.g., for generic arrangements of at least $2d-2$ real hyperplanes), this complex has dimension $2d-2$.

In the following, a cell in a regular cell complex will be called *zonotopal* if its boundary complex is combinatorially isomorphic to a zonotope. For facts about zonotopes, and in particular the polarity between real arrangements and zonotopes, see [BLSWZ, Section 2.2].

Proposition 5.7. *Let* Γ_{Sal} *be the Salvetti complex of an essential complexified arrangement* $\mathcal{A}^{\mathbb{C}}$ *in* $\mathbb{C}^d$*. Suppose that the number of* k*-dimensional cells in* Γ_{Sal} *is* f_k*, for* $0 \le k \le d$*. Then*

(a) $f_0 = f_d =$ *number of regions of* $\mathcal{A}$*.*
(b) *The* f*-vector* $(f_0, f_1, \ldots, f_d)$ *depends only on the underlying matroid.*
(c) $\chi(\Gamma_{Sal}) = f_0 - f_1 + f_2 - \ldots + (-1)^d f_d = 0$.

(d) $\binom{d}{k} f_0 \leq f_k < 2^k \binom{d-1}{k} f_0$, *for* $0 < k < d$.
(e) *If* $\mathcal{A}$ *is a simplicial arrangement, then* $f_k = \binom{d}{k} f_0$, *for* $0 \leq k \leq d$.
(f) *If* $\mathcal{A}$ *is a generic arrangement, then*

$$f_k = 2^{k+1} \binom{n}{k} \sum_{i=0}^{d-k-1} \binom{n-k-1}{i}, \qquad \text{for } 0 \leq k < d.$$

(g) *All cells of* Γ_{Sal} *are zonotopal.*
(h) *Every maximal cell in* Γ_{Sal} *has boundary complex isomorphic to* $\mathcal{Z}_{\mathcal{A}}$, *the zonotope polar to* $\mathcal{A}$, *and touches all the vertices.*

Proof. The statements about the face numbers f_k concern the enumeration of upper intervals $[Y, T]$ with $\operatorname{corank}(Y) = k$, in the face lattice $\mathcal{L}$ of $\mathcal{A}$ (Proposition 5.4).

(a) This was already mentioned.
(b) This is a consequence of the result of Bayer & Sturmfels [BS] that the flag vector of $\mathcal{L}$ only depends on the underlying matroid, see [BLSWZ, Corollary 4.6.3].
(c) For each fixed region T we have that $\sum_{0 \leq Y \leq T} (-1)^{\operatorname{corank}(Y)} = 0$, because this is the reduced Euler characteristic of a polytope (a region of $\mathcal{A}$). Summing this equation over T gives formula (c).
An alternative way to derive this is to substitute $t = -1$ in the formula of Theorem 7.1, and use the fact that $\chi_M(1) = 0$ for every matroid M.
(d) and (e) The lower bound stems form the fact that an interval $[0, T]$ in $\mathcal{L}$ has at least $\binom{d}{k}$ elements of corank k, with equality if and only if $[0, T]$ is boolean [BLSWZ, Exercise 4.4]. That $\mathcal{A}$ is simplicial means precisely that all intervals $[0, T]$ are boolean. The upper bound is a special case of a result of Varchenko [V] on the average number of k-faces of a d-cell in an arrangement, see also [BLSWZ, Proposition 4.6.9].
(f) That $\mathcal{A}$ is generic means that the intersection lattice L is a truncated boolean algebra (its matroid is the $(d-1)$-uniform matroid). The expression is therefore a special case of the formula in [BLSWZ, Exercise 4.32]. It can be computed from [BLSWZ, Theorem 4.6.2].
(g) Take a closed Salvetti cell $\sigma_{[Y,T]}$. Proposition 5.4 shows that there is a unique cell $\sigma_{[Y', Y' \circ T]} \subset \sigma_{[Y,T]}$ for every $Y' < Y$ in $\mathcal{L}^{op}$, that these cells are all the cells contained in $\sigma_{[Y,T]}$, and that the partial ordering of these cells is isomorphic to $\mathcal{L}^{op}_{<Y}$. Now use that $(\mathcal{L} \backslash 0)^{op}$ is the face poset of the zonotope $\mathcal{Z}_{\mathcal{A}}$, which shows that $\mathcal{L}^{op}_{<Y}$ is itself the face poset of a zonotope.
(h) Apply the argument in (g) to the case $Y = 0$. □

Proposition 5.7 remains true for the Salvetti complex of an arbitrary oriented matroid. The same proof applies. For part (h) the zonotope $\mathcal{Z}_{\mathcal{A}}$ must then be replaced by the "opposite big sphere" $\Gamma^{op}(\mathcal{L})$, see [BLSWZ, Corollary 4.3.4], and (g) must be similarly adjusted.

We close this section with a characterization of complexified arrangements in terms of their matroids M and $M^{\mathbb{R}}$, defined in Section 4. Actually, what we characterize are complex arrangements that are isomorphic to a complexified arrangement $\mathcal{A}^{\mathbb{C}}$ via an isomorphism of complex vector spaces. This is clearly the case if and only if all minimal dependencies between the forms $\ell_a \in (\mathbb{C}^d)^*$ can be scaled to have non-zero real coefficients.

Proposition 5.8. *Let $\mathcal{B} = \{H_1, \ldots, H_n\}$ be the arrangement in $\mathbb{C}^d$ defined by the linear forms $\ell_1, \ldots, \ell_n$, let $\mathcal{K}^{(1)}$ be the face poset of its $\mathbf{s}^{(1)}$-stratification, and let M and $M^{\mathbb{R}}$ be the associated matroids. Then the following conditions are equivalent:*

(i) *$\mathcal{B}$ is ($\mathbb{C}$-linearly isomorphic to) a complexified arrangement.*
(ii) *$\mathcal{K}^{(1)} = \mathcal{L} \circ i\mathcal{L}$, where $\mathcal{L} \subseteq \{+,-,0\}^n$ is the face poset of some real arrangement.*
(iii) *$M = M^{\mathbb{R}}$.*
(iv) *$r(M^{\mathbb{R}}) = r(M)$.*

Proof. The implication (i)$\Longrightarrow$(ii) was derived in Theorem 5.1. For this, observe that the $\mathcal{K}^{(1)}$-stratification is independent of change of coordinates in $\mathbb{C}^d$.

To see (ii)$\Longrightarrow$(iii), assume $\mathcal{K} = \mathcal{L} \circ i\mathcal{L}$ and let $\mathrm{Re}(Z)$ be some flat of $M^{\mathbb{R}}$. Now Z can be written as $Z = X \circ iY$ with $X, Y \in \mathcal{L} \subseteq \mathcal{K}$, and $iY = 0 \circ iY \in \mathcal{K}$. We get $\mathrm{Re}(Z) = \mathrm{Re}(X \circ iY) = \mathrm{Re}(iY) = \mathrm{Ze}(iY)$, so $\mathrm{Re}(Z)$ is also a flat of M. With Proposition 4.5 this implies $L = L^{\mathbb{R}}$, and hence (iii).

The equivalence (iii)$\Longleftrightarrow$(iv) follows from Proposition 4.5, because a strong map between two matroids on the same ground set is the identity if and only if they have the same rank [W].

Finally, assume that M and $M^{\mathbb{R}}$ are isomorphic, and assume that $\{1, \ldots, d\}$ is a basis for them. Then we can choose coordinates in $\mathbb{C}^d$ such that $\ell_a(\mathbf{z}) = z_a$, for $1 \le a \le d$. But now every other linear form ℓ_b ($b > d$) depends on $\{\ell_1, \ldots, \ell_d\}$ $\mathbb{R}$-linearly, that is, its defining equation is real. $\square$

6. Homology and Homotopy Type of the Link.

The cohomology algebra $H^*(C_{\mathcal{B}};\mathbb{Z})$ of the complement $C_{\mathcal{B}}$ of a complex arrangement has been computed in work of Arnol'd [A] and Brieskorn [Br]. Its combinatorial nature was shown by Orlik & Solomon [OS], see also Falk [F1]. Our approach to this problem differs from the by now standard one [O1] in that we first obtain an explicit description of the homology of the link, and then use Alexander duality. Our tools are quite elementary (everything that is needed can be learned from Munkres [Mu]), and very explicit, providing basis elements that are represented by geometric spheres in the link. We do not use any differentiable or algebraic structure, so that everything (except for the sign pattern of the relations) generalizes to the setting of "2-pseudoarrangements", see Section 8.

For the following, let $\mathcal{B} = \{H_1, \dots, H_n\}$ be a complex arrangement in $\mathbb{C}^d$, and let $\mathcal{K}$ be the $\mathbf{s}^{(1)}$-stratification (or any refinement). This induces on every minor (i.e., restriction of a subarrangement) of $\mathcal{B}$ a combinatorial stratification that refines the $\mathbf{s}^{(1)}$-stratifiction of the minor. Thus every $\mathbf{s}^{(1)}$-stratum of a minor intersects the unit sphere in a pure subcomplex (a ball or a sphere) of $\Gamma_{\mathcal{K}}$, so that it can be identified with the corresponding cellular chain in $\Gamma_{\mathcal{K}}$.

The computations will be in terms of cellular homology [Mu, §39], where they are most easily and economically done. As is well known, computations in simplicial homology (after barycentric subdivision) or singular theory yield isomorphic results.

For the broken circuit construction below we will need a linear ordering on the set of hyperplanes. The linear ordering of $\mathcal{B} = \{H_1, \dots, H_n\}$ is always given by the indices of its hyperplanes.

Let M be the matroid of $\mathcal{B}$, of rank $r = r(M)$, whose ground set $[n] : \{1, \dots, n\}$ we identify with $\mathcal{B}$. If $\mathcal{B}$ is not empty we can consider the *deletion*

$$\mathcal{B}' := \mathcal{B} \backslash H_n = \{H_1, \dots, H_{n-1}\},$$

whose matroid is the deletion $M' = M \backslash n$. $\mathcal{B}'$ is again an arrangement in $\mathbb{C}^d$, and it inherits the obvious linear order from $\mathcal{B}$.

Similarly, we consider the *restriction*

$$\mathcal{B}'' := \mathcal{B}|_{H_n} = \{H_a \cap H_n : 1 \le a < n\}.$$

This is an arrangement in $H_n \cong \mathbb{C}^{d-1}$, whose matroid is the simplification of the contraction M/n that is obtained by deletion of all but one element in each parallelism class. We prefer to describe this as

$$\mathcal{B}'' = \{H_a \cap H_n : 1 \le a < n;\ H_b \cap H_n \ne H_a \cap H_n \text{ for } b < a\},$$

so that the hyperplanes in $\mathcal{B}''$ and the elements of M'' are indexed by $\{a : H_b \cap H_n \ne H_a \cap H_n \text{ for } b < a\}$, and they inherit the obvious linear order.

The arrangement $\mathcal{B}$ is *essential* if $\bigcap \mathcal{B} = \{0\}$, that is, if its *rank* $r := \operatorname{codim}(\bigcap \mathcal{B}) = r(M)$ coincides with the dimension d. Usually, we do not assume that $\mathcal{B}$ is essential – this would make the inductions more complicated. Therefore, only $r \le d$ is guaranteed.

Note that $\mathcal{B}$, $\mathcal{B}'$ are arrangements in $\mathbb{C}^d$, whereas $\mathcal{B}''$ is an arrangement in H_n. Thus the corresponding links are subsets $D, D' \subseteq S^{2d-1}$, whereas $D'' \subseteq S^{2d-3}$ after an identification of H_n with $\mathbb{C}^{d-1}$.

The method of "deletion and contraction" can be used to derive properties of $\mathcal{B}$ recursively from knowledge of $\mathcal{B}'$ and $\mathcal{B}''$. In the following, we will construct $\widetilde{\mathrm{H}}_*(D;\mathbb{Z})$ this way. This enables us to give a combinatorial description of a basis of $\widetilde{\mathrm{H}}_*(D;\mathbb{Z})$ in terms of the "broken circuit complex", for which we need the definition, its recursive construction, and the face numbers.

Definition 6.1. *The broken circuits of M are the sets $C \setminus \min(C)$, formed by deleting the smallest element from a circuit C of M. The broken circuit complex $\mathtt{BC}(M)$ is the collection of all nonempty subsets of E that do not contain a broken circuit.*

The broken circuit complex is known to be a pure $(r-1)$-dimensional subcomplex of the matroid complex, that is, all its *facets* (maximal faces) are bases of the matroid. Furthermore, the number f_i of i-dimensional faces of $\mathtt{BC}(M)$ is given by the characteristic polynomial χ of M via the Whitney-Rota formula

$$1 + \sum_{i=1}^{r} f_{i-1} t^i = (-t)^r \chi(-\frac{1}{t}). \tag{6.1}$$

It can easily be derived from the simple recursive property given by the following lemma, which we also exploit in our homology computation. We refer to [Bry], [BZ] and [B2] for additional information about broken circuit complexes.

Lemma 6.2. (Brylawski [Bry]) *Let M be a simple matroid on the linearly ordered ground set $[n]$, $M' = M \setminus n$ the deletion of the largest element n, M'' the simplification of the contraction M/n obtained by deleting all elements with a smaller parallel element (as above). Then*

$$\mathtt{BC}(M) = \mathtt{BC}(M') \ \uplus \ \{I \cup n : I \in \mathtt{BC}(M'')\} \ \uplus \ \{\{n\}\}.$$

Let $\mathcal{B}$ be an arrangement in $\mathbb{C}^d$, let $\mathcal{K}$ be the $\mathbf{s}^{(1)}$-stratification or any combinatorial subdivision of it, and let M be its matroid. We will now construct the (cellular) cycles that generate the homology of D.

Let $A \subseteq [n]$ be an independent set of size $k > 0$. Then

$$L_A = \{\mathbf{z} \in \mathbb{C}^d : \mathbf{s}^{(1)}(\ell_a(\mathbf{z})) \in \{+,-,0\} \text{ for } a \in A\}$$

is a real subspace of dimension $2d-k$. In this subspace,

$$C_A := \{\mathbf{z} \in \mathbb{C}^d : \mathbf{s}^{(1)}(\ell_a(\mathbf{z})) \in \{0,+\} \text{ for } a \in A\}$$

is the intersection of k closed halfspaces whose orthogonal vectors ℓ_a $(a \in A)$ are linearly independent. Thus C_A is a closed, full-dimensional cone (of dimension $2d-k$) in L_A. It is a proper subset of L_A because $A \neq \emptyset$. The intersection of the cone with the unit sphere in $\mathbb{C}^d$ is

$$c_A := C_A \cap S^{2d-1}.$$

This c_A is a topological $(2d-k-1)$-ball, which in the $\mathbf{s}^{(1)}$-stratification may be subdivided. It determines (up to a sign) a cellular $(2d-k-1)$-chain which we will also denote by c_A. This chain is supported with (± 1)-coefficients by all $(2d-k-1)$-cells $Z \in \mathcal{K}$ which have $Z(A) = (+\ldots+)$. Here $Z(A)$ denotes the restriction of the sign vector Z to the positions indexed by A.

The boundary of the $(2d-k-1)$-ball c_A is the $(2d-k-2)$-sphere

$$d_A := \partial C_A \cap S^{2d-1}.$$

Since the boundary of the cone C_A is given by

$$\partial C_A = \{\mathbf{z} \in C_A : \ell_a(\mathbf{z}) = 0 \text{ for some } a \in A\},$$

this sphere is a subcomplex of D. The corresponding cellular chain will also be denoted by d_A. It is supported, with ± 1-coefficients, by all $(2d-k-2)$-cells W such that $W(A) \in \{+,0\}^A$ contains exactly one 0-entry.

In the following theorem we compute the homology of D with integer coefficients. For our computations we consistently use reduced homology, as derived from the augmented (cellular) chain complex, see [Mu]. This implies in particular that

$$\widetilde{\mathrm{H}}_{-1}(T;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } T = \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 6.3. *Let $\mathcal{B}$ be a complex arrangement in $\mathbb{C}^d$, $D = D_{\mathcal{B}}$ its link, and M its matroid. Then the homology of D is free, and*

$$\{\langle d_A \rangle : A \in \mathtt{BC}(M),\ |A| = 2d-2-i\}$$

is a $\mathbb{Z}$-basis of $\widetilde{\mathrm{H}}_i(D;\mathbb{Z})$, for $i \geq 0$.

Proof. We proceed by induction on $n = |\mathcal{B}|$, with a trivial start at $n = 0$, where $\mathcal{B} = \emptyset$, $D = \emptyset$, $\mathtt{BC}(M) = \emptyset$, $\widetilde{\mathrm{H}}_i(D) = 0$ for $i \geq 0$.

Now consider S^{2d-1} as a cell complex (in the $\mathbf{s}^{(1)}$-stratification), so that D, D' and $S := H_n \cap S^{2d-1} \cong S^{2d-3}$ are subcomplexes with $D' \cup S = D$ and $D' \cap S = D''$. For this we get the Mayer-Vietoris long exact sequence in reduced homology, with integer coefficients:

$$\cdots \longrightarrow \widetilde{\mathrm{H}}_i(D'') \xrightarrow{(j_*^1,-j_*^2)} \widetilde{\mathrm{H}}_i(D') \oplus \widetilde{\mathrm{H}}_i(S) \xrightarrow{j_*^3+j_*^4} \widetilde{\mathrm{H}}_i(D) \xrightarrow{\partial_*} \widetilde{\mathrm{H}}_{i-1}(D'') \longrightarrow \cdots$$

where $j_*^1, j_*^2, j_*^3, j_*^4$ are induced by inclusions.

For $i > 0$, we know by induction that $\widetilde{\mathrm{H}}_{i-1}(D'')$ is generated by

$$B''_{i-1} := \{\langle d''_A \rangle : A \in \mathtt{BC}(M''),\ |A| = 2(d-1) - 2 - (i-1) = 2d - 3 - i\}.$$

Now, $A \uplus n \in \mathtt{BC}(M)$ for all $\langle d_A \rangle \in B''_{i-1}$ by Lemma 6.2. We can write $d_{A \cup n} = c''_A + e'_{A,n}$, where c''_A is the chain corresponding to the independent set A in the restriction $\mathcal{B}''$, and $e'_{A,n}$ is a chain on D'. [The underlying spaces can be described by $c''_A = S \cap d_{A \cup n}$ and $e'_{A,n} = D' \cap d_{A \cup n}$.] Using $\partial e'_{A,n} = -\partial c''_A = \pm d''_A$, we can compute

$$\begin{aligned}
\partial_* \langle d_{A \cup n} \rangle &= \langle (j_*^1, -j_*^2)^{-1} \partial (j_*^3 + j_*^4)^{-1} d_{A \cup n} \rangle \\
&= \langle (j_*^1, -j_*^2)^{-1} \partial (e'_{A,n}, c''_A) \rangle \\
&= \langle (j_*^1, -j_*^2)^{-1} (\pm d''_A, \mp d''_A) \rangle \\
&= \langle \pm d''_A \rangle.
\end{aligned}$$

This implies that ∂_* is surjective, hence

$$(j_*^1, -j_*^2) : \widetilde{\mathrm{H}}_{i-1}(D'') \longrightarrow \widetilde{\mathrm{H}}_{i-1}(D') \oplus \widetilde{\mathrm{H}}_{i-1}(S)$$

is the zero-map for $i > 0$. Thus we get short exact sequences

$$0 \longrightarrow \widetilde{\mathrm{H}}_i(D') \oplus \widetilde{\mathrm{H}}_i(S) \longrightarrow \widetilde{\mathrm{H}}_i(D) \longrightarrow \widetilde{\mathrm{H}}_{i-1}(D'') \longrightarrow 0$$

for $i > 0$. Because $\widetilde{\mathrm{H}}_{i-1}(D'')$ is free by induction, we can use $\phi : \widetilde{\mathrm{H}}_{i-1}(D'') \longrightarrow \widetilde{\mathrm{H}}_i(D)$ with $\phi(\langle d''_A \rangle) := \langle d_{A \cup n} \rangle$ to split this sequence, and we get

$$\widetilde{\mathrm{H}}_i(D) \cong \widetilde{\mathrm{H}}_i(D') \oplus \widetilde{\mathrm{H}}_{i-1}(D'') \oplus \widetilde{\mathrm{H}}_i(S).$$

Thus, $\widetilde{\mathrm{H}}_i(D)$ is again free. A basis for it can be assembled from the bases of the summands: if $\langle S \rangle$ denotes a generator of $\widetilde{\mathrm{H}}_{2d-3}(S)$, then clearly $j_*^4(\langle S \rangle) = \pm \langle d_{\{n\}} \rangle$. Furthermore, if A is independent with $n \notin A$, then we get $j_*^3(\langle d'_A \rangle) = \pm \langle d_A \rangle$ by construction. Thus we get as a basis for $\widetilde{\mathrm{H}}_i(D)$ the set of all homology classes $\langle d_A \rangle$ with $|A| = 2d - 2 - i$ such that A lies in

$$\mathtt{BC}(M') \uplus \{A \uplus n : A \in \mathtt{BC}(M'')\} \uplus \{\{n\}\}.$$

By Lemma 6.2, this yields the right result.

The case of $i = 0$ is easily treated separately (by inspection of the Mayer-Vietoris sequence, or directly). $\square$

The Whitney-Rota formula (6.1) together with the formula

$$\mathrm{rank}(\widetilde{\mathrm{H}}_i(D; \mathbb{Z})) = f_{2d-3-i},$$

for the ranks of the free abelian groups $\widetilde{\mathrm{H}}_i(D; \mathbb{Z})$ leads to the following conclusion.

Corollary 6.4.

$$\sum_{i\geq 0} \operatorname{rank}(\widetilde{\mathrm{H}}_i(D;\mathbb{Z}))\ t^i \quad = \quad (-t)^{2d-r-2}\chi(-t) - t^{2r-2}.$$

Theorem 6.3 implies that D has "no homology below half the dimension": $\widetilde{\mathrm{H}}_i(D;\mathbb{Z}) = 0$ for $i < d-2$. This can be strengthened as follows.

Corollary 6.5. (Milnor [Mi])
The link D is homotopically $(d-3)$-connected.

Proof. This is vacuous for $d \leq 2$ and clear for $d = 3$. Since homology vanishes below dimension $d-2$, we need only check that D is simply-connected (by the Hurewicz theorem). This can be done by induction on n, just as in the proof of Theorem 6.3, using the Seifert-van Kampen theorem.

Alternatively, one can use a k-connectivity version of the Nerve Theorem, as in [B3, (10.6)] – this way one does not even use the homology computation of this section. □

Corollary 6.5 is a special case of a much more general result of Milnor [Mi], which he proves with methods from Morse theory. Theorem 8.15 below gives a generalization not covered by Milnor's result. We remark that for complexified arrangements $\mathcal{A}^{\mathbb{C}}$ the $(d-3)$-connectivity of $D_{\mathcal{A}^{\mathbb{C}}}$ can very easily be seen from the results of Section 5. Namely, observe that $\mathcal{K}^{(1)}_{link} = \mathcal{K}^{(1)} \backslash \mathcal{K}^{(1)}_{comp}$ in this case contains the complete $(d-2)$-skeleton of $\mathcal{K}^{(1)}$, since $\Gamma^{(1)}_{comp} = \Gamma_{Sal}$ is d-dimensional. Thus $\mathcal{K}^{(1)} \cong S^{2d-1}$ and its subcomplex $\mathcal{K}^{(1)}_{link}$ have the same $(d-2)$-skeleton. But $(d-3)$-connectivity is carried by the $(d-2)$-skeleton for any regular CW complex, so the $(d-3)$-connectivity of S^{2d-1} implies that of $\mathcal{K}^{(1)}_{link}$.

For arrangements in $\mathbb{C}^d$, $d \geq 4$, we can say more about the topology of the singularity link.

Theorem 6.6. *The link D has the homotopy type of a wedge of spheres, if $d \geq 4$.*

Proof. Let $\mathtt{BC}(M)$ be the broken circuit complex of the matroid M of the arrangement. For each $A \in \mathtt{BC}(M)$ we have constructed a $(2d-2-|A|)$-sphere d_A embedded in D, and we have shown that the spherical classes $\langle d_A \rangle$ give a free basis for $\widetilde{\mathrm{H}}_*(D;\mathbb{Z})$. Now construct a new space W as follows: create a new $(2d-2-|A|)$-sphere d'_A for each $A \in \mathtt{BC}(M)$, then take the disjoint union of these spheres and one extra point x, then finally join x to each one of the spheres by a path from x to some point x_A on d'_A. This space W clearly has the homotopy type of a wedge of spheres (just contract the paths) and its homology is isomorphic to $\widetilde{\mathrm{H}}_*(D;\mathbb{Z})$ with a basis given by the classes $\langle d'_A \rangle$. We define a mapping $f : W \longrightarrow D$ by choosing a homeomorphism $f_A : d'_A \longrightarrow d_A$ for each $A \in \mathtt{BC}(M)$, setting $f(x) = y$ for some arbitrary point $y \in D$, and then arbitrarily mapping the paths in W from x to x_A to paths in D from y to $f(x_A)$. This mapping induces an isomorphism $f_* : \widetilde{\mathrm{H}}_*(W;\mathbb{Z}) \cong \widetilde{\mathrm{H}}_*(D;\mathbb{Z})$, and since D is simply connected by Corollary

6.5 an application of Whitehead's theorem [Sp, pp. 405-406] gives that f induces homotopy equivalence between W and D. $\square$

Theorem 6.6 implies that the cohomology of the link has trivial multiplication. For arrangements in $\mathbb{C}^2$ the link D is simply a disjoint union of copies of S^1 embedded in S^3. We do not know the homotopy type of D for arrangements in $\mathbb{C}^3$.

7. Cohomology of the Complement.

From our description in Theorem 6.3 of the homology of the link of a complex arrangement, Alexander duality [Mu, §71] immediately yields the structure of the cohomology groups of the complement:

Theorem 7.1. (Brieskorn [Br], Orlik & Solomon [OS])
Let $C = C_{\mathcal{B}}$ be the complement of a complex arrangement $\mathcal{B}$ of rank r in $\mathbb{C}^d$. Then the cohomology $\mathrm{H}^*(C;\mathbb{Z})$ of C is free, and its ranks are given by

$$\sum_{i\geq 0} \mathrm{rank}(\widetilde{\mathrm{H}}^i(C;\mathbb{Z}))t^i = (-t)^r \chi(-\frac{1}{t}) - 1.$$

In particular, $\widetilde{\mathrm{H}}^i(C;\mathbb{Z}) = 0$ for $i > r$.

The goal of the present section is to establish three additional pieces of information:

(1) we construct explicit (simplicial) cocycles c_Δ^A on $\Delta(\mathcal{K}_{comp})$ (following Gel'fand & Rybnikov [GR]) such that $\{\langle c_\Delta^A\rangle : A \in \mathtt{BC}(M)\}$ is a $\mathbb{Z}$-basis of $\widetilde{\mathrm{H}}^*(C;\mathbb{Z})$,

(2) we give a proof that the cocycles c_Δ^A and the cycles d_A are Alexander duals of each other,

(3) we determine the structure of the relations in the cohomology algebra $\mathrm{H}^*(C;\mathbb{Z})$ (first achieved by Orlik & Solomon [OS]), and comment on the role of complex structure for these relations.

Let $\mathcal{B} = \{H_1, \ldots, H_n\}$ again be a complex arrangement in $\mathbb{C}^d$. Let $\mathcal{K}^{(1)}$ be the face poset of the $\mathbf{s}^{(1)}$-stratification of $\mathcal{B}$. Thus the elements of $\mathcal{K}^{(1)}\backslash 0$ represent both sign vectors and cells of the $\mathbf{s}^{(1)}$-stratification of S^{2d-1}. It will be useful to switch freely between the two interpretations.

In the following we again allow that the cell complex under consideration is possibly a subdivision of $\mathcal{K}^{(1)}$. In this case the sign vector $Z \in \{i, j, +, -, 0\}^n$ denotes a possibly subdivided cell, or cellular chain, in $\mathcal{K}^{(1)}$.

We may consider $\mathcal{K}^{(1)} \uplus \{\hat{1}\}$ to be the augmented face poset of a $2d$-ball. To construct the cellular chain complex $D_*(\mathcal{K}^{(1)};\mathbb{Z})$ [Mu, §39] and to compute cellular (co)homology with it, we choose an orientation for every cell of $\mathcal{K}^{(1)}$. This provides a function λ that assigns a sign $\lambda(Z{:}W) \in \{+1, -1\}$ to every covering relation $Z \lessdot W$, so that cellular homology on subcomplexes of $\Gamma_{\mathcal{K}^{(1)}}$ can be computed with the boundary operator

$$\partial W = \sum_Z \lambda(Z{:}W)\ Z.$$

At the same time, a coboundary operator to compute cohomology on subcomplexes of $\Gamma_{\mathcal{K}^{(1)}}$ is then given by

$$\delta Z = \sum_W \lambda(Z{:}W)\ W.$$

Note that δ is at the same time a boundary operator for subcomplexes of $\Gamma^{op}_{\mathcal{K}^{(1)}}$.

Such an orientation function λ can explicitly be constructed for every regular cell complex, see [CF]. The condition which characterizes λ is that

$$\lambda(Z{:}W')\lambda(W'{:}U) + \lambda(Z{:}W'')\lambda(W''{:}U) = 0$$

for $r(U) = r(Z) + 2$, where W', W'' are the two elements W that satisfy $Z \lessdot W \lessdot U$.

We set $\lambda(Z{:}W) = 0$ whenever W does not cover Z. By reorienting the vertices and the maximal cells of S^{2d-1}, we may also assume that $\lambda(0{:}W) = +1$ for $0 \lessdot W$ and $\lambda(Z{:}\hat{1}) = +1$ for $Z \lessdot \hat{1}$.

(1) Construction of a basis for cohomology. Recall that the order complex $\Delta(\mathcal{K}^{(1)}_{comp})$ yields the barycentric subdivision of the cell complex Γ_{comp}. Following the Russian preprint version of [GR], we consider for $a \in [n]$ the 1-cochain $c^{\{a\}}_{\Delta}$ on $\Delta(\mathcal{K}_{comp})$ defined by

$$c^{\{a\}}_{\Delta}(Z < Z') = \begin{cases} 1 & \text{if } Z_a = + \text{ and } Z'_a = i, \\ 0 & \text{otherwise.} \end{cases}$$

The coboundary of $c^{\{a\}}_{\Delta}$ vanishes:

$$\begin{aligned} \delta c_{\Delta}(Z < Z' < Z'') &= c^{\{a\}}_{\Delta}(Z < Z') - c^{\{a\}}_{\Delta}(Z < Z'') + c^{\{a\}}_{\Delta}(Z' < Z'') \\ &= \begin{cases} 1 - 1 + 0 = 0 & \text{if } (Z_a, Z'_a, Z''_a) = (+, i, i), \\ 0 - 1 + 1 = 0 & \text{if } (Z_a, Z'_a, Z''_a) = (+, +, i), \\ 0 - 0 + 0 = 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We can now take cup products of these cocycles, to define

$$c^{A}_{\Delta} \ := \ c^{\{a_1\}}_{\Delta} \smile \ldots \smile c^{\{a_k\}}_{\Delta} \in C^k(\Delta(\mathcal{K}_{comp}))$$

for every (ordered) subset $A = \{a_1, \ldots, a_k\}_< \subseteq [n]$. Since cup products of cocycles are cocycles, we have

$$\delta c^{A}_{\Delta} = 0 \qquad \text{for all } A \subseteq [n].$$

Explicitly, the cocycles c^{A}_{Δ} are given (using the simplicial description of cup product) by

$$c^{A}_{\Delta}(Z^0 < Z^1 < \ldots < Z^k) = \begin{cases} 1 & \text{if } Z^s(a_t) = \begin{cases} + & \text{for } s < t, \\ i & \text{for } s \geq t, \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem combines results from Orlik & Solomon [OS], Björner [B1] (or Jambu & Leborgne [JL]) and Gel'fand & Rybnikov [GR].

Theorem 7.2. *Let $\mathcal{B}$ be a complex arrangement in $\mathbb{C}^d$ with complement $C = C_{\mathcal{B}}$ and matroid M. Then*

$$\{\langle c^{A}_{\Delta}\rangle : A \in \mathtt{BC}(M),\ |A| = i\}$$

is a $\mathbb{Z}$-basis of $\widetilde{\mathrm{H}}^i(C_{\mathcal{B}}; \mathbb{Z})$ for all $i \geq 0$.

Proof. In the following D^* will denote cellular and C^* will denote simplicial cochain complexes. Recall that $\Delta(\mathcal{K}_{comp})$ is the barycentric subdivision of Γ_{comp}. The corresponding cochain map

$$\mathrm{sd}^\sharp : C^i(\Delta(\mathcal{K}_{comp})) \longrightarrow D^i(\Gamma_{comp})$$

is the dual of the subdivision operator [Mu, §17], and induces an isomorphism in cohomology. Therefore it suffices to show that the cellular cochains $c^A := \mathrm{sd}^\sharp(c^A_\Delta)$ give rise to a $\mathbb{Z}$-basis

$$\{\langle c^A\rangle : A \in \mathtt{BC}(M)\}$$

for $\widetilde{\mathrm{H}}^*(\Gamma_{comp};\mathbb{Z})$ in cellular cohomology. This will be achieved by Alexander duality in the following part **(2)**, together with Theorem 6.3. $\square$

(2) Alexander duality $\langle c^A\rangle \longleftrightarrow \langle d_A\rangle$. Using the orientation function λ, we can express the subdivision operator $\mathrm{sd}_\sharp$ by

$$\mathrm{sd}_\sharp(Z) = \sum \lambda(Z^0{:}Z^1)\cdot\ldots\cdot\lambda(Z^{k-1}{:}Z^k)\ [Z^0, Z^1, \ldots, Z^k],$$

when Z is a cell of rank $\rho(Z) = 2d-k$, where the sum is over all chains of the form $Z = Z^0 \lessdot Z^1 \lessdot \ldots \lessdot Z^k \lessdot \hat{1}$. To verify this formula, observe that the right hand side expression involves every maximal simplex in the barycentric subdivision of the dual cell of Z exactly once, with coefficient ±1. By direct calculation we see that this $\mathrm{sd}_\sharp$ satisfies $\partial\mathrm{sd}_\sharp Z = \mathrm{sd}_\sharp\delta Z$. Thus the algebraic subdivision operator is a chain map, so its dual $\mathrm{sd}^\sharp$ is a cochain map. We use it to define for every ordered independent set $A = \{a_1,\ldots,a_k\}_<$ a k-cochain

$$c^A := \mathrm{sd}^\sharp(c^A_\Delta) \ \in\ D^k(\Gamma_{comp};\mathbb{Z}),$$

which is given by

$$\begin{aligned} c^A(Z) &= c^A_\Delta(\mathrm{sd}_\sharp(Z)) \\ &= \begin{cases} \lambda(Z^0{:}Z^1)\cdot\ldots\cdot\lambda(Z^{k-1}{:}Z^k) \\ \qquad \text{if } \rho(Z)=2d-k,\ Z(A)=(+\ldots+) \text{ and} \\ \qquad Z=Z^0,Z^1,\ldots,Z^k \text{ is the associated } A\text{-chain of Lemma 4.6(ii)}, \\ 0 \quad \text{otherwise.} \end{cases} \end{aligned}$$

Alexander duality [Mu, §71] asserts an isomorphism

$$\widetilde{\mathrm{H}}^k(C_{\mathcal{B}};\mathbb{Z}) \ \cong\ \widetilde{\mathrm{H}}_{2d-2-k}(D_{\mathcal{B}};\mathbb{Z})$$

for all k. Note that both groups vanish for $k \le 0$. For $k > 0$, the Alexander duality isomorphism is given by

$$\widetilde{\mathrm{H}}^k(\Gamma_{comp};\mathbb{Z}) \ \overset{\mathbf{L}}{\cong}\ \mathrm{H}_{2d-1-k}(\Gamma,\Gamma_{link};\mathbb{Z}) \ \overset{\partial_*}{\cong}\ \widetilde{\mathrm{H}}_{2d-2-k}(\Gamma_{link};\mathbb{Z}).$$

The first isomorphism $\mathbf{L}$ is Lefschetz duality for the relative manifold (Γ, Γ_{link}) [Mu, Theorem 70.2], and maps

$$c^A \in D^k(\Gamma_{comp}; \mathbb{Z})$$

to the relative homology class of

$$c'_A := \sum_{Z \in \Gamma_{comp}} c^A(Z)\ Z \in D_{2d-1-k}(\Gamma, \Gamma_{link}; \mathbb{Z}).$$

Here we use that Γ_{comp} is the complex of dual blocks of Γ that do not meet the link, and orientations for the cellular complexes $D^*(\Gamma_{comp}; \mathbb{Z})$ and $D_*(\Gamma, \Gamma_{link}; \mathbb{Z})$ are chosen compatibly (by the same orientation function λ), as in [Mu, proof of Theorem 65.1]. The second isomorphism ∂_* is the connecting homomorphism of the long exact sequence in reduced homology for the pair (Γ, Γ_{link}). It maps $\langle c'_A \rangle$ to the homology class of

$$d'_A := \partial c'_A \in D_{2d-2-k}(\Gamma_{link}; \mathbb{Z}).$$

We claim that $c_A = \pm c'_A$ and $d_A = \pm d'_A$, so that the homology class of d_A is Alexander dual (up to a sign) to the cohomology class of c^A. To see this, observe that for every a the formula for $c_\Delta^{\{a\}}$ defines a cocycle on the complement of the hyperplane H_a, that is, on $\Delta(\{Z \in \mathcal{K}^{(1)} : Z_a \neq 0\})$. Taking cup products, we get that c_Δ^A is a cocycle on $\Delta(\{Z \in \mathcal{K}^{(1)} : Z_a \neq 0 \text{ for all } a \in A\})$. Applying $\mathrm{sd}^\sharp$, we get that c^A is a cocycle on $\{Z \in \mathcal{K}^{(1)} : Z_a \neq 0 \text{ for all } a \in A\}$, which is the complement poset corresponding to the subarrangement $\mathcal{B}(A) = \{H_a : a \in A\}$.

This implies that $\partial c'_A$ is supported only on $\{Z \in \mathcal{K}^{(1)} : Z_a = 0 \text{ for some } a \in A\}$, that is, the support of $\partial c'_A$ is in $D(A) = \bigcup_{a \in A} S_a$. Now note that c'_A has ± 1-coefficients exactly on the $(2d-k-1)$-cells with $Z(A) = (+\ldots+)$. Thus c'_A is (up to a sign) the chain of the subdivided cell c_A described in Section 6, and $\partial c'_A$ is therefore the chain of the $(2d-k-2)$-sphere d_A. □

(3) Relations in $\mathrm{H}^*(C_{\mathcal{B}}; \mathbb{Z})$.

We will now use Theorem 7.2 to derive "most of" the multiplicative structure of the cohomology algebra $\mathrm{H}^*(C_{\mathcal{B}}; \mathbb{Z})$. Here we pass to non-reduced cohomology in order to obtain a unit.

First we note that by Theorem 7.2, the cohomology algebra $\mathrm{H}^*(C_{\mathcal{B}}; \mathbb{Z})$ is generated by its unit together with the 1-dimensional classes $\langle c_\Delta^{\{a\}} \rangle$ $(1 \leq a \leq n)$, since

$$\langle c_\Delta^A \rangle = \langle c_\Delta^{\{a_1\}} \rangle \smile \ldots \smile \langle c_\Delta^{\{a_k\}} \rangle$$

by definition.

Using the anti-commutativity of the cup product in cohomology, this implies that the cohomology algebra can be written as a quotient of the exterior algebra $\Lambda^* \mathbb{Z}^n$, if a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{Z}^n$ is identified with the basis $\{\langle c_\Delta^{\{1\}} \rangle, \ldots, \langle c_\Delta^{\{n\}} \rangle\}$ of $\widetilde{\mathrm{H}}^1(C_{\mathcal{B}}; \mathbb{Z})$.

For the following, let $A = \{a_1, \ldots, a_k\}_<$ be a circuit of the matroid M. The broken circuit complex of the corresponding submatroid is

$$\mathtt{BC}(M(A)) \;=\; 2^A \setminus \{A, A\backslash\{a_1\}, \emptyset\}\,.$$

Now consider the subarrangement $\mathcal{B}(A) = \{H_{a_1}, \ldots, H_{a_k}\}$ of $\mathcal{B}$, whose complement C_A contains $C = C_{\mathcal{B}}$. By Theorem 7.2, we get that

$$\big\{\langle c_\Delta^A\rangle : A \in \mathtt{BC}(M(A))\big\}$$

is a $\mathbb{Z}$-basis for $\widetilde{\mathrm{H}}^*(C_{\mathcal{B}(A)}; \mathbb{Z})$. From this we deduce that $\widetilde{\mathrm{H}}^k(C_{\mathcal{B}(A)}; \mathbb{Z}) = 0$. Thus the cohomology class of the cochain c_Δ^A vanishes, and the same holds for its restriction to $C_{\mathcal{B}}$. We conclude that

$$\langle c_\Delta^A\rangle = 0 \qquad \text{for every circuit } A \text{ of } M.$$

[Note that by Lemma 4.6(i), we get in fact $c^A = \mathrm{sd}^\sharp c_\Delta^A = 0$ if A is dependent in $M^{\mathbb{R}}$. However, if A is only dependent in M, then the cocycle $c^A = \mathrm{sd}^\sharp c_\Delta^A$ need not vanish: see Example 5.6.]

But even more is true: Since $\langle c_\Delta^{A\backslash\{a_1\}}\rangle$ is not in the basis, it can be written as a linear combination of the cohomology classes $\langle c_\Delta^{A\backslash\{a_i\}}\rangle$ with $2 \le i \le k$. However, changing the numbering of the hyperplanes/elements, we find that each of the k classes $\langle c_\Delta^{A\backslash\{a_i\}}\rangle$ with $1 \le i \le k$ can be written as a linear combination of the others, while every subset of $k-1$ of them is linearly independent over $\mathbb{Z}$. Hence there is a linear dependence of the form

$$\sum_{i=1}^{k} \epsilon_i \; \langle c_\Delta^{A\backslash\{a_i\}}\rangle,$$

with $\epsilon_i \in \{+1, -1\}$.

Corollary 7.3. *Let $\mathcal{B} = \{H_1, \ldots, H_n\}$ be a complex arrangement in $\mathbb{C}^d$. Then the cohomology algebra of the complement is generated by the classes $\langle c_\Delta^{\{a\}}\rangle$, $1 \le a \le n$. It has a presentation*

$$0 \longrightarrow I_C \longrightarrow \Lambda^*\mathbb{Z}^n \stackrel{\pi}{\longrightarrow} \mathrm{H}^*(C_{\mathcal{B}}; \mathbb{Z}) \longrightarrow 0,$$

defined by $\pi(\mathbf{e}_a) := c_\Delta^{\{a\}}$.

The relation ideal I_C is generated by elements of the form

$$\sum_{i=1}^{k} \epsilon_i \; \mathbf{e}_{a_1} \wedge \ldots \wedge \widehat{\mathbf{e}_{a_i}} \wedge \ldots \wedge \mathbf{e}_{a_k}, \qquad (*)$$

for circuits $A = \{a_1, \ldots, a_k\}_<$, with $\epsilon_i \in \{+1, -1\}$.

Proof. We have seen above that $\mathrm{H}^*(C; \mathbb{Z})$ has a presentation as stated. To see that the relations $(*)$ generate the ideal I_C, observe that they allow us to take any class c_Δ^A for which A contains a broken circuit and write it as a $\mathbb{Z}$-linear combination of classes of lexicographically smaller sets $A' <_{lex} A$. Iterating the procedure, we can write every c_Δ^A in terms of the basis given by Theorem 7.2. Thus the relations $(*)$ generate the ideal. (This is the standard "straightening" technique to show that the broken circuit complex yields a basis of the Orlik-Solomon algebra, see [BZ] [B2]). □

The precise form of the relations, namely $\epsilon_i = (-1)^i$, was given by Orlik & Solomon [OS]. This does not follow directly from the combinatorics of the $\mathbf{s}^{(1)}$-stratification, for reasons that we will soon explain. However, it is easily derived in deRham cohomology. Under the isomorphism between simplicial and deRham theory, the generators $c_\Delta^{\{a\}}$ are mapped to the logarithmic differential forms

$$\frac{1}{2\pi i}\frac{\mathrm{d}\ell_a}{\ell_a}.$$

Now if $A = \{a_1, \ldots, a_k\}$ is a circuit, then a simple computation with differential forms shows

$$\sum_{i=1}^{k}(-1)^i \frac{\mathrm{d}\ell_{a_1}}{\ell_{a_1}} \wedge \ldots \wedge \widehat{\frac{\mathrm{d}\ell_{a_i}}{\ell_{a_i}}} \wedge \ldots \wedge \frac{\mathrm{d}\ell_{a_k}}{\ell_{a_k}} \quad = \quad 0,$$

so that in fact the coefficients in Corollary 7.3 can be chosen as $\epsilon_i = (-1)^i$.

At first sight it may seem surprising that one needs a detour to deRham theory to derive this. However, this can be explained by the observation that in a more general setting (Section 8), the result of Theorem 7.2 (and hence Corollary 7.3) stays valid, whereas the sign patterns of the relations $(*)$ change. In fact, one can show [Z2] that if one considers arrangements of four 2-dimensional real subspaces in $\mathbb{R}^4$ for which any two have intersection $\{0\}$, then the pattern of the coefficients ϵ_i in a presentation of the cohomology algebra does not in general coincide with that of an arrangement with a complex structure. Nevertheless, such an arrangement admits a stratification of the $\mathbf{s}^{(1)}$-type, as we will see in the next section.

8. 2-Pseudoarrangements.

The techniques we have used for the analysis of complex arrangements are quite elementary, and thus quite general. They show that key results (construction of the Salvetti complex, structure of the cohomology algebra) extend to more general situations. Two earlier results in this direction are:

- Gel'fand & Rybnikov [GR] construct the Salvetti complex of an oriented matroid, and
- Goresky & MacPherson [GM] compute the cohomology groups of arrangements of codimension 2 subspaces with even-dimensional intersections.

In the following, we outline a concept of 2-*pseudoarrangements*, which contains the case of complex arrangements, and also the just mentioned generalizations, as special cases. The main results of this paper (construction of cell complexes for the complements, $\mathbf{s}^{(1)}$-stratifications, homology and connectivity of the link, cohomology of the complement) all have straightforward extensions to this greater generality – with the same proofs.

The axiomatic theory of 2-pseudoarrangements (and their combinatorial counterpart: 2-matroids) is not complete and will be discussed in a separate paper [Z2]. We use, however, the axiomatic theory of (real) 1-pseudoarrangements and their equivalence with oriented matroids. The key result there is the *PL* Topological Representation Theorem of Folkman & Lawrence [FL] and its *PL* version by Edmonds & Mandel [EM] (see [BLSWZ, Chapter 5]), which implies that every 1-pseudoarrangement has the structure of a *PL* cell complex.

Definition 8.1. (see [BLSWZ, Section 5.1])

(1) *Let S be a (homeomorphic) embedding of S^{k-1} in S^k. Then the complement $S^k \backslash S$ has two connected components S^+ and S^-, called the sides of S.*

S is a pseudosphere if its embedding is tame, that is, if the closed sides $\overline{S^+}$ and $\overline{S^-}$ are homeomorphic to closed k-balls.

(2) *A 1-pseudoarrangement (or, arrangement of pseudospheres) is a finite set $\mathcal{A} = \{S_a : a \in [n]\}$ of pseudospheres in S^k, such that*

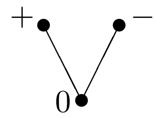
(A1) *$S_A := \bigcap_{a \in A} S_a$ is homeomorphic to a sphere, for all $A \subseteq [n]$ (where $S^{-1} = \emptyset$ is allowed), and*

(A2) *If $S_A \nsubseteq S_a$ for some $A \subseteq [n]$, $a \in [n] \backslash A$, then $S_A \cap S_a$ is a pseudosphere in S_A whose sides are $S_A \cap S_a^+$ and $S_A \cap S_a^-$.*

(3) *The pseudoarrangement $\mathcal{A} = \{S_a : a \in [n]\}$ is essential if $\cap_{a \in [n]} S_a = \emptyset$.*

(4) *A signed pseudoarrangement is a pseudoarrangement $\mathcal{A} = \{S_a : a \in [n]\}$ with a fixed choice of a positive side S_a^+ and a negative side S_a^- for every $S_a \in \mathcal{A}$.*

(5) *The position of $\mathbf{x} \in S^k$ with respect to a 1-pseudoarrangement $\mathcal{A} = \{S_a : a \in [n]\}$ is given by a position vector $\mathbf{s}_{\mathcal{A}}^{\mathbb{R}}(\mathbf{x}) \in \{+, -, 0\}^n$ whose a-th entry tells whether $\mathbf{x} \in S_a^+$, $\mathbf{x} \in S_a^-$ or $\mathbf{x} \in S_a$. The set of all such position vectors $\mathcal{L}_{\mathcal{A}} = \mathbf{s}_{\mathcal{A}}^{\mathbb{R}}(S^k)$ is a poset under the coordinate-wise ordering of signs induced by:*



(6) *The cells of $\mathcal{A}$ are the maximal subsets of S^k on which the map $\mathbf{s}^{\mathbb{R}}_{\mathcal{A}} : S^k \longrightarrow \mathcal{L}_{\mathcal{A}}$ is constant. The cells whose position vectors have no zero entry are called topes (or regions).*

Theorem 8.2. (Edmonds & Mandel [EM])
Every signed 1-pseudoarrangement $\mathcal{A}$ in S^k induces a PL cell decomposition $\Gamma_{\mathcal{A}}$ of S^k.
Its face poset is naturally identified with the poset $\mathcal{L}_{\mathcal{A}} \subseteq \{+, -, 0\}^n$.

The $\mathbf{s}^{(1)}$- and $\mathbf{s}^{(2)}$-stratifications of complex arrangements that we have used in previous sections are for their definitions strongly dependent on the choice of linear forms. In the following definition of a 2-pseudoarrangement this is reflected by the need to be able to represent a codimension two object as the intersection of two objects of codimension one.

Definition 8.3. *A 2-pseudoarrangement is a finite set $\mathcal{B} = \{S_1, \ldots, S_n\}$ of $(2d-3)$-spheres in S^{2d-1} satisfying the following two properties:*
(i) $\bigcap_{a \in A} S_a$ *is a subsphere of even codimension, for all $A \subseteq [n]$,*
(ii) *there exists a 1-pseudoarrangement $\{T_1, \ldots, T_n, U_1, \ldots, U_n\}$ in S^{2d-1} such that $S_a = T_a \cap U_a$, for $1 \leq a \leq n$.*
A 1-pseudoarrangement such as in (ii) *is called a real frame for $\mathcal{B}$.*

Definition 8.3 can be extended to "k-pseudoarrangements", see Remark 9.3.

Example 8.4. For every complex arrangement $\mathcal{B} = \{H_1, \ldots, H_n\}$ in $\mathbb{C}^d$, there is an associated real arrangement

$$\mathcal{B}^{\mathbb{R}} = \{H_1^{\mathrm{Re}}, \ldots, H_n^{\mathrm{Re}}, H_1^{\mathrm{Im}}, \ldots, H_n^{\mathrm{Im}}\}$$

in $\mathbb{C}^d = \mathbb{R}^{2d}$, whose hyperplanes are defined by

$$H_a^{\mathrm{Re}} = \{\mathbf{z} \in \mathbb{C}^d : \mathrm{Im}(\ell_a(\mathbf{z})) = 0\},$$
$$H_a^{\mathrm{Im}} = \{\mathbf{z} \in \mathbb{C}^d : \mathrm{Re}(\ell_a(\mathbf{z})) = 0\}.$$

With this set-up, we get that

$$H_a = H_a^{\mathrm{Re}} \cap H_a^{\mathrm{Im}}$$

Thus every complex arrangement yields a 2-pseudoarrangement (after intersection with S^{2d-1}), and every choice of linear forms ℓ_a yields a real frame.

Example 8.5. Goresky & MacPherson [GM, p. 257] consider arrangements of subspaces of codimension 2 in $\mathbb{R}^{2d}$ with the condition that all intersections of subfamilies have even codimensions. They observe that such arrangements can have non-representable matroids, so they are much more general than complex arrangements.

Subspace arrangements with the "even intersection condition" are examples of 2-pseudoarrangements: for this we again write every subspace of codimension 2 as the intersection of two real hyperplanes and intersect with S^{2d-1}.

Example 8.6. If $\mathcal{L}$ is the face poset of a 1-pseudoarrangement in S^{d-1}, then $\mathcal{L}\times\mathcal{L}$ is the face poset of the real frame of a 2-pseudoarrangement in S^{2d-1}, which we will refer to as a *complexified pseudoarrangement*. To see this, one can use the Topological Representation Theorem for oriented matroids: $\mathcal{L}$ is the covector span of an oriented matroid, and $\mathcal{L}\times\mathcal{L}$ is the covector span of the direct sum of the oriented matroid with itself. Geometrically, the complexified pseudoarrangement can be constructed as the join [Mu, §62] of the complex $\Gamma_{\mathcal{L}}$ with itself.

The preceding has a combinatorial reformulation for oriented matroids, because of the one-to-one correspondence between 1-pseudoarrangements and oriented matroids. This associates with an oriented matroid $\mathcal{L} \subseteq \{+,-,0\}^n$ (in terms of covectors) its *complexification* $\mathcal{K} = \mathcal{L}\circ i\mathcal{L}$. Just as for real arrangements in Section 5, this gives the $\mathcal{K}^{(1)}$ face poset of the complexification of the corresponding 1-pseudoarrangement, as can be seen from Lemma 8.8 below.

To perform the Salvetti construction in this context, we have to show (Theorem 8.11) that $\mathcal{K} = \mathcal{K}^{(1)}$ is the face poset of a *PL* cell complex (then the Salvetti complex is a subcomplex of $\Gamma(\mathcal{K}^{(1)})^{op}$). The resulting "Salvetti complex of an oriented matroid" is the complex considered by Gel'fand & Rybnikov [GR]. We will see that it is homotopy equivalent to the complement $S^{2d-1}\backslash(\bigcup_{a=1}^{n} S_a)$, where $\{S_1, ..., S_n\}$ is the complexification of the 1-pseudoarrangement corresponding to $\mathcal{L}$.

After these examples we will now start to develop the theory of 2-pseudoarrangements. This will be done only up to a point where it is evident that the proofs for complex arrangements from the earlier sections generalize.

Definition 8.7. *Let $\mathcal{B} = \{S_1, \ldots, S_n\}$ be a 2-pseudoarrangement in S^{2d-1}, and choose a signed real frame $\mathcal{A} = \{T_1, \ldots, T_n, U_1, \ldots, U_n\}$.*

(1) *Let*

$$\mathbf{s}^{(1)}_{\mathcal{B}} : S^{2d-1} \longrightarrow \{i, j, +, -, 0\}^n$$

be defined (for $1 \le a \le n$, $\mathbf{x} \in S^{2d-1}$) by:

$$\left(\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{x})\right)_a = \begin{cases} i, & \text{if } \mathbf{x} \in T_a^+, \\ j, & \text{if } \mathbf{x} \in T_a^-, \\ +, & \text{if } \mathbf{x} \in T_a \cap U_a^+, \\ -, & \text{if } \mathbf{x} \in T_a \cap U_a^-, \\ 0, & \text{if } \mathbf{x} \in T_a \cap U_a. \end{cases}$$

(2) *Let*

$$\mathbf{s}^{(2)}_{\mathcal{B}} : S^{2d-1} \longrightarrow \{+, -, 0\}^{2n}$$

be defined (for $1 \le a \le 2n$, $\mathbf{x} \in S^{2d-1}$) by:

$$\left(\mathbf{s}^{(2)}_{\mathcal{B}}(\mathbf{x})\right)_a = \begin{cases} +, & \text{if } 1 \le a \le n \text{ and } \mathbf{x} \in T_a^+, \\ -, & \text{if } 1 \le a \le n \text{ and } \mathbf{x} \in T_a^-, \\ 0, & \text{if } 1 \le a \le n \text{ and } \mathbf{x} \in T_a, \\ +, & \text{if } n+1 \le a \le 2n \text{ and } \mathbf{x} \in U_{a-n}^+, \\ -, & \text{if } n+1 \le a \le 2n \text{ and } \mathbf{x} \in U_{a-n}^-, \\ 0, & \text{if } n+1 \le a \le 2n \text{ and } \mathbf{x} \in U_{a-n}. \end{cases}$$

(3) *Let $\mathcal{K}^{(1)}_{\mathcal{B}} := \mathbf{s}^{(1)}_{\mathcal{B}}(S^{2d-1})$ and $\mathcal{K}^{(2)}_{\mathcal{B}} := \mathbf{s}^{(2)}_{\mathcal{B}}(S^{2d-1})$. These are posets with the ordering induced as subsets of $\{i, j, +, -, 0\}^n$ and $\{+, -, 0\}^{2n}$. (See Figures 2.1 and 2.2).*

(4) *The decomposition of S^{2d-1} into fibers of $\mathbf{s}^{(1)}_{\mathcal{B}}$ will be called the $\mathbf{s}^{(1)}$-stratification of $\mathcal{B}$, and similarly for $\mathbf{s}^{(2)}$.*

Let $\mathcal{L}_{\mathcal{A}}$ be the covector lattice of the oriented matroid of the chosen real frame $\mathcal{A}$. It is clear that $\mathcal{K}^{(2)} = \mathcal{L}_{\mathcal{A}} \backslash 0$, so the topological theory of such covector lattices is available (see Chapters 4 and 5 of [BLSWZ]).

In the sequel we will assume that the 2-pseudoarrangement $\mathcal{B}$ is *essential*, i.e., that $\bigcap_{a=1}^n S_a = \emptyset$. This implies that $\mathcal{A}$ is essential, and hence that $\mathcal{L}_{\mathcal{A}} = 0 \uplus \mathcal{K}^{(2)}$ is a ranked poset of total rank $2d$. Also, we will often suppress the subscript "$\mathcal{B}$" from the notation.

It follows from the connection with oriented matroids that $\mathcal{K}^{(2)}$ is the face poset of a regular cell decomposition of S^{2d-1} having $\bigcup \mathcal{B} = S_1 \cup \ldots \cup S_n$ as a subcomplex. It is not so obvious that the same is true about $\mathcal{K}^{(1)}$. This will be shown in Theorem 8.11.

We will write (Y, X) for sign vectors in $\mathcal{K}^{(2)}$, letting $Y \in \{+, -, 0\}^n$ denote the first half (the "T-part") and $X \in \{+, -, 0\}^n$ the second half (the "U-part") of the vector. Also, the composition of complex sign vectors (Definition 4.1) will be used.

Lemma 8.8. *The assignment $(Y, X) \longmapsto X \circ iY$ defines a mapping*

$$\phi : \mathcal{K}^{(2)} \longrightarrow \mathcal{K}^{(1)}$$

that is order-preserving and surjective. In particular,

$$\mathcal{K}^{(1)} = \{X \circ iY : (Y, X) \in \mathcal{K}^{(2)}\}.$$

Proof. One sees by coordinate-wise checking that

$$\phi(\mathbf{s}^{(2)}(\mathbf{x})) = \mathbf{s}^{(1)}(\mathbf{x}), \tag{8.1}$$

for all $\mathbf{x} \in S^{2d-1}$. Hence, ϕ is well-defined and surjective. That ϕ is order-preserving is shown precisely as in the proof of Theorem 5.1(vi). $\square$

Proposition 8.9. *For $Z \in \mathcal{K}^{(1)}$, let $B_Z = \{\mathbf{x} \in S^{2d-1} : \mathbf{s}^{(1)}(\mathbf{x}) \leq Z\}$. Then B_Z is a closed ball with boundary $\partial B_Z = \{\mathbf{x} \in S^{2d-1} : \mathbf{s}^{(1)}(\mathbf{x}) < Z\}$.*

Proof. It follows from formula (8.1) that

$$B_Z = \{\mathbf{x} \in S^{2d-1} : \mathbf{s}^{(2)}(\mathbf{x}) \in \phi^{-1}(\mathcal{K}^{(1)}_{\leq Z})\}. \tag{8.2}$$

The idea is now to write this in terms of $\mathcal{K}^{(2)} = \mathcal{L}_{\mathcal{A}} \backslash 0$, and then to use oriented matroid theory to conclude that B_Z is a ball.

Define a real sign vector Y^Z and a set of such $\mathcal{F}^Z$ as follows:

$$Y^Z_a \;=\; \begin{cases} +, & \text{if } Z_a = i, \\ -, & \text{if } Z_a = j, \\ 0, & \text{otherwise,} \end{cases}$$

$$X \in \mathcal{F}^Z \quad \Longleftrightarrow \quad X_a = Z_a, \quad \text{for all } a \in \mathrm{Re}(Z).$$

Then the definition of ϕ shows that

$$\phi^{-1}(Z) \;=\; \{(Y^Z, X) \in \mathcal{K}^{(2)} : X \in \mathcal{F}^Z\}, \tag{8.3}$$

$$\phi^{-1}(\mathcal{K}^{(1)}_{\leq Z}) \;=\; \{(Y, X) \in \mathcal{K}^{(2)} : Y \leq Y^Z \text{ and } X \leq X' \text{ for some } X' \in \mathcal{F}^Z\}. \tag{8.4}$$

Note that the surjectivity of ϕ implies the existence of such $(Y^Z, X) \in \mathcal{K}^{(2)}$, and choosing a point $\mathbf{x} \in S^{2d-1}$ such that $\mathbf{s}^{(1)}(\mathbf{x}) = Z$ generically we see that such $(Y^Z, X) \in \mathcal{K}^{(2)}$ exists with $\mathrm{Ze}(X) = \mathrm{Ze}(Z)$. This last observation is needed for the forward set inclusion in (8.4).

Combining (8.2) with (8.4) we see that the set B_Z is described by the following inequalities and equalities in S^{2d-1}:

$$\begin{cases} Y_a \leq +, & \text{if } Z_a = i \\ Y_a \leq -, & \text{if } Z_a = j \\ Y_a = 0, & \text{if } Z_a \in \{+, -, 0\} \\ X_a \leq Z_a, & \text{if } Z_a \in \{+, -\} \\ X_a = 0, & \text{if } Z_a = 0 \end{cases}$$

Furthermore, by (8.3) there exist feasible points that achieve equality in all inequalities. Hence, by Lemma 5.1.8 of [BLSWZ] applied to the subarrangement

$$\mathcal{A}' = \mathcal{A} \backslash \{U_a : a \in [n] \backslash \mathrm{Re}(Z)\}$$

it follows that B_Z is a ball. The description of the boundary ∂B_Z is a consequence of the same lemma. ∎

Theorem 8.10. *Let $\mathcal{B}$ be a 2-pseudoarrangement in S^{2d-1}, and let $\mathcal{K}^{(1)}$ be the poset of sign vectors of its $\mathbf{s}^{(1)}$-stratification (with respect to some real frame). Then*

(i) *the strata of the $\mathbf{s}^{(1)}$-stratification are the open cells of a PL regular CW decomposition $\Gamma^{(1)}$ of S^{2d-1}.*

(ii) *$\mathcal{K}^{(1)}$ is the face poset of $\Gamma^{(1)}$.*

(iii) *$\bigcup \mathcal{B} = S_1 \cup \ldots \cup S_n$ is a subcomplex of $\Gamma^{(1)}$, with face poset*

$$\mathcal{K}^{(1)}_{link} \quad = \quad \{Z \in \mathcal{K}^{(1)} : Z_a = 0 \text{ for some } a \in [n]\}.$$

Proof. Proposition 8.9 shows that $\{B_Z : Z \in \mathcal{K}^{(1)}\}$ gives a covering of S^{2d-1} with closed balls whose interiors partition S^{2d-1}. Furthermore, it shows that the boundary of each such ball of positive dimension is a union of balls from the same family. Hence, using the "ball complex" characterization of regular *CW* complexes (see Definition 4.7.4 and associated comments in [BLSWZ]), part (i) except for "*PL*" and part (ii) already follow. Part (iii) is clear from the construction.

The $\mathbf{s}^{(2)}$-stratification gives a regular cell complex $\Gamma^{(2)}$ with face poset $\mathcal{K}^{(2)} = \mathcal{L}_{\mathcal{A}} \backslash 0$ which refines $\Gamma^{(1)}$. But $\Gamma^{(2)} = \Gamma_{\mathcal{A}}$ is *PL*, as shown by Theorem 8.2, hence so is $\Gamma^{(1)}$. □

Corollary 8.11. *The poset $(\mathcal{K}^{(1)} \backslash \mathcal{K}^{(1)}_{link})^{op}$ is the face poset of a regular CW complex Γ_{comp} which is homotopy equivalent to $S^{2d-1} \backslash (\bigcup \mathcal{B})$.*

Proof. This follows from the theorem together with Proposition 3.1. □

With this the results for complex arrangements in this paper up to Theorem 3.5 have been generalized to 2-pseudoarangements (for the $\mathbf{s}^{(1)}$-stratification). To be able to continue with the homology and cohomology computations we have to first develop the combinatorics of $\mathcal{K}^{(1)}$-posets a little more in this general setting.

Proposition 8.12. *Let $\mathcal{K}^{(1)} \subseteq \{i, j, +, -, 0\}^n$ be the face poset of an $\mathbf{s}^{(1)}$-stratified 2-pseudoarrangement. Then*

(i) *$0 \uplus \mathcal{K}^{(1)}$ satisfies the properties of* Theorem 4.3,

(ii) *the poset $0 \uplus \mathcal{K}^{(1)}$ satisfies the properties of* Proposition 4.4.

Proof. We will use the representation $\mathcal{K}^{(1)} \; = \; \{X \circ iY : (Y, X) \in \mathcal{K}^{(2)}\}$ given by Lemma 8.9, and the fact that the real sign vector system $\mathcal{K}^{(2)} \subseteq \{+, -, 0\}^{2n}$ satisfies the covector axioms $(L1)$, $(L2)$, $(L3)$ and $(L3')$ for oriented matroids [EM] [BLSWZ, Section 4.1]. These axioms, applied to $\mathcal{K}^{(2)}$, are:

$(L1)$ $X \in \mathcal{K}^{(2)}$ implies that $-X \in \mathcal{K}^{(2)}$,

$(L2)$ $X, X' \in \mathcal{K}^{(2)}$ implies that $X \circ X' \in \mathcal{K}^{(2)}$,

$(L3)$ if $X, X' \in \mathcal{K}^{(2)}$ and $a \in S(X, X')$, then there exists $X'' \in \mathcal{K}^{(2)}$ such that $X''_a = 0$ and $X''_b = (X \circ X')_b = (X' \circ X)_b$ for all $b \notin S(X, X')$.

$(L3')$ if $X, X' \in \mathcal{K}^{(2)}$ with $X' \not\leq X$ and $X \circ X' = X$, then there exists $X'' \in \mathcal{K}^{(2)}$ such that $X'' < X$ and $X''_a = X_a$ for all $a \notin S(X, X')$.

Each property (k), $1 \leq k \leq 3'$, of Theorem 4.3 can now be derived for $\mathcal{K}^{(1)}$ from the corresponding property (Lk) for $\mathcal{K}^{(2)}$. We will give the details for two cases.

Case $k = 2$:

$$\begin{aligned} Z \circ W &= (X \circ iY) \circ (X' \circ iY') = X \circ (iY \circ X') \circ iY' = \\ &= X \circ (X' \circ iY) \circ iY' = (X \circ X') \circ (iY \circ iY') = \\ &= (X \circ X') \circ i(Y \circ Y'), \quad \text{using Lemma 4.2.} \end{aligned}$$

Case $k = 3$: Let $Z = X \circ iY$ and $W = X' \circ iY'$. If $Z_a = \pm i$ and $W_a = -Z_a$, then eliminate in position a between the two vectors $(Y, X), (Y', X') \in \mathcal{K}^{(2)}$ to get $(Y'', X'') \in \mathcal{K}^{(2)}$. If $Z_a = \pm 1$ and $W_a = -Z_a$, then similarly eliminate in position $a+n$. In either case the vector $U := X'' \circ iY'' \in \mathcal{K}^{(1)}$ will have the right properties.

For property $(1')$, a separate argument is needed. Let $A = \mathrm{Ze}(Z)$ and $S_A := \bigcap_{a \in A} S_a$. One sees from Definition 8.3 that the intersections $S_A \cap S_b$ form a 2-pseudoarrangement in the sphere S_A, whose $\mathcal{K}^{(1)}$ sign vectors are the restrictions to positions in $[n] \backslash A$ of the sign vectors in the given $\mathcal{K}^{(1)}$. Hence we may assume $A = \emptyset$, in which case property $(1')$ is trivial.

For part (ii) the proof given for Proposition 4.4 applies. □

Proposition 8.13. *Let $\mathcal{K}^{(1)}$ be the face poset of an $\mathbf{s}^{(1)}$-stratified 2-pseudoarrangement. Then the set families*

$$\begin{aligned} L &= \{\mathrm{Ze}(W) : W \in \mathcal{K}^{(1)}\} \cup \{[n]\}, \text{ and} \\ L^{\mathbb{R}} &= \{\mathrm{Re}(W) : W \in \mathcal{K}^{(1)}\} \cup \{[n]\}, \end{aligned}$$

ordered by inclusion, are geometric lattices.

Proof. For $L^{\mathbb{R}}$ this is easy to deduce from existing theory. Namely, $L^{\mathbb{R}}$ determines the underlying matroid of the oriented matroid $\mathcal{M}^{\mathbb{R}}$ of the 1-pseudoarrangement $\{T_1, \ldots, T_n\}$, by Proposition 4.1.13 of [BLSWZ]. (Recall that $\{T_1, \ldots, T_n, U_1, \ldots, U_n\}$ is the chosen real frame.)

For the analysis of L we will go back to Definition 8.3. Let $L_{\mathcal{B}}$ denote the intersection lattice of $\mathcal{B}$, that is, the set of all intersections of subfamilies $S_A = \bigcap_{a \in A} S_a$, ordered by reverse inclusion. This is an atomic lattice with least element $S_{\emptyset} = S^{2d-1}$ and greatest element $S_{[n]} = \emptyset$. Let $r(S_A) := \frac{1}{2}\mathrm{codim}(S_A)$. By the even codimension condition this rank function has range $[d]$, and the atoms $S_1, \ldots, S_n$ of $L_{\mathcal{B}}$ are exactly the elements of rank 1.

It remains only to check the semimodularity condition: If $S_A \not\subseteq S_k$, then $r(S_A \cap S_k) = r(S_A) + 1$. This is a consequence of

$$\mathrm{codim}(S_A \cap S_k) \;=\; \mathrm{codim}(S_A \cap T_k \cap U_k) \;\leq\; \mathrm{codim}(S_A \cap T_k) + 1 \;\leq\; \mathrm{codim}(S_A) + 2,$$

which follows from axiom $(A2)$ in the Definition 8.1(2) of a 1-pseudoarrangement.

We have shown that $L_{\mathcal{B}}$ is a geometric lattice. Now use that the mapping $A \mapsto S_A$ determines an isomorphism $L \longrightarrow L_{\mathcal{B}}$, as is easy to see by choosing points generically on each subsphere and taking the zero-set of their $\mathbf{s}^{(1)}$-vectors. □

With a 2-pseudoarrangement $\mathcal{B}$ we can associate two matroids $M = M(\mathcal{B})$ and $M^{\mathbb{R}}$, defined as the matroids of the geometric lattices L and $L^{\mathbb{R}}$. The proof of Proposition 8.13 shows that M depends only on $\mathcal{B}$, whereas $M^{\mathbb{R}}$ depends on the choice of a real frame for $\mathcal{B}$.

The concrete meaning of the two matroids M and $M^{\mathbb{R}}$ for the case of complex arrangements was discussed in Section 4. The example of [GM, p. 257] shows that the matroid M may be non-representable over every field, even if $\mathcal{B}$ comes from an arrangement of codimension 2 real linear subspaces. Also, the matroid $M^{\mathbb{R}}$ may fail to be realizable.

Lemma 8.14. *Let $\mathcal{K}^{(1)}$ be the face poset of an $\mathbf{s}^{(1)}$-stratified 2-pseudoarrangement. Then the statements in Lemma 4.6 remain valid.*

Proof. For part (i) let $Z = Z^0 < Z^1 < \ldots < Z^k$ be an A-chain. Then $\mathrm{Re}(Z^1)$ is a flat of the matroid $M^{\mathbb{R}}$ with $a_1 \notin \mathrm{Re}(Z^1)$ and $a_t \in \mathrm{Re}(Z^1)$ for $t > 1$. Thus a_1 is not in the $M^{\mathbb{R}}$-closure of $\{a_2, \ldots, a_k\}$. By induction, we find that A is independent.

For part (ii) one can use the same proof as for Lemma 4.6(ii), where the hyperplanes $H^{\mathrm{Re}}_{a_j}$ are now replaced by the pseudospheres T_{a_j}. □

We have now assembled enough of the general picture to assert that the results of Sections 6 and 7 are valid for every 2-pseudoarrangement. All necessary components for the proofs in those sections have been established in this greater generality.

Using the matroid $M = M(\mathcal{B})$ and the $\mathbf{s}^{(1)}$-stratification induced by a real frame, we can construct spherical cycles d_A in the link $\mathcal{K}^{(1)}_{link}$ as in Section 6, and cocycles c^A_Δ on the chain complex of the complement $\mathcal{K}^{(1)}_{comp}$, as in Section 7.

Theorem 8.15. *Let $\mathcal{B} = \{S_1, \ldots, S_n\}$ be a 2-pseudoarrangement in S^{2d-1}, $D_{\mathcal{B}} = S_1 \cup \ldots \cup S_n$ its link, $C_{\mathcal{B}} = S^{2d-1} \backslash D_{\mathcal{B}}$ its complement, and M its matroid. Then*

(i) *the homology of $D_{\mathcal{B}}$ is free, and*

$$\{\langle d_A \rangle : A \in \mathtt{BC}(M),\ |A| = 2d - 2 - i\}$$

is a $\mathbb{Z}$-basis of $\widetilde{\mathrm{H}}_i(D_{\mathcal{B}}; \mathbb{Z})$, for $i \geq 0$,

(ii) *$D_{\mathcal{B}}$ is $(d-3)$-connected, and $D_{\mathcal{B}}$ has the homotopy type of a wedge of spheres, if $d \geq 4$,*

(iii) *the cohomology of $C_{\mathcal{B}}$ is free, and*

$$\{\langle c^A_\Delta \rangle : A \in \mathtt{BC}(M),\ |A| = i\}$$

is a $\mathbb{Z}$-basis of $\widetilde{\mathrm{H}}^i(C_{\mathcal{B}}; \mathbb{Z})$ for all $i \geq 0$. Furthermore, $\mathrm{H}^*(C_{\mathcal{B}}; \mathbb{Z})$ has a presentation as in Corollary 7.3.

We remark that Gel'fand & Rybnikov [GR] have shown that if $\mathcal{B}$ corresponds to a complexified oriented matroid, then the Orlik-Solomon sign patterns are valid for the relations $(*)$ in $\widetilde{\mathrm{H}}^*(C_{\mathcal{B}}; \mathbb{Z})$.

9. Greater Generality.

The setting of Section 8 does not fully reveal the generality of the methods developed in this paper. In this section we will therefore mention a few extensions. Since the ideas are the same, but the necessary notation for their formulation in the more general settings may obscure their simplicity, we have chosen to keep the discussion very informal here.

9.1. Arrangements of Polyhedra. A *polyhedron* in $\mathbb{R}^d$ is by definition the solution set of a feasible finite set of linear equations and non-strict linear inequalities. Equivalently, it is a non-empty intersection of finitely many affine hyperplanes and closed half-spaces. For instance, every affine subspace is a polyhedron. We will call a finite set $\mathcal{P} = \{P_1, \ldots, P_n\}$ of polyhedra P_a an *arrangement.*

Suppose that we want to construct a finite regular CW complex having the homotopy type of the complement $C_{\mathcal{P}} = \mathbb{R}^d \backslash (P_1 \cup \ldots \cup P_n)$. This can be done as follows.

First take an arrangement $\mathcal{A} = \{H_1, \ldots, H_s\}$ of affine hyperplanes in $\mathbb{R}^d$ such that every P_a can be obtained as an intersection of some hyperplanes and half-spaces coming from $\mathcal{A}$. For instance, $\mathcal{A}$ can be put together as the union of minimal such arrangements chosen individually for each P_a, $a \in [n]$. Next, embed $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ by the mapping $\mathbf{x} \mapsto (\mathbf{x}, 1)$, and let $\mathcal{A}' = \{H_0', H_1', \ldots, H_s'\}$ be the central hyperplane arrangement in $\mathbb{R}^{d+1}$ defined by $H_0' = \{\mathbf{x} \in \mathbb{R}^{d+1} : x_{d+1} = 0\}$ and $H_a' = \operatorname{span} H_a$, for $1 \leq a \leq s$. Finally, let $\mathcal{L} = \mathcal{L}_{\mathcal{A}'}$ be the face poset of the regular cell decomposition $\Gamma_{\mathcal{A}'}$ of S^d induced by the intersections $H_a' \cap S^d$, $0 \leq a \leq s$. This cell complex is *PL*, since it is polytopal. Polytopality follows from known facts, as in the proof of Theorem 2.6.

Now, let $\Gamma_0 := \{\mathbf{x} \in S^d : x_{d+1} \leq 0\} \cup \{\mathbf{x} \in S^d : x_{d+1} > 0 \text{ and } \mathbf{x}/x_{d+1} \in P_1 \cup \ldots \cup P_n\}$. This is by construction a subcomplex of $\Gamma_{\mathcal{A}'}$, and $S^d \backslash \Gamma_0$ is by radial projection homeomorphic to $C_{\mathcal{P}}$. Let $\mathcal{L}_0$ be the face poset of Γ_0, $\mathcal{L}_0 \subseteq \mathcal{L}$. It follows from Proposition 3.1 that $(\mathcal{L} \backslash \mathcal{L}_0)^{op}$ is the face poset of a regular CW complex Γ_{comp} which is homotopy equivalent to $C_{\mathcal{P}}$.

Essentially the same construction has been considered for arrangements of subspaces by Orlik [O2]. He obtains that the chain complex $\Delta(\mathcal{L} \backslash \mathcal{L}_0)$, i.e., the barycentric subdivision of Γ_{comp}, is homotopy equivalent to $C_{\mathcal{P}}$.

There is a still more general version of the preceding which models complements of unions of polyhedra in 1-pseudoarrangements [BLSWZ, Section 10.1].

9.2. Arrangements of Subspaces. To achieve greater economy with the number of cells needed for the cell complex Γ_{comp} one can use the idea of the $\mathbf{s}^{(1)}$-stratification, suitably generalized. In Section 9.3 this will be described in a way that covers a special class of arrangements of linear subspaces. The formulation for general arrangements of linear subspaces (with no restrictions on their dimensions) should be clear. Arrangements of affine subspaces can be treated in the same way after a reduction to the linear case via an embedding of $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$, as explained in Section 9.1.

Given an arrangement $\mathcal{A} = \{L_1, \ldots, L_n\}$ of linear subspaces in $\mathbb{R}^d$, the $\mathbf{s}^{(1)}$-stratification requires the choice of a flag of subspaces

$$\mathbb{R}^d = N^{(0)} \supset N^{(1)} \supset \ldots \supset N^{(k_a)} = L_a,$$

with $k_a = \operatorname{codim}(L_a)$, for each $1 \leq a \leq n$. Then there is a poset Σ_{k_a} of signs recording the position of $\mathbf{x} \in \mathbb{R}^d$ with respect to L_a, and providing the entry in position a in the sign vector $\mathbf{s}^{(1)}(\mathbf{x})$, see Section 9.3. The poset of such sign vectors $\mathbf{s}^{(1)}(\mathbb{R}^d)$ has a filter which via Proposition 3.1 provides a regular cell complex Γ_{comp} having the homotopy type of $C_{\mathcal{A}} = \mathbb{R}^d \backslash (L_1 \cup \ldots \cup L_n)$.

The method for (co)homology computations in this paper breaks down in this generality. For this we need some matroid structure. In Section 9.3 a suitable framework with such structure will be described. A combinatorial formula for the linear cohomology structure of $C_{\mathcal{A}}$ for *any* subspace arrangement was given by Goresky & MacPherson [GM]. It shows that the cohomology of $C_{\mathcal{A}}$ is in general not torsion-free.

9.3. k-Pseudoarrangements. Define a *k-pseudoarrangement* to be a finite set $\mathcal{B} = \{S_1, \ldots, S_n\}$ of $(k(d-1)-1)$-spheres in S^{kd-1} such that:

(i) $S_A := \cap_{a \in A} S_a$ is a subsphere and $\operatorname{codim}(S_A) \equiv 0 \pmod{k}$, for all $A \subseteq [n]$,

(ii) there exists a 1-pseudoarrangement $\{T_{aj}\}_{1 \leq a \leq n, 1 \leq j \leq k}$ in S^{kd-1} such that $S_a = T_{a1} \cap \ldots \cap T_{ak}$, for all $1 \leq a \leq n$.

This specializes to Definition 8.3 for $k = 2$. The T_{aj}'s need not be pairwise distinct. Examples of 4-pseudoarrangements are given by hyperplane arrangements in quaternionic vector spaces.

Most of what is done in Section 8 goes through in this setting. The subspheres S_A ordered by reverse inclusion form a geometric lattice, and this provides the matroid $M(\mathcal{B})$.

Let us describe the construction of the $\mathbf{s}^{(1)}$- and $\mathbf{s}^{(k)}$-stratifications. For this choose a real frame $\{T_{aj}\}$ as in part (ii). The $\mathbf{s}^{(k)}$-stratification is the cell decomposition of S^{kd-1} induced by this 1-arrangement, and its corresponding poset of sign vectors $\mathcal{K}^{(k)} \subseteq \{+, -, 0\}^{kn}$.

The $\mathbf{s}^{(1)}$-stratification requires the flag of intermediate spheres

$$S^{kd-1} = T^{(0)} \supset T^{(1)} \supset \ldots \supset T^{(k)} = S_a,$$

where $T^{(j)} = T_{a1} \cap \ldots \cap T_{aj}$. The set of signs in this case is

$$\Sigma_k = \{0, +1, -1, +2, -2, \ldots, +k, -k\}$$

with partial order: $s \leq t \iff |s| \leq |t|$, for all $s, t \in \Sigma_k$. The position of a point $\mathbf{x} \in S^{kd-1}$ with respect to the subsphere S_a is given by 0 if $\mathbf{x} \in S_a$ and by $+s$ (resp. $-s$) if $\mathbf{x}$ is on the positive (resp. negative) side of $T^{(k-s+1)}$ in $T^{(k-s)}$. Finally, the position vector $\mathbf{s}^{(1)}(\mathbf{x}) \in {\Sigma_k}^n$ is the vector which in the a-coordinate records the position of $\mathbf{x}$ with respect to S_a.

The poset of position vectors $\mathbf{s}^{(1)}(S^{kd-1}) \subseteq {\Sigma_k}^n$ is a CW poset which can be split into a part giving a cell complex for $\bigcup\mathcal{B} = S_1 \cup \ldots \cup S_n$ as well as a cell complex Γ_{comp} that is homotopy equivalent to $S^{kd-1}\backslash\bigcup\mathcal{B}$. The topological properties of the "link" $\bigcup\mathcal{B}$ and the complement $S^{kd-1}\backslash\bigcup\mathcal{B}$ derived for the $k = 2$ case in this paper generalize.

For linear pseudoarrangements (for which every S_a is an intersection of S^{kd-1} with a real vector space of codimension k in $\mathbb{R}^{kd}$) the freeness and ranks of cohomology follow from the work of Goresky & MacPherson [GM, p. 239]. Our approach adds some information also in the linear case, namely combinarially constructed $\mathbb{Z}$-bases for homology and cohomology; a presentation of the cohomology algebra of the complement as in Corollary 7.3; the fact that the link $\bigcup\mathcal{B}$ is $(d-3)$-connected, independently of k.; and that $\bigcup\mathcal{B}$ has the homotopy type of a wedge of spheres, if $d \geq 4$.

Figure 9.1 shows the poset Σ_4 of quaternionic signs, as used for the $\mathbf{s}^{(1)}$-stratification of quaternionic hyperplane arrangements. We have here relabeled $2 \mapsto i$, $3 \mapsto j$, $4 \mapsto k$ in accordance with the customary notation for a basis of the quaternions.

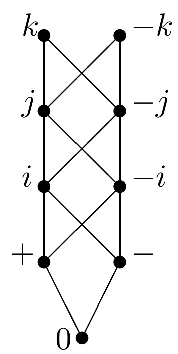


Figure 9.1. Quaternionic sign poset Σ_4.

9.4. ***k*-Matroids.** Let Σ_k denote the poset of signs defined in Section 9.3. For sign vectors $Z, W \in {\Sigma_k}^n$ define the *composition* $Z \circ W$ and the *separation set* $S(Z, W)$ as in parts (iii) and (iv) of Definition 4.1. We define a *k-matroid* to be a family of sign vectors $\mathcal{K} \subseteq {\Sigma_k}^n$ satisfying the following axioms:

(0) $0 \in \mathcal{K}$.

(1) $Z \in \mathcal{K}$ implies that $-Z \in \mathcal{K}$.

(1′) $Z \in \mathcal{K}$, $Z_a \neq 0$ and $\sigma \in \Sigma_k\backslash 0$ implies that there exists $W \in \mathcal{K}$ such that $W_a = \sigma$ and $\mathrm{Ze}(W) = \mathrm{Ze}(Z)$.

(2) $Z, W \in \mathcal{K}$ implies that $Z \circ W \in \mathcal{K}$.

(3) $Z, W \in \mathcal{K}$ with $a \in S(Z, W)$ implies that there exists a vector $U \in \mathcal{K}$ such that $U_a < Z_a, W_a$ and $U_b = (Z \circ W)_b = (W \circ Z)_b$ for all $b \notin S(Z, W)$.

Thus, 1-matroids are the same thing as oriented matroids (given in terms of their covectors). The $\mathcal{K}^{(1)}$ face posets of complex arrangements (Theorem 4.3), and also of 2-pseudoarrangements (Proposition 8.12), are examples of 2-matroids. More generally, the $\mathcal{K}^{(1)}$ augmented face posets of k-pseudoarrangements are k-matroids.

The most interesting case of k-matroids (except for $k = 1$, i.e., oriented matroids) is that of $k = 2$. The relationship between 2-pseudoarrangements and 2-matroids, with or without extra "complex" structure, will be further studied in [Z2].

We remark that the process of complexification of an oriented matroid $\mathcal{L} \longrightarrow \mathcal{L} \circ i\mathcal{L}$, described in Example 8.6, can be generalized so that it converts a k-matroid into a pk-matroid, for any $k, p \geq 1$.

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