Revised version of (??).

Posets with
Maximal Möbius Function

by

Günter M. Ziegler

(Universität Augsburg)

Abstract

The absolute value of the Möbius function of a bounded poset P with $n+2$ elements satisfies

$$|\mu(P)| \leq \max_{r\geq 0} \max_{\substack{p_1+\ldots+p_r=n \\ p_i\geq 1}} \prod_{i=1}^{r}(p_i-1).$$

This bound is sharp. The posets achieving it are classified.
The same problem is solved for graded posets of given rank and attacked for finite lattices.

Contents

1. Introduction

The Möbius function of a finite poset was recognized by G.-C. ROTA [6] in 1964 as a central tool in combinatorial enumeration. Its theory has since been systematically explored and developed by various authors.

We refer to R. STANLEY's exposition in [7, Chapter 3] for a thorough treatment, containing all the basic results and the poset terminology used in the following.

Throughout this paper we are dealing with finite posets only.

Recall that the *sum* and the *ordinal sum* of two posets P and Q are formed such that P and Q are (induced) subposets, where for $x \in P$ and $y \in Q$, x and y are always incomparable in $P + Q$, whereas $x < y$ in $P \oplus Q$. In particular, the finite chain with n elements arises as an ordinal sum $\mathbf{n} = \mathbf{1} \oplus \ldots \oplus \mathbf{1}$, while the n element antichain is a sum $n\mathbf{1} = \mathbf{1} + \ldots + \mathbf{1}$. The *length* of a chain is defined such that $\ell(\mathbf{n}) = n - 1$. The *length* $\ell(P)$ of an arbitrary poset P is the maximal length of a chain in P. Its *width* is the maximal size of an antichain in P.

We will denote the number of chains respectively maximal chains in a poset by $\text{C}(P)$ respectively $\text{MC}(P)$. Similarly, $\text{A}(P)$ and $\text{MA}(P)$ are the number of antichains and the number of maximal antichains in the poset P.

We will write $x \lessdot y$ for $x, y \in P$ if y *covers* x, that is, if $x < y$ and there is no $z \in P$ such that $x < z < y$. In this case y is a *cover* of x, and x is a *cocover* of y.

The *Möbius function* on the (closed) intervals of a poset is defined recursively by

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum\limits_{x \le z < y} \mu(x, z) & \text{otherwise.} \end{cases}$$

A poset is *bounded* if it contains a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. For every poset P, $\widehat{P} = \hat{0} \oplus P \oplus \hat{1}$ is bounded. The inverse operation produces the *proper part* $\overline{P} = P - \{\hat{0}, \hat{1}\}$ of a bounded poset. We will also use $\acute{P} = P - \{\hat{1}\}$.

The *Möbius function of a bounded poset* P is $\mu(P) = \mu(\hat{0}, \hat{1})$.

The question for the maximum (absolute) value of the Möbius function of a bounded poset with $n + 2$ elements was posed by R. STANLEY as [7, Exercise 3.41a]. Our answer was announced with the solutions to the exercises [7, p.187].

In Section 2 of this paper we give a short and simple proof for this result and determine the extreme examples (Theorem 2.5).

With a similar approach we study in Section 3 the case of graded posets of given length. For this we agree that a poset with $\hat{0}$ is *ranked* if for all $x \in P$, all maximal chains of $[\hat{0}, x]$ have the same length $r(x)$. P is *graded* if it is bounded and ranked, that is, if all maximal chains in P have the same length $\ell(P) = r(\hat{1})$.

In the graded case (Theorem 3.2) the structure of the extreme examples is more complicated and correspondingly the induction used for the proof needs more care.

In both cases the extreme examples tend to be *series parallel posets* [7, p.100]: they arise from one element posets $\mathbf{1}$ by successive sums and ordinal sums. (Equivalently, they do not contain the N-shaped four element poset as a subposet [8].) This motivates to put the Möbius function of the series and parallel connection of bounded posets on record – they, as given in the following Lemma, are the analogues of sums and ordinal sums in the category of bounded posets.

Lemma 1.1:

Let P and Q be two bounded posets. Then the Möbius function of their parallel connection $\widehat{\overline{P}+\overline{Q}} = \hat{0}\oplus(\overline{P}+\overline{Q})\oplus\hat{1}$ *and their series connection* $\widehat{\overline{P}\oplus\overline{Q}} = \hat{0}\oplus\overline{P}\oplus\overline{Q}\oplus\hat{1}$ *are given by*

$$\mu(\hat{0}\oplus(\overline{P}+\overline{Q})\oplus\hat{1}) = \mu(P)+\mu(Q)+1,$$
$$\mu(\hat{0}\oplus(\overline{P}\oplus\overline{Q})\oplus\hat{1}) = -\mu(P)\cdot\mu(Q).$$

In Section 4 we will discuss the case of posets with given length in which every element has at most k covers and cocovers, corresponding to [7, Exercise 3.42]. We develop a technique to construct posets of this type with high Möbius function (Proposition 4.4), which in particular for $k=2$ yields non-trivial lower bounds.

Finally in Section 5 we discuss the maximal Möbius function for finite lattices and give a proof, due to J. Kahn and P. Edelman, for a subexponential upper bound in this case.

Our methods are not quite as *ad hoc* as they may seem. In fact they are closely related to the compression techniques of extremal set theory, as surveyed in [4]. They lead naturally, as a topological counterpart, to the study of maximal f- and β-vectors of simplicial complexes, as pursued in [2].

As a warm-up exercise in *extremal poset theory*, we ask for the maximal numbers of (maximal) chains and antichains in n element posets.

Already for these simple questions, the keys to the proofs are the corresponding monotonicity properties. More precisely, (weak) monotonicity is needed to allow for induction proofs, whereas strict monotonicity allows to characterize all the extremal examples.

Lemma 1.2:

Let P be a poset and $x \in P$.

(1) $\mathrm{C}(P-x) < \mathrm{C}(P)$
$\mathrm{MC}(P-x) \le \mathrm{MC}(P)$

(2) $\mathrm{A}(P-x) < \mathrm{A}(P)$
$\mathrm{MA}(P-x) \le \mathrm{MA}(P)$

Proposition 1.3:

Let P be a poset of length $\ell-1$ with n elements ($0<\ell\le n$).

(1) *P contains at most*

$$\mathrm{C}(n,\ell) \ := \max_{\substack{p_1+\ldots+p_\ell=n\\ p_i\ge 1}} \prod_{i=1}^{\ell}(p_i+1) \tag{1.1}$$

chains, with equality if and only if P is an ordinal sum of antichains

$$P \simeq p_1\mathbf{1}\oplus\ldots\oplus p_\ell\mathbf{1}$$

such that the p_i achieve the maximum in (1.1).

(2) *P contains at most*

$$\mathrm{MC}(n,\ell) \;:=\; \max_{\substack{p_1+\ldots+p_\ell=n\\ p_i\geq 1}} \prod_{i=1}^{\ell} p_i \tag{1.2}$$

maximal chains, with equality if (but not only if) P is an ordinal sum of antichains

$$P \simeq p_1\mathbf{1} \oplus \ldots \oplus p_\ell \mathbf{1}$$

that achieves the bound in (1.2).

Proof:

Let P be a poset of cardinality n, and let $T := \max(P)$ be its set of maximal elements. From P we form a new poset $P' := (P-T)\oplus T$. It has the same length $\ell(P) = \ell(P')$. For the number $\mathrm{C}(P)$ of chains in P we get, denoting the number of chains that contain the element x by $\mathrm{C}(x)$,

$$\mathrm{C}(P) = \sum_{x\in T} \mathrm{C}(x) + \mathrm{C}(P-T) \leq \mathrm{C}(P-T)(|T|+1) = \mathrm{C}(P'),$$

with equality if and only if $P = P'$. Similarly for the number of maximal chains

$$\mathrm{MC}(P) = \sum_{x\in T} \mathrm{MC}(x) \leq \mathrm{MC}(P-T)|T| = \mathrm{MC}(P'),$$

where equality holds if $P \simeq (P-T)\oplus T = P'$, but also, for example, for $P = \mathbf{1}+\mathbf{2}$ with $n=3$ (and $\ell = 2$).

By induction on size or length, the claims now follow. □

For fixed ℓ, the expressions for $\mathrm{C}(n,\ell)$ and $\mathrm{MC}(n,\ell)$ grow like polynomials in n of degree ℓ. If we do not restrict the length, we get exponential formulas for the maximal number of (maximal) chains in a poset of given size.

Namely, if P has n elements, then it contains at most 2^n chains, with equality if and only if P is a chain. This (trivial) result follows from Proposition 1.3 with

$$\max_{0<\ell\leq n} \mathrm{C}(n,\ell) = 2^n,$$

where the maximum is achieved only for $p_1 = \ldots = p_\ell = 1$ in formula (1.1).

Similarly, if P is an n element poset, then it has at most $\max_{0<\ell\leq n} \mathrm{MC}(n,\ell)$ maximal chains. To evaluate this, we observe $2(n-2) \geq n$ for $n \geq 4$ (with equality only for $n=4$), such that without loss of generality we may assume $p_i \in \{2,3\}$ for $n>1$. From $2^3 < 3^2$ we see that every optimal product in (1.2) contains the factor '2' at most twice. Thus for $n \geq 2$ we get

$$\mathrm{MC}(n) = \max_{0<\ell\leq n} \mathrm{MC}(n,\ell) = \begin{cases} 3^{\frac{n}{3}} & \text{for } n \equiv 0 \bmod 3,\\ 4\cdot 3^{\frac{n-4}{3}} & \text{for } n \equiv 1 \bmod 3,\\ 2\cdot 3^{\frac{n-2}{3}} & \text{for } n \equiv 2 \bmod 3.\end{cases}$$

In particular we get $\frac{4}{3^{\frac{4}{3}}} \cdot 3^{\frac{n}{3}} \leq \textsc{mc}(n) \leq 3^{\frac{n}{3}}$ for $n \geq 2$, and thus

$$\lim_{n\to\infty} \frac{\log \textsc{mc}(n)}{n} = \frac{1}{3}\log 3.$$

$\textsc{mc}(n)$ is the maximal number of maximal complete subgraphs of a comparability graph on n nodes. Moon & Moser [5] prove that the same upper bound holds for arbitrary graphs.

Similarly, we can treat the maximal number of (maximal) antichains in a poset of given width (maximal size of an antichain) w.

Proposition 1.4:

Let P be a poset of width w with n elements ($0 < w \leq n$).

(1) *Then P contains at most $\textsc{c}(n, w)$ antichains, with equality if and only if P is a sum of chains*

$$P \simeq \mathbf{p}_1 + \ldots + \mathbf{p}_w$$

such that the p_i achieve the maximum in (1.1).

(2) *P contains at most $\textsc{mc}(n, w)$ maximal antichains, with equality if (but not only if) P is a sum of chains*

$$P \simeq \mathbf{p}_1 + \ldots + \mathbf{p}_w$$

that achieves the bound in (1.2).

Proof:

Let P be a poset of cardinality n and width w. By Dilworth's Theorem [1, Theorem 8.14] P can be written as a disjoint union $P = C_1 \uplus \ldots \uplus C_w$ of w chains. We will compare P to $P' = C_1 + \ldots + C_w$.

Every antichain of P is an antichain of P', and thus we get for the numbers of antichains

$$\textsc{a}(P) \leq \textsc{a}(P') = \prod_{i=1}^{w} (p_i + 1)$$

for $p_i = |C_i|$, with equality if and only if every antichain in P' is an antichain in P, that is, $P' \simeq P$.

Now consider a fixed antichain A_0 in P that has maximal size $w = |A_0|$, so that $A_0 \cap C_i = \{a_i^0\}$ ($1 \leq i \leq w$). With this for every maximal antichain A of P we define a maximal antichain A' of P' by

$$A' := A \uplus \{a_i^0 : A \cap C_i = \emptyset\}.$$

Now consider maximal antichains A_1, A_2 of P such that $A_1' = A_2'$. We get for these $A_1 - A_0 = A_1' - A_0 = A_2' - A_0 = A_2 - A_0$. Thus $A_1 \triangle A_2 \subseteq A_0$ and this implies that $A_1 \cup A_2$ is again an antichain in P, hence $A_1 = A_2$. From this we can conclude

$$\textsc{ma}(P) \leq \textsc{ma}(P') = \prod_{i=1}^{w} p_i.$$

which proves the result.

Considering $P = \mathbf{1} \oplus (\mathbf{1} + \mathbf{1})$ for $n = 3$ (and $w = 2$), we see that the sums of chains are *not* the only extremal posets. $\square$

It may seem curious that the extremal numbers coincide for the cases of chains and of antichains. However, this fits nicely into the duality between chains and antichains that appears throughout much of extremal set theory and may to a certain extent be explained in linear programming terms.

In fact the reader may observe that the extremal posets, although different, are in both cases series parallel posets. For every series parallel poset P, there is another series parallel poset P' on the same ground set, such that the (maximal) chains of P are (maximal) antichains of P', and *vice versa.* It is easy to see how the extremal posets for the Theorems 1.3 and 1.4 correspond to each other in this way. (The fact that for a poset P there is a poset P' whose chains are the antichains of P extends to, and in fact characterizes, posets of order dimension at most 2; see [7, Exercise 3.10] for further references.)

The same phenomena appear in the analysis of posets with minimal numbers of (maximal) chains or antichains, which we leave to the interested reader to work out.

2. Posets with maximal Möbius function

Definition 2.1:

For $n \geq 0$, let

$$\mu_{\mathrm{P}}(n) := \max\{|\mu(P)| : \ P \text{ a bounded poset}, |\overline{P}| = n\},$$

and for $0 \leq r \leq n$,

$$\mu_{\mathrm{GP}}(n, r) := \max\{|\mu(P)| : \ P \text{ a graded poset of rank } r+1, |\overline{P}| = n\}.$$

Lemma 2.2:

(1) $\mu_{\mathrm{P}}(0) = 1, \quad \mu_{\mathrm{P}}(1) = 0, \quad \mu_{\mathrm{P}}(2) = 1.$
$\mu_{\mathrm{P}}(n+1) \geq \mu_{\mathrm{P}}(n)$ *for* $n \geq 1$.

(2) $\mu_{\mathrm{GP}}(n, r) = 0$ *for* $n < 2r$, $\mu_{\mathrm{GP}}(2r, r) = 1$.
$\mu_{\mathrm{GP}}(n, 1) = n - 1$,
$\mu_{\mathrm{GP}}(n+1, r) \geq \mu_{\mathrm{GP}}(n, r)$ *for* $n \geq r$.

Proof:

If P is any bounded poset, $n = |\overline{P}| > 0$, then we can form a new poset P' by adding to P a new element x that covers a unique element $y \in \overline{P}$. Then $\mu(\hat{0}, x) = 0$, hence $\mu(P') = \mu(P)$, which proves $\mu_{\mathrm{P}}(n+1) \geq \mu_{\mathrm{P}}(n)$ for $n \geq 1$.
If P is graded, and $r(P) > 2$, then we can construct P' such that y has corank 2, and such that $y \lessdot x \lessdot \hat{1}$. Then P' is again graded of the same rank, which proves the monotonicity of $\mu_{\mathrm{GP}}(n, r)$ in n. ∎

Our goal will be to determine $\mu_{\mathrm{P}}(n)$ and $\mu_{\mathrm{GP}}(n, r)$. In this Section we will deal with the first case of $\mu_{\mathrm{P}}(n)$, which turns out to be easy ("[3–]" in R. Stanley's [7] classification) because the posets achieving the bound have very simple structure.

Definition 2.3:

A bounded poset has level type if it is an ordinal sum of antichains.

If P has level type, rank $\ell(P) = r + 1$, and rank generating function $\sum_{i \geq 0} p_i t^i$, then

$$P = \hat{0} \oplus p_1 \mathbf{1} \oplus \ldots \oplus p_r \mathbf{1} \oplus \hat{1}.$$

In this case we compute

$$\mu(P) = (-1)^{r+1} \prod_{i=1}^{r} (p_i - 1),$$

for example by induction on r, with Lemma 1.1.
Let $\mu_{\mathrm{P}}^{\mathrm{L}}(n)$ and $\mu_{\mathrm{GP}}^{\mathrm{L}}(n, r)$ be the maximal Möbius functions achieved by posets (respectively, by graded posets of given length) of level type. For these we can make explicit computations.

Lemma 2.4:

$$\mu^{\mathrm{L}}_{\mathrm{GP}}(n,r) = \max_{\substack{p_1+\ldots+p_r=n\\ p_i\geq 1}} \prod_{i=1}^{r}(p_i-1) \text{ for } n \geq r.$$

$$\mu^{\mathrm{L}}_{\mathrm{P}}(n) = \max_{0\leq r\leq n} \mu^{\mathrm{L}}_{\mathrm{GP}}(n,r).$$

In particular, $\mu^{\mathrm{L}}_{\mathrm{P}}(n)$ *is strictly increasing in* n *for* $n \geq 1$.

With this we obtain our first main result.

Theorem 2.5:

Let P *be a bounded poset of length* $\ell + 1$ *and* $|\overline{P}| = n$. *Then the Möbius function of* P *is bounded by*

$$|\mu(P)| \;\leq\; \max_{0\leq r\leq \ell} \max_{\substack{p_1+\ldots+p_r=n\\ p_i\geq 1}} \prod_{i=1}^{r}(p_i-1). \tag{2.1}$$

If P *satisfies this with equality, then* P *has level type.*
Furthermore, for every poset P *there is a bounded poset* P^* *with* $|P| = |P^*|$ *and* $\ell(P^*) \leq \ell(P)$ *which achieves the bound in* (2.1).
In particular, this means that

$$\mu_{\mathrm{P}}(n) \;=\; \max_{r\geq 0} \max_{\substack{p_1+\ldots+p_r=n\\ p_i\geq 1}} \prod_{i=1}^{r}(p_i-1). \tag{2.2}$$

Proof:

We use induction on n, the claim being trivial for $n = 0, 1, 2$ with $\mu_{\mathrm{P}}(0) = 1$, $\mu_{\mathrm{P}}(1) = 0$ and $\mu_{\mathrm{P}}(2) = 1$ (cf. Lemma 2.2(1)).
Let P be a poset with maximal Möbius function, $|\overline{P}| \geq 2$. Recall $\acute{P} = P - \{\hat{1}\}$. By $T := \max(\overline{P}) = \{x_1, \ldots, x_m\}$ we denote the elements in P that are covered by $\hat{1}$, that is, T is the set of maximal elements of $\acute{P}$.
Now from [7, Lemma 3.14.4] we get

$$\mu(P) = \mu(P - T) - \sum_{j=1}^{m} \mu(\hat{0}, x_j).$$

Here we assume that the elements $x_j \in T$ are labelled so that x_1 makes a maximal contribution to $|\mu(P)|$, that is,

$$-\sigma\mu(\hat{0}, x_1) = \max_{1\leq j\leq m} (-\sigma\mu(\hat{0}, x_j)),$$

where $\sigma = \mathrm{sign}(\mu(P))$. This allows to construct a poset P' from P as follows:

$$P' = (P - T) \uplus \{y_1, \ldots, y_m\}$$

where for $x \in \acute{P} - T$:

$$x <_{P'} y_j \Longleftrightarrow x \leq_P x_1$$

and the y_j are (incomparable) maximal elements of $\overline{P'}$, that is, if $T' = \max(\overline{P'})$ denotes the elements of P' covered by $\hat{1}$, then

$$\{y_1, \ldots, y_m\} \subseteq T'.$$

In analogy with the "compression" technique of extremal set theory (see [4, Sec. 8]), we could call P' a *compression* of P.

The poset P' is constructed so that $\ell(P') \leq \ell(P)$ and

$$\mu(P') = \mu(P - T) - m\mu(\hat{0}, x_1),$$

where $P - T = P' - T$, and thus

$$|\mu(P')| \geq |\mu(P)|,$$

because of our special choice of x_1. Because $|\mu(P)|$ was maximal, this implies that $\mu(\hat{0}, x_j) = \mu(\hat{0}, x_1)$ for $1 \leq j \leq m$.

Now as long as $\{y_1, \ldots, y_m\} \subset T'$ we can iterate this construction, producing a sequence of posets $P, P', P'' \ldots$ with $|\mu(P)| \leq |\mu(P')| \leq \ldots$ and $|\ell(P)| \geq |\ell(P')| \geq \ldots$. Furthermore $|T| < |T'| < \ldots$, and thus the sequence has to stop because $|T^{(i)}| \leq n$.

Thus it suffices to study the case where P, P' are as above and $\{y_1, \ldots, y_m\} = T'$; we have to show that then $P \simeq P'$.

In this case we have $x_1 > x$ for all $x \in \acute{P} - T$, and hence

$$P' \simeq (\acute{P} - T) \oplus m\mathbf{1} \oplus \hat{1}.$$

We conclude that $P - T \simeq [\hat{0}, x_1]$, hence $\mu(P - T) = \mu(\hat{0}, x_1)$ and

$$\mu(P) = \mu(P') = -(m-1)\mu(\hat{0}, x_1).$$

This implies, by induction, that $[\hat{0}, x_1]$ has level type. Because $\mu_{\mathrm{P}}^{\mathrm{L}}$ is strictly monotone by Lemma 2.4, we have $|[\hat{0}, x_j]| = |[\hat{0}, x_1]|$ for all $1 \leq j \leq m$ and hence every x_j satisfies $x_j > x$ for every $x \in \acute{P} - T$. Thus

$$P \simeq (\acute{P} - T) \oplus m\mathbf{1} \oplus \hat{1}$$

and hence P has level type, with $P \simeq P'$. □

To illustrate the proof technique of Theorem 2.5, we show in Figure 2.1 the posets P' and P'' derived from a given poset P.

Figure 2.1: Two alternative sequences of posets $P \longrightarrow P' \longrightarrow P''$ as generated in the proof of Theorem 2.5, with $\mu(P) = \mu(P') = 0$ and $\mu(P'') = -2$ (top) respectively $\mu(P'') = 4$ (bottom).

Of course it is interesting to evaluate the expression for $\mu_{\mathrm{P}}(n)$ given by Theorem 2.5. The same technique we used to compute $\mathrm{MC}(n)$ in Proposition 1.3 here yields

$$\mu_{\mathrm{P}}(n) = \begin{cases} n-1 & \text{for } 1 \leq n \leq 7, \\ 4^{\frac{n}{5}} & \text{for } n \equiv 0 \bmod 5, \\ 5 \cdot 4^{\frac{n-6}{5}} & \text{for } n \equiv 1 \bmod 5,\ n \geq 6, \\ 3^i \cdot 4^{\frac{n-4i}{5}} & \text{for } n \equiv -i \bmod 5,\ i \in \{1,2,3\},\ n \geq 8, \end{cases}$$

In particular, this yields $5 \cdot 4^{\frac{n-6}{5}} \leq \mu_{\mathrm{P}}(n) \leq 4^{\frac{n}{5}}$ for all $n \geq 8$ and thus

$$\lim_{n\to\infty} \frac{\log \mu_{\mathrm{P}}(n)}{n} = \frac{2}{5} \log 2.$$

3. Ranked posets with maximal Möbius function

Since the posets with maximal Möbius function are all of level type and hence graded, one would assume that the posets of level type again maximize the Möbius function among the graded posets of given rank and cardinality. This is in fact the case.

However, our proof for Theorem 2.5 does not solve the problem for graded posets of given rank because the P' constructed there may not be a graded poset even if P is, and the length may drop (see Figure 2.1).

Instead, we will now work with an induction on n that depends on removing a suitable element from P. The problem we face here is that even if P is graded with all p_i large, there may not be an $x \in \overline{P}$ such that $P - x$ is again graded. This motivates the attack on the more general problem for posets P such that $\acute{P} = P - \{\hat{1}\}$ is ranked. This property is preserved under removal of maximal elements of $\acute{P}$.

One more problem arises: even if we only consider graded posets (of given rank), not all extremal posets have level type. The smallest example for this arises is $n = 4$ and $r = 2$, with $\mu_{\text{GP}}(4,2) = 1$ and $P = \hat{0} \oplus (\mathbf{2}+\mathbf{2}) \oplus \hat{1}$. If we only require that $\acute{P}$ be ranked, then there may not even be an extremal graded poset of level type (e.g., for $n = 3$ and $r = 2$).

The following results contain a complete analysis of the extremal cases that can arise.

Definition 3.1:

A bounded poset P of length $\ell(P) = r+1$ has generalized level type if $\overline{P}$ is an ordinal sum of antichains and of copies of posets $P_k = \mathbf{k}+\mathbf{k}$ ($k \geq 2$).

Equivalently, P is of generalized level type if it is graded and every rank selection $P_{\{i,i+1\}}$ for $1 \leq i < r$ is isomorphic either to the ordinal sum $m\mathbf{1} \oplus n\mathbf{1}$ of two antichains, or to the sum $\mathbf{2}+\mathbf{2}$.

Theorem 3.2:

Let P be a bounded poset of length $r+1$ such that $\acute{P} = P - \{\hat{1}\}$ is ranked with rank generating function $1 + \sum_{i=1}^{r} p_i t^i$ ($p_i \geq 1$). Let $s := \min\{i \geq 1 : p_i = 1\}$ (where we set $p_{r+1} := 1$). Then

$$|\mu(P)| \;\leq\; \prod_{i=1}^{s-1} (p_i - 1). \qquad (3.1)$$

Furthermore, if $t := \max\{i < s : p_i \geq 3\}$ (with $t = 1$ if this set is empty) and P achieves the bound in (3.1), then there is a unique t' satisfying $t \leq t' \leq s$ such that

(i) the rank selection

$$P_{[t']} = \{\hat{0}\} \cup \{x \in \overline{P} : 1 \leq r(x) \leq t'\} \cup \{\hat{1}\}$$

of P has generalized level type, with $\mu(\hat{0},x) \neq 0$ for all x of rank t', and

(ii) the elements in $P - P_{[t']}$ do not contribute to the Möbius function of P, that is, $\mu(\hat{0},y) = 0$ for $y \in \acute{P}$ with $r(y) > t'$.

Proof:

Assume that P is a bounded poset with maximal $|\mu(P)|$ among the bounded posets for which $\acute{P}$ is ranked with a given rank generating function $1+\sum_{i=1}^{r} p_i t^i$. We may assume $s>1$ and hence $p_1 \geq 2$. Thus $|\mu(P)| \geq \prod_{i=1}^{s-1}(p_i - 1)$, which is achieved taking $P_{[t]}$ of level type (cf. Lemma 2.4) and inserting the other elements $y \in \acute{P}$ with $r(y) > t$ so that $\mu(\hat{0}, y) = 0$. In particular we have $\mu(P) \neq 0$.

We proceed by induction on $n = \sum_{i=1}^{r} p_i$.

(i) We may assume that $s \in \{r, r+1\}$ (that is, $p_i \geq 2$ for $1 \leq i < r$), because $p_i = 1$ implies $\mu(\hat{0}, y) = 0$ for all $y \in \acute{P}$ with $r(y) > i$, so that these elements can be deleted from P without changing $\mu(P)$, and we are done by induction.

(ii) Now, as the **first case** to consider, assume that there is an element $x \in \max(\acute{P})$ with $p_{r(x)} \geq 3$. By (i) we have $r(x) < s$.

Then from $\mu(P) = \mu(P-x) - \mu(\hat{0}, x)$

with

$$|\mu(P)| \geq \prod_{i=1}^{s-1}(p_i - 1), \tag{3.2}$$

$$|\mu(\hat{0}, x)| \leq \prod_{i=1}^{r(x)-1}(p_i - 1), \tag{3.3}$$

$$|\mu(P-x)| \leq (p_{r(x)} - 2) \cdot \prod_{\substack{1 \leq i < s \\ i \neq r(x)}} (p_i - 1), \tag{3.4}$$

we get

$$\prod_{i=r(x)+1}^{s-1}(p_i - 1) = 1,$$

hence $r(x) = s-1$, and equality holds in (3.2), (3.3) and (3.4). Thus $P - x$ and $[\hat{0}, x]$ have the described structure by induction, and the rest is routine to check.

(iii) For our **second case**, assume that there is an element $x \in \max(\acute{P})$ with $p_{r(x)} = 2$ and $r(x) < r$.

Then we get $\mu(\hat{0}, y) = 0$ for all $y \in \acute{P}$ with $r(y) > r(x)$ as in (i), considering the poset $P - x$. This implies that we can delete these elements y from P, and the induction proceeds.

(iv) Now all we are left with is the **third case** where $\max(\acute{P}) = T = \{y \in \acute{P} : r(y) = r\}$ and $|T| \leq 2$.

If $T = \{x_1\}$, then this implies $x \leq x_1 \lessdot \hat{1}$ for all $x \in \acute{P}$ and thus $\mu(P) = 0$.

If $T = \{x_1, x_2\}$, then with the "compression technique" of Theorem 2.5 we can modify P so that both top elements x_1, x_2 cover the same set of elements in P. The argument there in particular implies $\mu(\hat{0}, x_1) = \mu(\hat{0}, x_2)$. If now $x_1, x_2 > x$ for all $x \in \acute{P} - \{x_1, x_2\}$, then $\acute{P} = (\acute{P} - T) \oplus T$ and we are done by induction. Otherwise we get new maximal elements in $\acute{P}$ and have thus reduced the problem to case (ii) or (iii). □

Figure 3.1: Two ranked posets $\acute{P}$ with maximal $|\mu(\acute{P} \oplus \hat{1})| = 2$ (and $t' = 4$ respectively $t' = 3$) for given rank generating function $1 + 2p + 2p^2 + 3p^3 + 2p^4 + 2p^5$.

Corollary 3.3:

Let P be a graded poset of rank $r + 1$, with $|\overline{P}| = n \geq r$. Then

$$|\mu(P)| \ \leq \ \mu_{\text{GP}}(n, r) \ = \max_{\substack{p_1 + \ldots + p_r = n \\ p_i \geq 1}} \prod_{i=1}^{r} (p_i - 1),$$

and this bound is sharp.
Furthermore, if P achieves this bound and $n \geq 2r$, then P has generalized level type.
If $n \geq 3r - 1$, then P has level type.

Proof:

The first part is clear from Theorem 3.2. Note that if $n < 2r$, then $p_i = 1$ for some $1 \leq i \leq r$, which for a graded poset implies $\mu(P) = 0$. For $n \geq 2r$ we can choose $p_i \geq 2$ for all i, hence $\mu_{\text{GP}}(n, r) > 0$ in this case.
If $p_i = 2$ for some i in an optimal poset, then $p_j \leq 3$ for all j, because $p_j - 1 < 2(p_j - 2)$ for $p_j \geq 4$. Thus any optimal poset that contains P_0 as a rank selection has $n \leq 2 \cdot 2 + (r - 2) \cdot 3 = 3r - 2$. □

4. Posets with bounded degrees

In [7, Exercise 3.42] R. Stanley asks for the maximum Möbius function of a bounded poset of length $\ell+1$ in which every element is covered by at most k other elements.

P. Edelman, who originally posed this problem, noted (see [7, p.187]) that the answer is *not* the obvious one: one can do considerably better than $|\mu(P)| = (k-1)^\ell$, which is achieved by the poset $P = \hat{0} \oplus k\mathbf{1} \oplus \ldots \oplus k\mathbf{1} \oplus \hat{1}$ of level type.

In this Section we will systematically construct examples for this. Our construction technique will then be employed to yield examples for an even more restricted version, in which we require that not only the number of covers, but also the number of cocovers of every element in P is bounded by k, that is, so that in the Hasse diagram of P (viewed as a directed graph), every element has not only its outdegree, but also its indegree bounded by k.

Definition 4.1:

Fix integers $k \geq 1$ and $\ell \geq 0$.

(1) *A finite bounded poset P has updegree bounded by k if every element in P has at most k covers.*
The maximal absolute value of the Möbius function on bounded posets of length $\ell+1$ with updegrees bounded by k will be denoted by $\mu_{\text{UD}}(k,\ell)$.

(2) *A finite bounded poset P has degrees bounded by k if every element in P covers at most k other elements, and it is covered by at most k other elements, that is, every $x \in P$ has at most k covers and at most k cocovers.*
The maximal absolute value of the Möbius function on bounded posets of length $\ell+1$ with degrees bounded by k will be denoted by $\mu_{\text{BD}}(k,\ell)$.

We start by collecting some simple cases and lower bounds.

Lemma 4.2:

(i) $\mu_{\text{UD}}(1,\ell) = \mu_{\text{BD}}(1,\ell) = \begin{cases} 1 & \textit{for } \ell = 0, \\ 0 & \textit{else.} \end{cases}$

(ii) $\mu_{\text{UD}}(k,i) = \mu_{\text{BD}}(k,i) = (k-1)^i \quad$ *for* $\; i = 0,1,2.$

(iii) $\mu_{\text{UD}}(k,\ell) \geq \mu_{\text{BD}}(k,\ell) \geq (k-1)^l \quad$ *for all* $\; \ell \geq 0, k \geq 1.$

Proof:

(i) is trivial and (iii) is clear with the above. The only part needing proof is $\mu_{\text{UD}}(k,2) \leq (k-1)^2$, which follows from elementary counting in the possible length 3 posets. □

Lemma 4.3:

$$\mu_{\text{UD}}(k,\ell_1+\ell_2) \;\geq\; \mu_{\text{UD}}(k,\ell_1) \cdot \mu_{\text{UD}}(k,\ell_2),$$

$$\mu_{\text{BD}}(k,\ell_1+\ell_2) \;\geq\; \mu_{\text{BD}}(k,\ell_1) \cdot \mu_{\text{BD}}(k,\ell_2).$$

In particular $\mu_{\text{UD}}(k,\ell)$ and $\mu_{\text{BD}}(k,\ell)$ are monotone in ℓ for $k=2$ and strictly monotone in ℓ for $k \geq 3$.

Proof:

This follows from the series connection construction, see Lemma 1.1. ∎

The lower bound of Lemma 4.2 can be improved considerably. One construction technique is given by the proof of following Proposition. We think that the lower bounds produced by it may well be optimal.

Proposition 4.4:

$$\mu_{\mathrm{UD}}(k,\ell) \geq \max_{s\geq 0} \max_{\substack{t_1+\ldots+t_s=\ell-1\\ t_i\geq 1}} (k-1)\cdot\prod_{i=1}^{s}(k^{t_i}-1), \tag{4.1}$$

$$\mu_{\mathrm{BD}}(k,\ell) \geq \max_{s\geq 0} \max_{\substack{t_1+\ldots+t_s=\frac{1}{2}(\ell+s)-1\\ t_i\geq 1}} (k-1)\cdot\prod_{i=1}^{s}(k^{t_i}-1), \tag{4.2}$$

Proof:

Let $s \geq 0$ and $t_1,\ldots,t_s \geq 1$. For $p_i := k^{t_i}$ we define the bounded poset P of level type (with $\ell(P) = s+2$) by

$$P = \hat{0} \oplus k\mathbf{1} \oplus p_1\mathbf{1} \oplus \ldots \oplus p_s\mathbf{1} \oplus \hat{1},$$

with

$$|\mu(P)| \;=\; (k-1)\cdot\prod_{i=1}^{s}(p_i-1).$$

We will now construct from P a new poset P' with the same Möbius function and updegree bounded by k.

For this, we insert between every element $x > \hat{0}$ of P and its $p_i = k^{t_i}$ covers a complete k-ary tree (of depth t_i). This yields, from P, a new poset P' with updegrees bounded by k so that $\mu(P) = \mu(P')$, (because we have only inserted elements that cover a unique element larger than $\hat{0}$ in P and thus do not contribute to the Möbius function), and $\ell(P') = 2 + \sum_{i=1}^{s} t_i = \ell + 1$. This proves (4.1).

See Figure 4.1 for an example that derives $\mu_{\mathrm{UD}}(2,3) \geq 3$ from the case $s = 1$, $t_1 = 2$.

Figure 4.1: Construction of a poset demonstrating $\mu_{\text{UD}}(2,3) \geq 3$.

For (4.2), we start with the poset P

$$P = \hat{0} \oplus k\mathbf{1} \oplus p_1\mathbf{1} \oplus \ldots \oplus p_s\mathbf{1} \oplus k\mathbf{1} \oplus \hat{1},$$

of level type with $\ell(P) = s + 3$ and

$$|\mu(P)| \;=\; (k-1)^2 \cdot \prod_{i=1}^{s} (p_i - 1).$$

Now for every element that is covered by or covers k^{t_i} elements we insert a complete k-ary tree upwards respectively downwards, so that we get a new poset P' of length $\ell(P') = (s+3) + 2\sum_{i=1}^{s}(t_i - 1) = 3 + \sum_{i=1}^{s}(2t_i - 1)$ and Möbius function $\mu(P') = \mu(P)$. Figure 4.2 shows, for example, how the case $s = 1$, $t_1 = 2$ gives $\mu_{\text{BD}}(2,5) \geq 3$. ∎

Figure 4.2: Construction of a poset demonstrating $\mu_{\text{BD}}(2,5) \geq 3$.

We can at least partially evaluate the bounds in (4.1) and (4.2). In fact, for $k \geq 2$ the function $f_1(t) = \frac{1}{t}\log(k^t - 1)$ is strictly increasing in t, which implies that the maxima in the right-hand side of (4.1) are achieved only for $s = 1$ and $t_1 = \ell - 1$. Thus (4.1) is equivalent to

$$\mu_{\text{UD}}(k,\ell) \geq (k-1)\cdot(k^{\ell-1}-1) \quad \text{for } \ell \geq 2. \tag{4.1$'$}$$

Similarly, we find that for $k \geq 3$, the function $f_2(t) = \frac{1}{2t-1}\log(k^t - 1)$ is maximized on $\mathbb{N}$ by $t = 1$, corresponding to $s = \ell - 2$ and $t_1 = \ldots = t_s = 1$. Thus for $k \geq 3$, (4.2) is equivalent to

$$\mu_{\text{BD}}(k,\ell) \;\geq\; (k-1)^{\ell}, \tag{4.2$'$}$$

which is not too exciting, because it does not beat the trivial lower bound of Lemma 4.2(iii).

However, for $k = 2$ we do better. In this case $f_2(t)$ is maximized on $\mathbb{N}$ by $t = 3$, which (for $s = \frac{1}{5}(\ell - 2)$ and $t_i = 3$) yields that, for $k = 2$ and $\ell \equiv 2 \bmod 5$, (4.2) is equivalent to

$$\mu_{\text{BD}}(2, 5s+2) \;\geq\; 7^{s}. \tag{4.2$''$}$$

The precise evaluation of the bound (4.2) for $\mu_{\text{BD}}(2,\ell)$ in the other cases is tedious. Instead, we will be content with the observation that a combination of (4.2$''$) with Lemma 4.3 implies

$$\mu_{\text{BD}}(2,\ell) \;\geq\; 7^{\frac{\ell-4}{5}} \qquad \text{for } \ell \geq 1 \tag{4.2$'''$}$$

and thus

Corollary 4.5:

$$\lim_{\ell\to\infty} \frac{\log \mu_{\text{BD}}(2,\ell)}{\ell} \;\geq\; \frac{1}{5}\log 7.$$

We will end this Section with a (rather trivial) upper bound for $\mu_{\text{UD}}(k,\ell)$ and $\mu_{\text{BD}}(k,\ell)$. For this we note the following Lemma, which does not seem to be well known. A. Björner has noted that it does *not* extend to the topological setting of simplicial complexes.

Lemma 4.6:

For every bounded poset P with $\overline{P} \neq \emptyset$,

$$|\mu(P)| \;\leq\; \text{MC}(P) - 1.$$

Furthermore, equality holds if and only if $\overline{P}$ is a sum of chains.

Proof:

Induction on $n = |\overline{P}|$. Let $T = \max(\overline{P})$ and choose $x \in T$.

Case 1: If x covers $\hat{0}$, then $\mu(P) = \mu(P-x)+1$ and $\text{MC}(P) = \text{MC}(P-x)+1$, and we are done by induction.

Case 2: If x covers an element $y \in \overline{P}$ such that x is its *only* cover, then $\mu(P) = \mu(P-y)$. Also we have $\text{MC}(P) \geq \text{MC}(P-y)$ (since every maximal chain C of $P-y$ extends to a unique maximal chain of P, which is $C \cup y$ if this is a chain, and C otherwise). Thus the inequality holds by induction:

$$|\mu(P)| = |\mu(P-y)| \leq \text{MC}(P-y) - 1 \leq \text{MC}(P) - 1.$$

For equality to hold, we need $|\mu(P-y)| = \text{MC}(P) - 1$, so that $\overline{P-y}$ is a sum of chains, and $\text{MC}(P) = \text{MC}(P-y)$, that is, every maximal chain of P containing y extends a maximal chain of $P-y$. Thus the maximal chain C of $P-y$ that contains x yields the maximal chain $C \cup \{y\}$ of $\overline{P}$. Now y has no other cover than x, which implies that $\overline{P}$ is a sum of chains as well.
Case 3: From $\mu(P) = \mu(P-x) - \mu(\hat{0}, x)$, and by induction, we get

$$\begin{aligned} |\mu(P)| &\leq |\mu(P-x)| + |\mu(\hat{0}, x)| \\ &\leq \text{MC}(P-x) + \text{MC}(\hat{0}, x) - 2. \end{aligned}$$

Now every maximal chain C of $\overline{P}$ either is a maximal chain of $\overline{P} - x$ (if $x \notin C$), or it is a maximal chain of $(\hat{0}, x]$ (where $x \in C$). This yields the "$\leq$" part of

$$\text{MC}(P) = \text{MC}(P-x) + \text{MC}(\hat{0}, x).$$

Here equality holds because every maximal chain in $P-x$ also is a maximal chain in P, that is, there is no $y \in P$ for which x is the only cover: otherwise we would be in Case 2.
Thus we can conclude

$$|\mu(P)| \leq \text{MC}(P) - 2$$

in this Case 3, which completes the proof. □

From this Lemma we get

$$\mu_{\text{BD}}(k, \ell) \ \leq \ \mu_{\text{UD}}(k, \ell) \ \leq \ k^{\ell} - 1,$$

where the second inequality holds with equality only for $\ell = 1$ and for $k = 1$.

Corollary 4.7:

$$\log k \ \geq \ \lim_{\ell \to \infty} \frac{\log \mu_{\text{UD}}(k, \ell)}{\ell} \ \geq \ \lim_{\ell \to \infty} \frac{\log \mu_{\text{BD}}(k, \ell)}{\ell} \ \geq \ \begin{cases} \frac{1}{5} \log 7 & \textit{for } k = 2, \\ \log(k-1) & \textit{for } k \geq 3. \end{cases}$$

5. Lattices with maximal Möbius function

Surprisingly, the question for the maximal Möbius function on the *lattices* with $n+2$ elements [7, Exercise 3.41b] turns out to be much harder than the same question for posets.

One reason is that the conjectured extreme examples, namely, the subspace lattices $L(k,q)$ of finite vector spaces, do not exist for arbitrary n, so that proofs by induction on the size become impossible.

Definition 5.1:

For $n \geq 0$, $\mu_{\mathrm{L}}(n)$ is the maximal absolute value of the Möbius function attained on a lattice with $n+2$ elements.

As long as some of the $(n+2)$-element posets with maximal Möbius function are lattices, we get, of course, that $\mu_{\mathrm{L}}(n) = \mu_{\mathrm{P}}(n)$; as soon as there is no lattice among the posets achieving $\mu_{\mathrm{P}}(n)$, we get $\mu_{\mathrm{L}}(n) < \mu_{\mathrm{P}}(n)$.

Lemma 5.2:

(i) $\mu_{\mathrm{L}}(0) = 1$
$\mu_{\mathrm{L}}(n) = \mu_{\mathrm{P}}(n) = n-1$ *for* $1 \leq n \leq 7$
$\mu_{\mathrm{L}}(n) < \mu_{\mathrm{P}}(n)$ *for* $n \geq 8$

(ii) $\mu_{\mathrm{L}}(n+1) \geq \mu_{\mathrm{L}}(n)$ *for* $n \geq 1$.

Proof:

(i) Theorem 2.5.

(ii) See Lemma 2.2. □

However, $\mu_{\mathrm{L}}(n)$ is probably not even strictly increasing. It seems that reasonable asymptotics are the most one can hope for in this case.

Conjecture 5.3:

(1) $\mu_{\mathrm{L}}(n) = n-1$ *for* $1 \leq n \leq 25$
$\mu_{\mathrm{L}}(26) = \mu_{\mathrm{L}}(27) = 27.$

(2) $\lim_{n\to\infty} \frac{\mu_{\mathrm{L}}(n)}{n^2} = 0.$

This conjecture arises from the analysis of the linear lattices $L_k(q)$, as suggested by R. Stanley. For these $|\mu(L_k(q))| = q^{\binom{k}{2}}$ [7, Example 3.10.2], whereas $|L_k(q)|$ is a polynomial in q of degree $\lfloor \frac{k^2}{4} \rfloor$. Thus the exponent 2 in Conjecture 5.3(2) cannot be replaced by $2-\epsilon$ for any $\epsilon > 0$, that is, Conjecture 5.3(2) is essentially best possible.

However, if we denote by $\mu_{\mathrm{L}}(n,\ell)$ the maximal $|\mu(L)|$ attained by lattices with cardinality $|L| = n+2$ and length bounded as $\ell(L) \leq \ell+1$, then the linear lattices lead to the sharper conjecture

$$\mu_{\mathrm{L}}(n,\ell) = O\big(n^{2-\lceil\frac{\ell+1}{2}\rceil^{-1}}\big) \qquad \text{for} \qquad \ell \geq 1.$$

This is clear for $\ell = 1$ (with $\mu_{\text{L}}(n,1) = n-1$), and will follow in Proposition 5.4 and Corollary 5.5(2) below for $\ell = 2$.

Without length restriction, we consider $\mu_{\text{L}}(n) = \max_\ell \mu_{\text{L}}(n,\ell)$. For small values of n, the lattice of length 2 with $n+2$ elements is best possible. The smallest lattice we know of with $\mu(L) > |\overline{L}| - 1$ is the linear lattice $L_3(3)$ with $\mu(L_3(3)) = -27$ and $|\overline{L_3(3)}| = 26$. This leads to $\mu_{\text{L}}(26) \geq 27$, and a trivial modification of $L_3(3)$ as in Lemma 2.2(1) leads to $\mu_{\text{L}}(27) \geq 27$. In 5.3(1) we conjecture that this is best possible.

We can get close to this with graph theory methods.

Proposition 5.4:

Let L be a finite lattice with $n+2$ elements ($n \geq 1$) and of length at most 3. Then

$$n - 1 \geq \mu(L) \geq n - 1 - \frac{1}{4}\Big\{n + \sqrt{n^2 + 8\Big\lfloor \frac{n^2}{4} \Big\rfloor (n-2)}\Big\}, \qquad (5.1)$$

where the upper bound for $\mu(L)$ is sharp if and only if L has length 2, and the lower bound holds with equality if and only if L is the lattice of a projective plane.

Proof:

Let L be a lattice of length at most 3. Then $\acute{L}$ is ranked, with n_1 elements of rank 1 and $n_2 = n - n_1$ elements of rank 2.

We can consider $\overline{L}$ as a bipartite graph with color classes of sizes n_1 and n_2, and with e edges corresponding to the cover relations in $\overline{L}$.

Now the Möbius function of L is given by

$$\mu(L) = -1 + n - e,$$

which can be seen directly from the definitions or as a special case of [7, Prop. 3.8.5]. With this we can use methods of extremal graph theory to derive bounds on e. For this we note that $\overline{L}$ is a bipartite graph without quadrilateral (without $K_{2,2}$ subgraph), which means

$$0 \leq e \leq z(n_1, n_2, 2, 2)$$

in the notation of Bollobás [3, p.309].

With the argument from [3, Theorem VI.2.6(i)] we now derive the bounds $e(e - n_1) \leq n_1 n_2 (n_2 - 1)$ and $e(e - n_2) \leq n_2 n_1 (n_1 - 1)$. Adding up these two inequalities (with $n_1 + n_2 = n$ and $n_1 n_2 \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = \lfloor \frac{n^2}{4} \rfloor$) we get

$$e(2e - n) \leq \Big\lfloor \frac{n^2}{4} \Big\rfloor (n-2),$$

which implies (5.1). Equality in (5.1) implies $n_1 = n_2 = \frac{n}{2}$ and that L is a projective plane on n_1 points ($n_1 = q^2 + q + 1$), as in the proof of [3, Thm. VI.2.6(ii)]. □

Corollary 5.5:

(1)
$$\mu_L(n,2) = n-1 \quad \text{for } 1 \le n \le 24,$$
$$24 \le \mu_L(25,2) \le 25, \qquad \text{and}$$
$$\mu_L(26,2) = 27.$$

(2)
$$\lim_{n\to\infty} \frac{\mu_L(n,2)}{n^{\frac{3}{2}}} = \frac{1}{\sqrt{8}}.$$

Proof:

(1) follows from Proposition 5.4. For (2), see Bollobás [3, Cor. VI.2.7]. □

We are far from a proof of Conjecture 5.3(2). However, a clever argument by J. Kahn and P. Edelman (personal communication, 1988) at least yields that $\mu_L(n)$ grows subexponentially.

Proposition 5.6: (Kahn and Edelman)

$\mu_L(n) \le \mu_L(n-1) + \mu_L(\lfloor \frac{n}{2} \rfloor - 1) \qquad \text{for} \quad n \ge 6.$

Proof:

Let L be a finite lattice and x an atom of L. If $\mu(L) \ne 0$, then by Weisner's Theorem [7, Corollary 3.9.3] there is a coatom $y \in L$ such that $x \not\le y$. Then $[x,\hat{1}] \cap [\hat{0},y] = \emptyset$ and thus one of these two intervals has at most cardinality $\lfloor \frac{n}{2} \rfloor + 1$. Taking, if necessary, the order dual, we may therefore assume $|[x,\hat{1}]| \le \lfloor \frac{n}{2} \rfloor + 1$.

Now $L - x$ is again a finite lattice, with $\mu(L) = \mu(L-x) - \mu(x,\hat{1})$, from which we get the claim using the monotonicity of $\mu_L(n)$ (Lemma 5.2(ii)). □

From this, standard techniques of asymptotic analysis imply $\mu_L(n) = o(n^{\log_2 n})$, that is, $\lim_{n\to\infty} \frac{\mu_L(n)}{2^{(\log_2 n)^2}} = 0$ (and thus in particular $\lim_{n\to\infty} \frac{\log \mu_L(n)}{n} = 0$).

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