Abstract:

H. Terao has shown that the structure of the module of (rational) differential forms with at most logarithmic poles at an arrangement of hyperplanes (as defined by K. Saito) is very strongly controlled by the combinatorial structure of the arrangement.

In this paper we demonstrate how the existence of rational logarithmic forms with poles of high order depends on the existence of highly degenerate ('special position') subarrangements. The associated combinatorial structures are studied.

First a strong version of K. Saito's "Preparation Lemma" for logarithmic forms leads to a new, simple proof of Terao's celebrated "Addition-Deletion Theorem" for hyperplane arrangements with free module of logarithmic 1-forms ('free arrangements').

Then structural characterizations are developed in two extreme cases for the generation of the module of logarithmic differential 1-forms: this module has a triangular basis iff the arrangement is supersolvable (strictly linearly fibered), and it is generated by forms of degree -1 iff the arrangement is generic in codimension 3. Both conditions on the geometry of the arrangement are combinatorial in a very strong sense (determined by restricted data on the lattice of intersections of the hyperplanes.) However, examples show that the cardinality and degree sequence of a minimal set of generators for the module of logarithmic 1-forms are not in general determined by this intersection lattice.

Table of Contents

1. Introduction

The module $\Omega^*(X)$ of forms with at most logarithmic poles (*logarithmic forms*) at a reduced divisor X on a smooth complex manifold was introduced by K. Saito [Sa1] in 1980. Since then, H. Terao's study of the module $Der(X)$ of vector fields tangent to the divisor (which is the dual module to $\Omega^1(X)$) in the special case of a hyperplane arrangement X has lead to a remarkable sequence of results [T1,T2,T3,JT,T5,T6], chiefly centering around the question under what combinatorial conditions $Der(X)$ is a free module. However, the study of $\Omega^1(X)$, which is free whenever $Der(X)$ is free, offers the considerable advantage of both very geometric arguments studying the poles of forms in $\Omega^1(X)$, and the existence of many nontrivial elements that can be written down coordinate free in terms of defining equations l_H of the hyperplanes $H \in X$. This paper exploits this geometric point of view to gain new insight into the structure of $\Omega^1(X)$.

Section 2 starts to define hyperplane arrangements and to describe the combinatorial structures associated with it. Section 3 defines the module of logarithmic differential forms at an arrangement, reviews basic properties and gives criteria for certain logarithmic forms to form a basis. In Section 4 we briefly describe a category of hyperplane arrangements which supports these structures, making the module of logarithmic forms into a covariant functor. In Section 5, we prove the "Strong Preparation Lemma" for logarithmic 1-forms and use it to give a new, simple proof of H. Terao's Addition-Deletion Theorem, avoiding the extensive commutative algebra arguments of [T1]. In Section 6 to 8 we study generators for the module $\Omega^1(X)$ and their combinatorial description. The following two extreme cases are characterized combinatorially: the case of free arrangements admitting a basis for $\Omega^1(X)$ that is upper triangular in suitable coordinates (Section 6), and the case where $\Omega^1(X)$ is already generated by the "obvious" logarithmic forms of degree -1 (Section 7). In Section 8 we finally develop a study of the minimal arrangements that support logarithmic differential forms of given negative degrees, and these "critical arrangements" are used to construct an example to show that in general the cardinality and degrees of a minimal set of homogeneous generators for $\Omega^1(X)$ is not determined by the combinatorial invariants of the arrangement.

2. Combinatorial Invariants of an Arrangement

Hyperplane arrangements are mathematical objects arising in a large variety of geometric, algebraic and (last, not least) combinatorial situations. For the theory of logarithmic differential forms [Sa1] they provide a sufficiently complex model for the behaviour localized at a point of a divisor. The linearized case of a hyperplane arrangement has the additional advantage of allowing generalization to an arbitrary field without technical difficulties [T6].

Definition 2.1:

Let V be an n-dimensional vector space over an arbitrary field $\mathbf{k}$. An arrangement in V is a finite set $X = \{H_1, ..., H_m\}$ of linear hyperplanes (linear subspaces of dimension $n-1$) in V. The order of X is $|X| = m$.

Note that the arrangements we consider are assumed to be *central*, that is, all hyperplanes contain the origin of V.

Definition 2.2:

Let X be an arrangement of hyperplanes in $V = \mathbf{k}^n$. For $H \in X$, let $l_H \in V^*$ be a linear form with kernel H, i.e., $H = \{x \in V : l_H(x) = 0\}$. Every l_H is well-defined up to a constant non-zero factor. We call $Q = \Pi_{H \in X} l_H$ a defining polynomial for X. If we choose a basis $\{x_1, \ldots, x_n\}$ of V^*, then Q is a homogeneous polynomial in $S = \mathbf{k}[x_1, \ldots, x_n]$ of degree m, well-determined by X up to a constant factor.

In the following we will study how and to what extent the "algebraic structure" of an arrangement (specifically, the module of logarithmic differential forms) at the arrangement is determined or encoded by the "combinatorial structure" of an arrangement (as encoded by the intersection lattice, given by the following definition.)

Definition 2.3: [Za]

Let X be an arrangement in V. The intersection lattice of X is

$$L = L_X = \Big\{ \bigcap_{H \in Y} H : Y \subseteq X \Big\},$$

the collection of intersections of hyperplanes in X. L is ordered by reverse inclusion, that is, $W_1 \leq W_2 \iff W_1 \supseteq W_2$. The rank of X is

$$r(X) = r(L) = \operatorname{codim}_V(\bigcap X).$$

The relevance and importance of the intersection lattice to the combinatorial structure of hyperplane arrangements was first realized and exploited by T. Zaslavsky [Za], who developed an enumeration theory for hyperplane arrangements in this framework. The reason for ordering by *reverse* inclusion is combinatorial, and follows from the following theorem.

Theorem 2.4:

Let X be an arrangement of hyperplanes, and L its lattice of flats.

(a) *L is a geometric lattice, with minimal element $\hat{0} = V$ and maximal element $\hat{1} = \bigcap X$. Its rank function is given by*

$$r(W) = \operatorname{codim}_V(W) = \dim(V) - \dim(W).$$

The lattice operations in L are

$$W \vee W' = W \cap W',$$

$$W \wedge W' = \bigcap \{H \in X : H \supseteq W \cup W'\}.$$

(b) *L is the lattice of flats for the linear matroid $\{l_H : H \in X\}$ represented in V^*. The corresponding abstract matroid will be denoted by $\mathcal{M}(X)$.*

Definition 2.5:

An arrangement X in $V \cong \mathbf{k}^n$ is essential if $r(L) = n$, that is, if $\bigcap X = \{0\}$. For an arrangement X in V, the associated essential arrangement is the arrangement $X/\bigcap X = \{H/\bigcap X : H \in X\}$ in $V/\bigcap X$.

For the algebraic scenario as well as for induction proofs (starting with the "empty arrangement") and other constructions it is *not* natural to always assume that the arrangement considered is essential, and therefore we will not do it.

The "associated essential arrangement" can be described as a "localization" by the following definition, which demonstrates how basic matroid constructions translate into constructions of hyperplane arrangements.

Definition 2.6:

(a) **Restriction to a hyperplane**

Let X be an arrangement, H a hyperplane. The restriction of X to H is the arrangement $X\Big|_H = \{K \cap H : K \in X - \{H\}\}$. $\mathcal{M}(X\Big|_H)$ is the contraction of $\mathcal{M}(X \cup \{H\})$ by H. The intersection lattice of $X\Big|_H$ is the interval (segment) $[H; \hat{1}]$ of $L_{X\cup\{H\}}$.

(b) **Restriction**

Generalizing (a), let X be an arrangement in V, W an arbitrary subspace of V, of dimension $\dim(W) = k$. Then the restriction of X to W is the arrangement $X\Big|_W = \{K \cap W : K \in X, K \not\supseteq W\}$ in W. If W is a flat in L_X, then the intersection lattice of $X\Big|_W$ is the interval $[W; \hat{1}]$ of L_X. If X is essential and W is generic in the sense that W contains no flat $U \in L_X$ except for $\hat{1} = \{0\}$, then the intersection lattice of $X\Big|_W$ is the k-truncation of L_X:

$$L_{X|_W} = L_X^{[k]} = \{U \in L_X : r(U) < k\} \cup \{\hat{1}\}.$$

(We put $L_X^{[k]} = L_X$ if $k \geq r(L_X)$.)

(c) **Localization**

Let X be an arrangement, $W \in L$ a flat of V. The localization of X at W is the arrangement $X/W = \{H/W : H \in X, H \supseteq W\}$ in the quotient space V/W. This arrangement is the essential arrangement associated to the subarrangement $X^W = \{H \in X : H \supseteq W\}$ of X in V. Its intersection lattice is the interval $[\hat{0}, W]$ of L_X. $\mathcal{M}(X/W)$ is the matroid restriction of $\mathcal{M}(X)$ to the flat W.

(d) **Sum**

Let X be an arrangement in V, X' and arrangement in V'. Then the (direct) sum of X and X' is the arrangement

$$X \times X' = \{H \oplus V' : H \in X\} \cup \{V \oplus H' : H' \in X'\}$$

in $V \oplus V'$. The arrangement $X \times X'$ has order $|X \times X'| = |X| + |X'|$. Its intersection lattice is the product $L_{X \times X'} = L_X \times L_{X'}$. $\mathcal{M}(X \times X')$ is the matroid sum of $\mathcal{M}(X)$ and $\mathcal{M}(X')$.

(e) **Irreducibility**

An arrangement X in V is reducible if it is isomorphic to a direct sum $X_0 \times X_1$, where X_i $(i = 0, 1)$ are nontrivial arrangements in V_i, that is, $\dim V_i \geq 1$. Otherwise X is irreducible. Every irreducible arrangement X is essential. An essential arrangement is reducible if and only if its lattice is a (nontrivial) product.

We will now introduce two important classes of examples.

Supersolvable arrangements can be defined by combinatorial or by geometric conditions. They are an important class of highly structured arrangements: they are relatively easy to construct, and will play a central role in the algebraic investigations of Section 6.

The supersolvable arrangements will be contrasted to generic (more precisely, 3-generic) arrangements, which in some sense represent the arrangements with the "least possible structure."

Definition 2.7: [St1, St2]

Let L be a finite geometric lattice of rank $r(L) = r$. An element $m \in L$ is called modular if $r(m \wedge m') + r(m \vee m') = r(m) + r(m')$ for every $m' \in L$. ($m \in L$ is modular if and only if its complements form an antichain [St1]. This is the case exactly if all complements of m have the same rank $r - r(m)$.)

L is supersolvable if it has a maximal chain $\hat{0} = m_0 < m_1 < \ldots < m_r = \hat{1}$ of modular elements, (called an M-chain of L).

We need some combinatorial facts about supersolvable geometric lattices.

Theorem 2.8: [St2]

Let L be a supersolvable geometric lattice of rank r, and $\hat{0} = m_0 < m_1 < \ldots < m_r = \hat{1}$ an M-chain in L.

For $1 \leq i \leq r$, let e_i be the number of atoms of L that lie below m_i but not below m_{i-1}, such that $\sum_{j=1}^{i}$ is the number of atoms of L in $[\hat{0}; m_i]$. Then $e_1 = 1$, $\sum_{i=1}^{r} e_i$ is the number of atoms of L, and

$$\chi_L(t) = \prod_{i=1}^{r}(t - e_i).$$

In particular the multiset $\{e_1, \ldots, e_r\}$ does not depend on the choice of an M-chain for L.

Definition 2.9:

An arrangement X is supersolvable *iff its intersection lattice L is a supersolvable lattice. The integers $e_1, e_2, \ldots, e_r$ are called the* (generalized) exponents *of X. An* M-chain *for a supersolvable arrangement in $\mathbf{k}^n$ is a maximal flag extending an M-chain in L.*

Supersolvable arrangements have first been considered by [St3] and [JT] in the context of *free* arrangements which we will develop in Section 3. Their combinatorics was studied in some detail in [BEZ, Section 4].

The following result, generalizing [JT, Theorem 5.4] to arrangements of arbitrary dimension, was developed in [BEZ] (there formulated for arrangements in $\mathbf{R}^n$) and independently by [Te8, Cor. 2.7] for complex arrangements. However, the proof from [BEZ] is independent of the field:

Theorem 2.10: [BEZ]

Every arrangement X of rank $r \leq 2$ is supersolvable. An arrangement X of rank $r \geq 3$ is supersolvable if and only if it can be written as a disjoint union $X = X_0 \cup X_1$, where X_0 is a supersolvable arrangement of rank $r-1$, and $X_1 \neq \emptyset$ is such that for $H', H'' \in X_1$ $(H' \neq H'')$, $H' \cap H'' \subseteq H$ for some $H \in X_0$.

For generic arrangements, we will use the following versions:

Definition 2.11:

An arrangement X in $V = \mathbf{k}^n$ is generic *if for any subarrangement $Y \subseteq X$ of size at most n, the intersection of the hyperplanes in Y has maximal codimension, that is, $\operatorname{codim}(\bigcap Y) = |Y|$. The arrangement X is* k-generic *($k \leq n$) if this holds for all subarrangements $Y \subseteq X$ of size at most k.*

Equivalently, X is k-generic (for $2 \leq k \leq n+1$) if every circuit of the associated matroid $M(X)$ has size at least $k+1$. X is generic if it is n-generic. In particular, every arrangement is 2-generic.

Thus an essential arrangement is generic if its intersection lattice is a truncated boolean algebra $B_m^{[r]}$. It is k-generic if the k-truncation $L^{[k]} = \{X \in L : r(X) < k\} \cup \{\hat{1}\}$ of its intersection lattice is isomorphic to the truncated boolean algebra $B_n^{[k]}$.

Definition 2.12:

A property or invariant of arrangements is called combinatorial if it can be decided (computed) from only the intersection lattice together with the dimension n. It is called 3-combinatorial if it can be determined from $L^{[3]}$ together with r and n.

For example, the number of hyperplanes (= number of atoms of L), being essential, irreducibility, and the rank $r(X) = r(L)$ of an arrangement are 3-combinatorial, as is the property of being 3-generic.

The property of being generic is combinatorial, but not 3-combinatorial. In the same way, the property of having a modular flat of rank $r(X) - 1$ (modular coatom of L) is combinatorial, but not 3-combinatorial. However, we have:

Theorem 2.13: [BZ, Corollary 2.8]

Supersolvability is a 3-combinatorial property.

The following sections will treat the combinatoriality of some algebraic invariants and constructions. It is quite surprising how rich a structure can be constructed from the mere knowledge of the intersection lattice of an arrangement.

Obviously, we cannot compute n from the intersection lattice, except for knowing $n \geq r(L)$. We have included n into Definition 2.12 because its knowledge is (in a rather superficial way) necessary to check or compute certain invariants. For many cases, it is possible and convenient to assume $n = r(L)$, only considering the essential arrangement associated to X.

3. Logarithmic Forms at an Arrangement

In this section, we develop basics of the theory of *free* hyperplane arrangements initiated by K. Saito [Sa1] and H. Terao [T1]. For this, the algebra $\Omega^*(X)$ of *forms with at most logarithmic poles* at an arrangement X will be defined. An arrangement X is called *free* if $\Omega^1(X)$ is a free S-module. The degrees of homogeneous basis elements are called the *exponents* of a free arrangement X.

The algebra $\Omega^*(X)$ is a natural setting to develop both Saito's basis criteria and Solomon-Terao's proof [ST] that exponents are combinatorial.

We will not give proofs for most of the elementary facts in this Section. Simple proofs for most of them can, e.g., be dualized from the arguments in [ST], see [Z1].

Definition 3.1:

Let X be an arrangement in $V = \mathbf{k}^n$, and $\{x_1, \ldots, x_n\}$ a basis for V^*. Let $S = \mathbf{k}[x_1, \ldots, x_n]$ be the ring of polynomial functions on V, and $S' = \mathbf{k}(x_1, \ldots, x_n)$ the ring of rational functions, its field of fractions.

The S-module of (algebraic) derivations (vector fields) on V is

$$Der(V) = \{\theta = \sum_{i=1}^{n} p_i \frac{\partial}{\partial x_i} : p_i \in S \text{ for } 1 \leq i \leq n\},$$

with the obvious S-module structure. Similarly, with $\mathrm{d}x_I$ denoting $\mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_s}$ for $I = \{i_1, \ldots, i_s\}_<$ (and $\mathrm{d}x_I = 1$ for $I = \emptyset$), the S-module of algebraic s-forms on V for $0 \leq s \leq n$ is

$$\Omega^s_{alg}(V) = \{\omega = \sum_{|I|=s} \omega^I dx_I : \omega^I \in S \text{ for } I \subseteq [n]\}.$$

In the same way, we denote the S-module of rational s-forms by

$$\begin{aligned}\Omega^s_{rat}(V) &= S' \otimes_S \Omega^s_{alg}(V) \\ &= \{\omega = \sum_{|I|=s} \omega^I dx_I : \omega^I \in S' \text{ for } I \subseteq [n]\}.\end{aligned}$$

The corresponding algebras of algebraic respectively rational forms will be denoted by

$$\Omega^*_{alg}(V) = \bigoplus_{s \geq 0} \Omega^s_{alg}(V)$$

and

$$\Omega^*_{rat}(V) = \bigoplus_{s \geq 0} \Omega^s_{rat}(V),$$

where $\Omega^s_{alg}(V) = \Omega^s_{rat}(V) = 0$ for all $s > n$.

These modules are certainly familiar objects. We will often treat them as graded modules, in the following canonical way.

Putting $\deg(x_i) = 1$ for all i, we get $S = \bigoplus_{k\geq 0} S_k$ (with $S_0 = \mathbf{k}$, $S_1 = V^*$) as a standard graded algebra. Similarily, we define the degree for homogeneous rational functions by

$$S'_k = \{\frac{f}{g} : f \in S_{l_1}, g \in S_{l_2}, l_1 - l_2 = k\}.$$

(Note that this does *not* make S' into a graded S-module: there is a strict inclusion $\bigoplus_{k\in\mathbf{Z}} S'_k \subset S$.)

Now let "$\deg(\frac{\partial}{\partial x_i}) = 0$" for all i, that is,

$$Der^k(V) = \{\theta = \sum_{i=1}^{n} p_i \frac{\partial}{\partial x_i} : p_i \in S_k \text{ for } 1 \leq i \leq n\},$$

to make $Der(V) = \bigoplus_{k\geq 0} Der^k(V)$ into a graded S-module.

Similarily, we define "$\deg(\mathrm{d}x_i) = 0$", thus

$$\Omega^{s,k}_{alg}(V) = \{\omega = \sum_{|I|=s} \omega^I dx_I : \omega^I \in S_k\},$$

for $k \geq 0$, such that $\Omega^s_{alg}(V) = \bigoplus_{k\geq 0} \Omega^{s,k}_{alg}(V)$; and

$$\Omega^s_{rat}(V) = \{\omega = \sum_{|I|=s} \omega^I dx_I : \omega^I \in S'_k\},$$

for $k \in \mathbf{Z}$, thus $\Omega^s_{rat}(V) \supset \bigoplus_{k\geq 0} \Omega^{s,k}_{rat}(V)$.

Definition and Lemma 3.2: [M, p. 281]

Let M be an S-module. The rank of M is

$$rk(M) := \dim_{S'}(S' \otimes_S M),$$

where S' is the quotient field of S, and $\dim_{S'}$ denotes vector space dimension. Equivalently, $rk(M)$ is the maximal size of a set of elements of M that are independent over S, or the size of every maximal independent subset of M.

Observe that $Der(V)$ and $\Omega^s_{alg}(V)$ $(s \geq 0)$ are free S-modules of rank n: $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ is a basis for the S-module $Der(V)$, and $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ is a basis both of the S-module $\Omega^1_{alg}(V)$ and the S'-module (vector space) $\Omega^1_{rat}(V)$. However, $\Omega^1_{rat}(V)$ as an S-module is *not* free – it has rank n, but it is not finitely generated, as S' contains homogeneous rational functions of arbitrarily small negative degree.

In the obvious way, bases for the module of 1-forms induce bases for the corresponding module of s-forms $(s \geq 0)$.

The following definition (due to [Sa1], [Sa2]) provides the central objects of study for the following:

Definition 3.3:

Let X be an arrangement in $V = \mathbf{k}^n$ with a defining equation Q. Then

$$(i) \qquad \begin{aligned} Der(X) &= \left\{\theta \in Der(V) :\ Q \middle| \theta(Q)\right\} \\ &= \left\{\theta \in Der(V) :\ l_H \middle| \theta(l_H) \text{ for all } H \in X\right\} \end{aligned}$$

is the S-module of logarithmic vector fields at X.

$$(ii) \qquad \begin{aligned} \Omega^s(X) =& \left\{\omega \in \Omega^s_{rat}(V) : Q\omega \in \Omega^s_{alg}(V), Qd\omega \in \Omega^{s+1}_{alg}(V)\right\} \\ =& \left\{\frac{\omega}{Q} : \omega \in \Omega^s_{alg}(V); l_H \middle| dl_H \wedge \omega \text{ for all } H \in X\right\} \end{aligned}$$

is the S-module of logarithmic differential s-forms at X (s-forms with at most logarithmic pole at X).

Of course this definition requires some checking. First, we see that the two definitions given in both cases are actually equivalent. For this, we compute

$$\theta(Q) = \theta \prod_{H\in X} l_H = \sum_{H\in X} \theta(l_H) \prod_{K\in X-\{H\}} l_K = Q \sum \frac{\theta l_H}{l_H} \qquad \text{for } (i)$$

and

$$Qd\frac{\omega}{Q} = d\omega - \frac{dQ}{Q} \wedge \omega = d\omega - \sum_{H\in X} \frac{dl_H}{l_H} \wedge \omega \qquad \text{for } (ii).$$

Second, we have to check that the sets $Der(X)$ and $\Omega^s(X)$ actually *are* S-modules, which is easy.

The *names* for these modules come from the fact that $\dfrac{dl_H}{l_H} =$ "$d \log l_H$" (for any $H \in X$) in some sense is a "typical" generator of $\Omega^1(X)$. Note that $\Omega^0(X) = S$.

From the definition, we get that $Der(X) = \bigcap_{H\in X} Der(\{H\})$. For a proper sub-arrangement $X' \subset X$, we observe that (with strict inclusions) $Der(X) \subset Der(X')$ and $\Omega^s(X') \subset \Omega^s(X)$ for $1 \le s \le n$: the inclusions are clear, and their strictness follows from $Q' \frac{\partial}{\partial l_H} \in Der(X) - Der(X')$, and $\frac{dl_H}{l_H} \in \Omega^1(X) - \Omega^1(X')$ for $H \in X - X'$.

Definition 3.4:

$\omega \in \Omega^s(X)$ has a pole at $H \in X$ if $\omega \notin \dfrac{l_H}{Q}\Omega^s_{alg}(X)$, that is, if $\omega \notin \Omega^s(X - \{H\})$.

The support of $\omega \in \Omega^s(X)$ is the set of hyperplanes in X at which ω has poles, that is,

$$\begin{aligned} \operatorname{supp}(\omega) =& \left\{H \in X : \frac{Q}{l_H}\omega \notin \Omega^s_{alg}(X)\right\} \\ =& \bigcap \left\{Y \subseteq X : Q_Y \omega \in \Omega^1_{alg}(X)\right\} \end{aligned}$$

Lemma 3.5: [Sa1]

We have the following inclusions of S-modules:

(i) $Q\ Der(V) \subseteq Der(X) \subseteq Der(V)$

(ii) $\frac{1}{Q}\Omega^s_{alg}(V) \supseteq \Omega^s(X) \supseteq \Omega^s_{alg}(V) \quad (0 \le s \le n).$

where our notation means $QDer(V) = \{Q\theta : \theta \in Der(V)\}$, *etc.*

These inclusions make $Der(X)$ and $\Omega^s(X)$ into *graded* S-modules

$$Der(X) = \bigoplus_{k \ge 0} Der^k(X)$$
$$\Omega^s(X) = \bigoplus_{k \in \mathbf{Z}} \Omega^{s,k}(X)$$

where

$$\begin{aligned} Der^k(X) &= Der^k(V) \cap Der(X) \\ \Omega^{s,k}(X) &= \frac{1}{Q}\Omega^{s,k}_{alg}(V) \cap \Omega^s(X) \\ &= \Omega^{s,k}_{rat}(V) \cap \Omega^s(X). \end{aligned}$$

$Der(V)$ is the module of polynomial vector fields that at every hyperplane are tangent to the hyperplane. This description also gives certain geometric intuition for $\Omega^1(X)$, with the following (Lemma and) Theorem:

Lemma 3.6: ("Preparation Lemma") [Sa1, (1.1)]

Let $\omega \in \Omega_{rat}(V)$. *Then* $\omega \in \Omega^s(X)$ *if and only if for every* $H \in X$, ω *can be written as*

$$\omega = \frac{dl_H}{l_H} \wedge \omega^1 + \omega^0$$

with $\frac{Q}{l_H}\omega^0 \in \Omega^s_{alg}(V)$, $\frac{Q}{l_H}\omega^1 \in \Omega^{s-1}_{alg}(V)$, *in particular such that* ω^0, ω^1 *are rational forms without a pole at* H.

Equivalently,

$$\Omega^s(X) = \left\{ \frac{\omega}{Q} : \omega \in \Omega^s_{alg},\ \omega\Big|_H = 0 \text{ for all } H \in X \right\}.$$

Proof.

For $\omega = \frac{dl_H}{l_H} \wedge \omega^1 + \omega^0$, we have $dl_H \wedge Q\omega = \mathrm{d}l_H \wedge Q\omega^0 = l_H \mathrm{d}l_H \wedge \frac{Q}{l_H}\omega^0$, and thus $l_H \Big| dl_H \wedge Q\omega$. This (for every $H \in X$) implies $\omega \in \Omega^*(X)$.

Conversely, let $\frac{\omega}{Q} \in \Omega^s(X)$, and $H \in X$. Introducing suitable coordinates $x_1, \ldots, x_n$ in V^*, we have $l_H = x_1$. Now we decompose

$$\omega = \mathrm{d}x_1 \wedge \sum_{|I|=s-1} q^I \mathrm{d}x_I + \sum_{|I|=s} r^I \mathrm{d}x_I,$$

where the sums are over index sets $I \subseteq \{2, 3, \ldots, n\}$, and $q^I, r^I \in S$ for all I. Hence the desired decomposition follows from $x_1 \Big| \mathrm{d}x_1 \wedge \omega$, i.e., $x_1 \Big| r^I$ for all I, putting $\omega^1 = \frac{l_H}{Q} \sum_{|I|=s-1} q^I \mathrm{d}x_I$ and $\omega^0 = \frac{1}{Q} \sum_{|I|=s} r^I \mathrm{d}x_I$. □

This Lemma is useful because it allows to treat $\Omega^s(X)$ as a module of algebraic forms, isomorphic to $Q\Omega^s(X) \subseteq \Omega^s_{alg}(V)$ (with a shift in the grading).

Theorem 3.7: [Sa1, (1.5,6)]

(a) *The natural inner product of S-modules*

$$i: \ Der(X) \times \Omega^1(X) \longrightarrow S$$
$$(\theta; \frac{\omega}{Q}) \longmapsto \frac{\omega(\theta)}{Q}$$

is well-defined and nondegenerate. Hence $Der(X)$ and $\Omega^1(X)$ are reflexive modules of rank n, and $Der(X)$ is free if and only if $\Omega^1(X)$ is free.

(b) *For $\theta \in Der(X)$ and $s > 0$, the interior product induces an S-module homomorphism*

$$\Omega^{s,k}(X) \longrightarrow \Omega^{s-1,k+1}(X)$$
$$\omega \longrightarrow i_\theta \omega.$$

(c) *$Der(X)$ and $\Omega^1(X)$ are noetherian.*

(d) *$\Omega^*(X)$ is closed under wedge product.*

(e) *$\Omega^*(X)$ is closed under exterior differentiation.*

Now we define "free" arrangements — an interesting class of arrangements that will be our object of study for the following. However, we wish to warn the reader in advance that the terminology is very misleading: free arrangements are going to be *very* far from being generic. In the contrary, they turn out to be a class of highly structured arrangements, for which Coxeter arrangements [Sa2] and supersolvable arrangements (Corollary 6.7) are prime examples.

Definition 3.8: [T1, p. 295] [T6, Definition 2]

Let X be an arrangement in $V \cong \mathbf{k}^n$. Then X is called free if $Der(X)$ is a free S-module.

By Theorem 3.7(a) we know that also X is free if and only if $\Omega^1(X)$ is free. Thus one can use whatever definition seems more natural or practical for the particular situation considered. However, it seems to us that the special geometric and combinatorial properties of the logarithmic forms make it more promising to develop intuition for $\Omega^1(X)$ than for $Der(X)$.

Main purpose of the rest of this section will be to develop criteria to decide freeness of arrangements and to apply them to several classes of arrangements considered before. First, the empty arrangement is always free, with $Q = 1$ in the proof of Theorem 3.7(a). Second, we get the following corollary:

Corollary 3.9: [Sa1, (1.7)]
If $n \leq 2$, then X is free.

Proof.
Over a ring S of homological dimension at most 2, every reflexive module M is free by routine homological algebra arguments. $\square$

Lemma 3.10:

$$\Omega^n(X) = \frac{1}{Q}\Omega^n_{alg}(V) = \left\{ p\frac{dx_1 \wedge \ldots \wedge dx_n}{Q} : p \in S \right\},$$

hence $\Omega^n(X)$ is a free module of rank 1.

Note that $\Omega^1(X)$ has rank n by Theorem 3.7(a), thus if it is free, then every basis has size n.

Proposition 3.11: ("Determinant Criterion") [Sa1, (1.8)]
The logarithmic forms $\omega_1 = \frac{1}{Q}\sum q_{1j}\mathrm{d}x_j, \ldots, \omega_n = \frac{1}{Q}\sum q_{nj}\mathrm{d}x_j \in \Omega^1(X)$ form a basis iff $\omega_1 \wedge \ldots \wedge \omega_n = c\frac{1}{Q^{n-1}}\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$, that is, iff $\det(q_{ij}) = cQ$ for some $c \in \mathbf{k}^*$.

If $\Omega^1(X)$ is free with basis $\omega_1, \omega_2, \ldots, \omega_n$, then for $1 \leq p \leq n$, $\Omega^p(X)$ is free with basis $\left\{\omega_{i_1} \wedge \ldots \wedge \omega_{i_p} : 1 \leq i_1 < \ldots < i_p \leq n\right\}$. In fact, we have more generally:

Theorem 3.12: ("Algebraic freeness criterion") [Sa1][ST, (3.4)][Z1, 3.2.9]
For every arrangement X in $V \cong \mathbf{k}^n$, the following are equivalent:
(i) *$Der(X)$ is a free S-module (that is, X is free)*
(ii) *$\Omega^1(X)$ is free*
(iii) *$\Omega^{n-1}(X) = \Lambda^{n-1}\Omega^1(X)$ ($\Omega^1(X)$ generates $\Omega^{n-1}(X)$)*
(iv) *$\Omega^n(X) = \Lambda^n\Omega^1(X)$ ($\Omega^1(X)$ generates $\Omega^n(X)$)*
(v) *$\Omega^*(X) = \Lambda^*\Omega^1(X)$*
(vi) *$\Omega^*(X)$ is free.*

The constructions of Definition 2.6 now produce further examples of free arrangements.

Theorem 3.13:
(a) *Let X_1 be an arrangement in V_1, and X_2 an arrangement in V_2. Then*
(i) *$\Omega^*(X_1 \times X_2) \cong \Omega^*(X_1) \wedge \Omega^*(X_2)$.*
(ii) *$X_1 \times X_2$ is free if and only if X_1 and X_2 are both free.*
(iii) *An arrangement X in V is free if and only if the associated essential arrangement $X/\cap X$ is free.*
(b) *Any localization of a free arrangement is again free.*

Proof.
Part *(a)* is dualized from [ST, Proposition (5.8)], see [Z1, Theorem 3.2.10]. Part *(b)* is in [T2], see also [Z1, Theorem 3.8.3]. $\square$

It has been conjectured ("Orlik's Conjecture", [T4, Problem 2]) that also every restriction of a free arrangement to one of its hyperplanes is again free. There is some evidence for this, but no proof so far.

Now we introduce certain integer sequences that come from the grading on modules such as $Der(X)$ and $\Omega^1(X)$.

Lemma 3.14:

Let M be a graded S-module that is free of rank n. Then M has a homogeneous basis. The degrees of basis elements do not depend on the particular basis chosen (up to permutation).

Proof.

This most easily seen from Lemma 3.18 below. The second claim can also be seen from the Poincaré series of the module M, which is $\dfrac{\Sigma x^{e_i}}{(1-x)^n}$ if M has a homogeneous basis with degrees e_i. □

Definition 3.15:

Let X be a free arrangement of hyperplanes, and let $\{\theta_1, \ldots, \theta_n\}$ be a homogenous basis for $Der(X)$. The degrees $e_i = \deg(\theta_i)$ are called the exponents of X. We write $\exp(X) = [e_1, \ldots, e_n]$ for the multiset of exponents, usually assuming $e_1 \leq \ldots \leq e_n$.

Lemma 3.16:

(i) Let X be free with $\exp(X) = [e_1, e_2, \ldots, e_n]$. Then $\sum_{i=1}^{n} e_i = m$.

(ii) If $\{\theta_1, \ldots, \theta_n\}$ is a homogeneous basis of $Der(X)$ with $\deg(\theta_i) = e_i (1 \leq i \leq n)$, then the dual basis $\{\omega_1, \ldots, \omega_n\}$ of $\Omega^1(X)$ satisfies $\deg(\omega_i) = -e_i (1 \leq i \leq n)$.

Proof.

This follows from the basis criterion Proposition 3.11, together with the fact that the coefficient matrices of dual bases are inverse matrices. □

The exponents will turn out to be combinatorial invariants of X, if X is free, with Theorem 3.20.

However, exponents are invariants that we can define in this form only for the (very special) case of free arrangements. Therefore, we first develop some theory about the *degree sequence* of an arrangement, which is an integer sequence defined for *every* arrangement that allows to decide freeness and reduces to the sequence of exponents in the case of free arrangements.

One notion of a degree sequence of an arrangement was defined (but hardly studied or used) by Terao in [T1] for the module $Der(X)$. However, we find it more rewarding and convenient to study the degree sequence for $\Omega^1(X)$ here, which does not have a direct relation to that of $Der(X)$. Hence we start with a slightly more general definition:

Definition 3.17:

Let M be a noetherian graded module of rank n over S. A sequence $(m_1, m_2, \ldots)$ of homogeneous elements of M is a generator sequence if for all i,

$$\deg(m_i) = \min\{\deg(m) : \ m \in M \ \ \textit{homogeneous and } m \notin Sm_1 + \ldots + Sm_{i-1}\}.$$

The degree sequence of M is the nondecreasing sequence of integers $[\deg(m_1), \deg(m_2), \ldots]$ given by a maximal generator sequence $(m_1, m_2, \ldots)$ in M.

We will show in Corollary 3.19 that this "degree sequence" is well-defined, i.e., independent of the choice of a maximal generator sequence.

Lemma 3.18:

Let M be as above. A multiset $\{m_1, m_2, \ldots\}$ of homogeneous elements of M is contained in a generator sequence iff its image in M/S_+M is linearly independent (over $\mathbf{k} = S/S_+$) in $\mathbf{k}\otimes_S M \cong M/S_+M$, where $S_+ = \bigoplus_{k>0} S_k$ is the irrelevant ideal of S (polynomials without constant term).

Proof.

This follows from the fact that the image of a homogeneous element of M vanishes in M/S_+M iff it can be written as $\Sigma p_i m_i$ for some $p_i \in S_+, m_i \in M$: by definition, every maximal generator sequence generates M.
For $\mathbf{k}\otimes_S M \cong M/S_+M$ we refer to [AM, p. 31]. □

Corollary 3.19:

(i) *The degree sequence for M does not depend on the choice of the maximal generator sequence of M, and is finite.*

(ii) *The degree sequence $[k_1, k_2, \ldots, k_l]$ has length $l = \dim_{\mathbf{k}} \mathbf{k}\otimes_S M \geq n$. M is free if and only if $l = n$.*

Proof.

(i) The sequence is finite because M is noetherian. A generator sequence can naturally be identified with a homogeneous basis of $\mathbf{k}\otimes_S M$.

(ii) By Lemma 3.17, $\dim_{\mathbf{k}} \mathbf{k}\otimes_S M$ is the size of any minimal set of generators, hence $l = \dim \mathbf{k}\otimes_S M$. Now M has rank n, thus $l \geq n$. If M is free, a basis of M induces a basis of $\mathbf{k}\otimes M$, hence $l = n$. Conversely, if $l = n$ then M is generated by n elements. Since $rk(M) = n$, the elements then cannot have linear relations, hence they form a basis for M. □

Theorem 3.20: [T2]

Let X be a free arrangement of rank n with $\exp(X) = [e_1, \ldots, e_n]$. Then the intersection lattice of X has characteristic polynomial $\chi(t) = \prod_{i=1}^{n} (t - e_i)$. Thus exponents are combinatorial.

The original proof of the remarkable Theorem 3.20 was very involved and difficult. A new and easier approach was recently taken by [ST]. Their main result, dualized to fit the framework of logarithmic forms, is the following Theorem. Its part *(a)* follows from a simple computation, part *(b)*, which clearly implies Theorem 3.20, is non-trivial.

Theorem 3.21:

Let X be an arrangement in $V = \mathbf{k}^n$. The Poincaré series of $\Omega^*(X)$ is (with nonzero terms in the summation only for $k \geq -m$, $0 \leq s \leq n$)

$$\mathrm{Poin}(\Omega^*(X); x, y) = \sum_k \sum_s \mathrm{dim}\Omega^{s,k}(X) x^k y^s.$$

The Ψ-function of X is

$$\Psi(X; x, t) = x^m \ \mathrm{Poin}(\Omega^*(X); x, t(1-x) - 1)$$

(a) *If X is free with* $\exp(X) = [e_1, \ldots, e_n]$*, then*

$$\mathrm{Poin}(\Omega^*(X); x, y) = \prod_{i=1}^{n} \frac{1 + yx^{-e_i}}{1 - x},$$

and

$$\Psi(X; x, t) = \prod_{i=1}^{n} (t - (1 + x + \ldots + x^{e_i - 1}))$$

(b) *For every arrangement X:*
  (i) *$\Psi(X; x, t)$ is a polynomial*
  (ii) *$\chi(t) = \Psi(X; 1, t)$.*

Example 3.22:

For a simple nontrivial example, let X be the generic arrangement of $m = 4$ planes in $\mathbf{R}^3$, defined, say, by $Q = x_1x_2x_3(x_1 + x_2 + x_3)$. We use $l_4 = x_1 + x_2 + x_3$ as an abbreviation.

Then $\Omega^1(X)$ is generated by

$$\frac{\mathrm{d}x_1}{x_1}, \frac{\mathrm{d}x_2}{x_2}, \frac{\mathrm{d}x_3}{x_3} \text{ and } \frac{\mathrm{d}l_4}{l_4},$$

as we will see with Corollary 7.5, and the degree sequence of $\Omega^1(X)$ is $[-1, -1, -1, -1]$. A single relation corresponds to $l_4 = l_1 + l_2 + l_3$, which allows to compute

$$\mathrm{Poin}(\Omega^1(X); x) = \frac{4/x - 1}{(1-x)^3}.$$

Trivially, we get

$$\mathrm{Poin}(\Omega^0(X); x) = \frac{1}{(1-x)^3}$$

and by Lemma 3.10,

$$\mathrm{Poin}(\Omega^3(X); x) = \frac{1/x^4}{(1-x)^3}.$$

The hard part is to find the generators (and relations) for $\Omega^2(x)$. Straightforward but tiresome calculations yield

$$\mathrm{Poin}(\Omega^2(X); x) = \frac{1}{(1-x)^3}\left(\frac{1}{x^3} + \frac{3}{x^2} - \frac{1}{x}\right),$$

thus

$$\mathrm{Poin}(\Omega^*(X); x, y) = \frac{1}{(1-x)^3}(1 + (\frac{4}{x} - 1)y + (\frac{1}{x^3} + \frac{3}{x^2} - \frac{1}{x})y^2 + \frac{1}{x^4}y^3).$$

and we get

$$\begin{aligned}\Psi(X; x, t) =& x^4 F(x; t(x-1) - 1) \\ =& t^3 + (x+1)(x-3)t^2 - (x^2 - 4x - 3)t - (2x+1)\end{aligned}$$

which correctly (cf. Theorem 3.20) reduces to

$$\chi(t) = \Psi(X; 1, t) = t^3 - 4t^2 + 6t - 3.$$

Detailed computations and more examples are recorded in [Z1].

4. Category of Hyperplane Arrangements

We will now describe a categorial framework in which hyperplane arrangements and the modules of logarithmic forms on them can be treated. We apply it to the study of restriction maps.

The intuition for the choice of a proper category to support the algebraic theory of hyperplane arrangements comes from several directions. We will here work parallel to a development of the category of matroids and strong maps [CR, Ch. 9], [K], and at the same time derive an algebraic treatment as it can be extracted and specialized from, e.g., Shafarevich's discussion of divisors and differential forms [Sh, Ch. III].

Definition 4.1:

The category $\mathcal{A}$ of arrangements (over $\mathbf{k}$) has as objects all arrangements over $\mathbf{k}$, that is, all pairs $(X; V_X)$ where X is an arrangement in the $\mathbf{k}$-vector space V_X.

A map

$$\Phi : (X; V_X) \longrightarrow (Y; V_Y)$$

is a linear map

$$\phi : V_Y \longrightarrow V_X$$

such that for $H \in X$, $\phi^{-1}(H) \in V_Y$. Thus a map Φ yields an application

$$\begin{aligned} \Phi_0 : &X \longrightarrow Y \\ &H \longmapsto \phi^{-1}(H). \end{aligned}$$

We call Φ injective (surjective) if Φ_0 is injective (surjective). By abuse of notation, we denote maps by $\Phi : X \longrightarrow Y$.

It is easy to check that this definition actually yields a category, in particular that the composition of two maps always is a map. Note that Definition 4.1 contains a non-degeneracy condition: if $\Phi : X \longrightarrow Y$ is a map then $\phi(V_Y)$ is not contained in a hyperplane of X. Thus, if $\mathbf{k}$ is a finite field and X_n is the arrangement of *all* hyperplanes in $V = \mathbf{k}^n$, then there is *no* map from X_n to X_k for $n > k$. This cannot happen over infinite fields.

Proposition 4.2:

(1) If Y is a subarrangement of X in V, then the inclusion $Y \hookrightarrow X$ is a map in $\mathcal{A}$, induced by the identity map $\phi : V \longrightarrow V$.

(2) Let $(X; V)$ be an arrangement and H a hyperplane in V, $H \notin X$. Then $\mathcal{A}$ contains the restriction $\Phi : (X; V) \longrightarrow (X\big|_H; H)$ to the hyperplane H induced by $H \hookrightarrow X$.

(3) The empty arrangement $(\emptyset; \{0\})$ in $\mathbf{k}^0 = \{0\}$ has a unique map into every arrangement: it is the initial object of $\mathcal{A}$. However, $\mathcal{A}$ has no terminal object.

(4) Sums exist: for $(X; V_X)$ and $(Y; V_Y)$, the sum is given as $(X \times Y; V_X \oplus V_Y)$, as defined in Definition 2.6(d). We get a commutative diagram (with the obvious maps):

$$\begin{array}{ccc} (\emptyset;\{0\}) & \longrightarrow & (X;V_X) \\ \downarrow & & \downarrow \\ (Y;V_Y) & \longrightarrow & (X\times Y;V_X\oplus V_Y) \end{array}$$

(5) *Let X be an arrangement in V, defined by $Q=\prod_{i=1}^m l_i$. Then there is a surjection*

$$\Phi : X_F \longrightarrow X,$$

unique up to automorphism of X_F, from the arrangement X_F of coordinate hyperplanes in $V_F=\mathbf{k}^m$ to X, defined by

$$\begin{aligned} \phi :& V \longrightarrow V_F \\ & \mathbf{x} \longmapsto (l_1(\mathbf{x}),\ldots,l_m(\mathbf{x})). \end{aligned}$$

(6) *Let $(X;V_X)$ be an arrangement, V_Y a vector space, and $\phi : V_Y \longrightarrow V_X$ a linear map whose image $\phi(V_Y)$ is not contained in a hyperplane of X. Then the image (of X via ϕ) is the arrangement $\phi^*X=Y$ in V_Y defined as the set*

$$\left\{\phi^{-1}(H) : H\in X\right\}.$$

Note that $\phi^{-1}(H)=ker(\phi\circ l_H)$, where $l_H\circ\phi\neq 0$ (and $\phi^{-1}(H)$ is a hyperplane in V_Y) iff $\phi(V_Y)\nsubseteq H$. $\phi^*(X)$ is the smallest arrangement in V_Y such that ϕ induces a map $\Phi : X \longrightarrow \phi^*(X)$.

(7) *If ϕ is an injective map, then ϕ^*X is the restriction of X to $\phi(V_Y)$. In this case we call $\Phi : X \longrightarrow \phi^*(X)$ a restriction. (This generalizes (2), where ϕ is the inclusion of $H=V_Y$ into V_X.)*
Thus a map $\Phi : X \longrightarrow Y$ is a restriction if and only if it is surjective and the associated $\phi : V_Y \longrightarrow V_X$ is injective.

(8) *Given $\phi : V_Y \longrightarrow V_X$, every polynomial function p on V_X can be pulled back to the polynomial $\phi^*(p) := p\circ\phi$ on V_Y. (Thus for an arrangement X in V_X, $\phi^*(Q)=Q\circ\phi$ defines a multiarrangement $\widetilde{\phi^*X}$, whose simplification is ϕ^*X, cf. [Z2])*
$\Phi : X \longrightarrow Y$ is an injection (that is, Φ_0 is injective) iff $Q_X\circ\phi$ is squarefree, that is, iff $\phi^*(Q_X)=Q_X\circ\phi$ divides Q_Y.

Proof.

Trivial. After all, this is category theory. □

Factorization Theorem 4.3:

Every map $\Phi : X \longrightarrow Y$ in $\mathcal{A}$ can be factored into an injection followed by a restriction.

Proof.

Factor $\phi : V_Y \longrightarrow V_X$ as

$$\phi = \psi\circ i : V_Y \xrightarrow{i} V_Z \xrightarrow{\psi} V_X,$$

where $V_0 \subseteq V_X$ such that $V_X \cong \phi(V_Y) \oplus V_0$, $V_Z = V_Y \oplus V_0$, i is the injection $i : \mathbf{y} \longmapsto (\mathbf{y}; 0)$ and ψ the surjection $\psi : (\mathbf{y}; \mathbf{x}) \longmapsto \phi(\mathbf{y}) + \mathbf{x}$.

Let Z be the set-union of the image of X via ψ and the sum of Y with the empty arrangement in V_0,

$$Z = \psi^*(X) \cup (Y \times \emptyset).$$

Then for $H \in X, \psi^{-1}(H) \in \psi^*(X) \subseteq Z$, hence ψ induces a map $\Psi : X \longrightarrow Z$ of arrangements. Since ψ is surjective, Ψ is an injection. For $H \oplus V_0 \in Y \times \emptyset$, $i^{-1}(H \oplus V_0) = H \in Y$. For $\psi^{-1}(H) \in \psi^*(X), i^{-1}(\psi^{-1}(H)) = \phi^{-1}(H) \in Y$. Thus i induces a map $I : Z \longrightarrow Y$.

I is surjective because $I\Big|_{Y \times \emptyset}$ is surjective, and i is injective. Thus, by 4.2(7), I is a restriction. $\square$

This Factorization Theorem is the analogue of the Higgs Factorization Theorem [CR; p. 9.41] [W] [K; p. 233] of matroid theory for our category of *represented matroids* (that is, hyperplane arrangements, via Theorem 2.4(b)) defined in 4.1. We feel that the proof becomes easier and more geometric in the represented setting. In the following we will construct a functor $\Omega^*(\cdot)$ from the category of hyperplane arrangements over $\mathbf{k}$ to the category of $\mathbf{k}$-vectorspaces. Note that for arrangements X and Y in different dimensions, $\Omega^*(X)$ and $\Omega^*(Y)$ are modules over different polynomial rings – hence we cannot expect to get module homomorphisms from maps of hyperplane arrangements. Here the proper formulation would require a sheaf-theoretic setting. Since we will have no use for this, the following elementary formulation will do:

Theorem 4.4:

Every map $\Phi : X \longrightarrow Y$ induces a $\mathbf{k}$-linear map

$$\begin{aligned} \Phi^\sharp : \Omega^*(X) &\longrightarrow \Omega^*(Y) \\ \omega &\longmapsto \phi^*(\omega) = \omega \circ \phi. \end{aligned}$$

With this $\Omega^*(\cdot)$ becomes a covariant functor from the category of hyperplane arrangements over $\mathbf{k}$ to the category of $\mathbf{k}$-vector spaces and linear maps or, more specifically, to the category of differential graded $\mathbf{k}$-algebras. This induces a contravariant functor $H^*\Omega^*(\cdot)$ from $\mathcal{A}$ to the category of finite dimensional $\mathbf{k}$-algebras.

Proof.

By Factorization Theorem 4.3, it is sufficient to show that $\phi^*(\omega)$ is logarithmic in the cases of injections and restrictions. Functoriality is clear.

(1) Let $I : X \longrightarrow Y$ be an injection, and $i : V_Y \longrightarrow V_X$ the associated linear map.
If Q_X and Q_Y are defining equations, then we know $i^*(Q_X) \Big| Q_Y$ by 4.2(8).
Now $Q_X\omega$ is algebraic for $\omega \in \Omega^*(X)$, thus $i^*(Q_X\omega)$ is algebraic. Similarly, from $Q_X d\omega$ algebraic we get that $i^*(Q_X d\omega) = i^*(Q_X) i^*(d\omega)$ divides $Q_Y d(i^*\omega)$ and thus is algebraic. Thus $i^*\omega \in \Omega^*(Y)$.

(2) It suffices to check the case where X is an arrangement in V and $H \subseteq V$ is a hyperplane not in X.

For this let $H_1, \ldots, H_k \in X (k \geq 2)$ such that

$$H_1 \cap H = \ldots = H_k \cap H \in X\Big|_H, \quad Y = \{H_1, \ldots, H_k\} \subseteq X,$$

and $Z = X - Y$. We assume $K \cap H \neq H_1 \cap H$ for $K \in Z$, that is, that Y was chosen maximal. Choose defining equations Q_X, Q_Y, Q_Z such that $Q_X = Q_Y Q_Z$. Now for $\omega \in \Omega^*(X)$ we get $Q_Z \omega \in \Omega^*(Y)$. Choosing coordinates in V such that $l_H = x_1$, $l_i = \alpha_i x_1 + x_2$ ($\alpha_i \in \mathbf{k}$), we get a basis for $\Omega^*(Y)$ as

$$\begin{aligned} \omega_1 &= \frac{\mathrm{d}l_1}{l_1} \\ \omega_2 &= \frac{1}{l_3 \ldots l_k} \left(\frac{\mathrm{d}l_1}{l_1} - \frac{\mathrm{d}l_2}{l_2} \right) \\ \omega_i &= \mathrm{d}x_i \quad \text{for } 3 \leq i \leq k. \end{aligned}$$

Now clearly $\omega_i\Big|_H \in \Omega^*(Y\Big|_H)$ for $i \neq 2$, and $\omega_2\Big|_H = 0 \in \Omega^*(Y\Big|_H)$. Thus for $Q_Z\omega \in \Omega^*(Y)$, we have $Q_Z \omega\Big|_H \in \Omega^*(Y\Big|_H)$. This means that $\omega\Big|_H$ has at most logarithmic pole at $H_1 \cap H$. $\square$

We will later use Theorem 4.4, mostly for the case of restriction maps.

Corollary 4.5:

Let X be an arrangement in V, $H \notin X$. Then there is an exact sequence of $\mathbf{k}$-vector spaces

$$0 \longrightarrow \Omega^1(X \cup \{H\}) \xrightarrow{\cdot l_H} \Omega^1(X) \xrightarrow{i^*} \Omega^1(X\Big|_H),$$

where "$\cdot l_H$" is multiplication by a defining equation l_H of H, and i^* is the restriction map induced by $i : H \hookrightarrow V$.

Proof.

'$\cdot l_H$' is obviously injective, and exactness at the middle term follows from Preparation Lemma 3.6. $\square$

Unfortunately, the restriction map i^* is not surjective in general. For example, let X in $\mathbf{k}^3$ be defined by $Q = x_1 x_2 x_3$, such that $\Omega^1(X)$ has a basis

$$\left\{ \frac{\mathrm{d}x_1}{x_1}, \frac{\mathrm{d}x_2}{x_2}, \frac{\mathrm{d}x_3}{x_3} \right\}.$$

Now if H is a generic plane (without loss of generality, $l_H = x_1 + x_2 + x_3$), then we have

$$\frac{1}{x_3} \left(\frac{\mathrm{d}x_1}{x_1} - \frac{\mathrm{d}x_2}{x_2} \right) \in \Omega^1(X\Big|_H),$$

which is a nonvanishing form of degree -2, hence not contained in the image of i^*. We will formalize this example in Proposition 7.2 to describe a necessary combinatorial condition for i^* to be surjective.

5. Strong Preparation Lemma

The final few sections will study how the combinatorial structure of an arrangement X determines the existence of particular logarithmic forms at X. Since we are interested mainly in generators for $\Omega^1(X)$ and their construction, we will develop in detail only the case of logarithmic 1-forms, leaving most of the (quite straightforward) generalizations to k-forms to the interested reader.

The main technical tools involved are the study of the supports of homogeneous logarithmic forms, and the strengthened version of the Preparation Lemma 3.6 given here as Theorem 5.1.

Theorem 5.1: ("Strong Preparation Lemma")

Let X be an arrangement in $V = \mathbf{k}^n$ with defining equation Q, and let $H \notin X$ be a hyperplane in V defined by $l_H \in V^*$. Let X' be a subarrangement of X (defined by Q') such that

$$X\Big|_H = X'\Big|_H \text{ and } \left|X'\Big|_H\right| = \left|X'\right|,$$

that is, X' is minimal such that its restriction to H coincides with that of X. Let $\overline{Q} = \dfrac{Q}{Q'}$ be a defining equation for $\overline{X} = X - X'$.

Then every $\omega \in \Omega^1(X \cup \{H\})$ can be written as

$$\omega = \frac{p\,\mathrm{d}l_H}{\overline{Q}\,l_H} + \omega^0,$$

where $p \in S$ and $\omega^0 \in \frac{1}{Q}\Omega^1_{alg}(V)$ does not have a pole at H. In particular,

$$-\deg(\omega) \leq \deg(l_H\overline{Q}) = 1 + \left|X\right| - \left|X\Big|_H\right|.$$

Furthermore, if ω is homogeneous with $\deg(\omega) = -\deg(l_H\overline{Q})$, then $p \in \mathbf{k}$ and ω^0 are unique.

Proof.

By Preparation Lemma 3.6, we can write ω as

$$\omega = \frac{q}{Q}\frac{\mathrm{d}l_H}{l_H} + \omega^0$$

with $q \in S$ and $\omega^0 \in \dfrac{1}{Q}\Omega^1_{alg}(V)$. Choose a representation

$$\omega = \frac{p}{Q^\star}\frac{\mathrm{d}l_H}{l_H} + \omega^0$$

such that $p \in S$, $Q\omega^0 \in \Omega^1_{alg}(V)$ and $Q^\star \Big| Q$ is of minimal degree, that is, $Q^\star$ defines a subarrangement $X^\star$ of X of minimal order.

Now let $H_1 \in X^\star$ such that *all* the hyperplanes $H_i \in X$ with $H_i \cap H = H_1 \cap H$ are in $X^\star$. We have to show that this cannot happen. Using the fact that the ideal $\langle \overline{Q}; l_H \rangle$ does not depend on the particular choice of $\overline{X}$, (because if $H_1 \cap H = H_2 \cap H$, then $l_1 \equiv l_2 \mod l_H$,) we are then done.

Let $\{H_1, \ldots, H_k\}$ be as described, and $l_1 = l_{H_1}$. Then

$$l_1 \frac{p}{Q^\star} \frac{\mathrm{d}l_H}{l_H}\Big|_{H_1}$$

has a pole of order k at $H \cap H_1$, whereas ω^0 does not have a pole at H, hence the pole at $H \cap H_1$ of $l_1\omega^0\Big|_H$ has at most order $k-1$. This contradicts $l_1\omega\Big|_H = 0$. $\square$

How much information about the freeness of arrangements is encoded in the degree sequence, respectively, the set of exponents? We start with a simple observation.

Proposition 5.2: [T1, p. 305]

If X and $X \cup \{H\}$ are both free for some $H \notin X$, and $\exp(X) = [e_1, \ldots, e_n]$, then $\exp(X \cup \{H\}) = [e_1, \ldots, e_i + 1, \ldots, e_n]$ for some $1 \le i \le n$.

Proof.

This follows from $\Omega^1(X \cup \{H\}) \supseteq \Omega^1(X)$: let $\{\omega_1, \ldots, \omega_n\}$ and $\{\omega'_1, \ldots, \omega'_n\}$ be homogeneous bases for $\Omega^1(X)$ and $\Omega^1(X \cup \{H\})$, respectively, with $\deg(\omega_i) = -e_i$. Then we can write

$$\omega_i = \sum_j p_{ij}\omega'_j$$

for some $p_{ij} \in S (1 \le i, j \le n)$. From the Basis Criterion 3.11, we get that $\det(p_{ij})$ is a defining equation for H. Now after rearranging the order of the ω_j, we can assume $p_{jj} \neq 0$ for all j. Thus p_{ii} is linear for some $i = i_0$ $(1 \le i \le n)$, and constant for $i \neq i_0$. But this means $\deg\ \omega_{i_0} = \deg\ \omega'_{i_0} - 1 = -(e_i + 1)$, and $\deg\ \omega_i = \deg\ \omega'_i = -e_i$ for all $i \neq i_0$. With this, $\{\omega_1, \ldots, \omega'_{i_0}, \ldots, \omega_n\}$ is a basis of $\Omega^1(X \cup \{H\})$. $\square$

Lemma 5.3:

Let X be an arrangement, and $H \notin X$. Let $X\Big|_H$ have degree sequence $[e_1, \ldots, e_k]_\le$. Then a sequence $\omega_1, \ldots, \omega_i \in \Omega^1(X)$ with $\deg(\omega_j) = -e_j$ for $1 \le j \le i \le k$ restricts to a generator sequence $\omega_1\Big|_H, \ldots, \omega_i\Big|_H$ in $\Omega^1(X\Big|_H)$ if and only if

$$\frac{1}{l_H}\Big(\omega_j - \sum_{l<j} p_l\omega_l\Big) \notin \Omega^1(X \cup \{H\}) \qquad (1)$$

for all j and for all choices of the p_l.

Proof.

(1) is equivalent to

$$\omega_j\Big|_H - \sum_{l<j} p_l\omega_l\Big|_H \neq 0,$$

that is, to

$$\omega_j\Big|_H \notin S/\langle l_H\rangle\, \omega_1\Big|_H + \ldots + S/\langle l_H\rangle\, \omega_{j-1}\Big|_H,$$

which proves the claim. □

Using the result Strong Preparation Lemma, we now get a new, simple proof for a fundamental result about the combinatorial structure of free arrangements, H. Terao's Addition-Deletion Theorem [T1], which by an observation of P. Cartier [C] can be stated as follows:

Theorem 5.4: ("Addition-Deletion Theorem") [T1]
Let X be an arrangement, and $H \notin X$. Then any two of the following statements imply the third:
(1) X is free with exponents $\exp(X) = [e_1, \ldots, e_n]$
(2) $X \cup \{H\}$ is free with exponents $\exp(X \cup \{H\}) = [e_1, \ldots, e_{i_0}+1, \ldots, e_n]$
(3) $X\Big|_H$ is free with exponents $\exp(X\Big|_H) = [e_1, \ldots, \widehat{e}_{i_0}, \ldots, e_n]$

The original proof in [T1] depends on involved commutative algebra arguments for the modules of logarithmic vector fields at the three arrangements. In the following proof, we will consider the $\Omega^1(X)$ instead of $Der(X)$, which allows to replace the commutative algebra by the geometric idea of the Strong Preparation Lemma 5.1.

Still a different proof can be seen from the following observation. In Terao's original proof, the hard part was to show that *(1)&(3) $\Longrightarrow$ (2)*. In the dualized version, as presented here, this part is actually easy, and this allows to give a simple proof by using both the original and the dualized viewpoint. However, since the "hard part" of our dualized version, *(2)&(3) $\Longrightarrow$ (1)*, is taken care of by the Strong Preparation Lemma 5.1, we present here this more "coherent" version, which entirely stays within the framework of logarithmic differential forms.

Proof.
(1)&(3) $\Longrightarrow$ (2): Let $\omega_1, \ldots, \omega_n$ be a maximal generator sequence of $\Omega^1(X)$, that is, a homogeneous basis such that $\deg(\omega_i) = -e_i$, $-e_1 \leq \ldots \leq -e_n$. Now if for some j_0 and polynomials p_j,

$$\frac{1}{l_H}\Big(\omega_{j_0} - \sum_{j<j_0} p_j\omega_j\Big) \in \Omega^1(X \cup \{H\}), \qquad (*)$$

then we are done, since then $\omega_1, \ldots, \frac{1}{l_H}(\omega_{j_0} - \sum_{j<j_0} p_j\omega_j), \ldots, \omega_n$ is a basis of $\Omega^1(X \cup \{H\})$, and $i_0 = j_0$, by Proposition 3.11 and Lemma 5.3.
With this we can now assume that $(*)$ never holds. Without loss of generality we have $-e_{i_0} < -e_{i_0+1}$ or $i_0 = n$. Then by Lemma 5.3, $\omega_1\Big|_H, \ldots, \omega_{i_0-1}\Big|_H$ is a generator

sequence, and

$$\omega_{i_0}\Big|_H \notin S/\langle l_H\rangle\, \omega_1\Big|_H + \ldots + S/\langle l_H\rangle\, \omega_{i_0-1}\Big|_H,$$

thus $\omega_1\Big|_H, \ldots, \omega_{i_0}\Big|_H$ is a generator sequence, contradicting *(3)*.

(2)&(3) $\Longrightarrow$ *(1)*: From the conditions *(2)* and *(3)*, we get $\Big|X\Big| - \Big|X\Big|_H\Big| = e_{i_0}$. Now assume that there is an element $\omega_0 \in \Omega^1(X - \{H\}) - \Omega(X)$ with $\deg(\omega_0) = -(e_{i_0}+1)$. Then (by Lemma 3.19) we can find a basis $\omega_1, \ldots, \omega_n$ of $\Omega^1(X \cup \{H\})$ such that $\omega_{i_0} = \omega_0$, since by Lemma 5.1 ω_0 is not in $S_+\Omega^1(X \cup \{H\})$. From this we can construct a basis $\omega_1', \ldots, \omega_{i_0-1}', l_H\omega_0, \ldots, \omega_n'$ of $\Omega^1(X)$, where $\omega_i' = \omega_i - p_i\omega_0$ and p_i is chosen such that ω_i' is homogeneous and $H \notin \mathrm{supp}(\omega_i')$ — the Strong Preparation Lemma 5.1 guarantees that this is possible.

But if such an ω_0 does not exist, then we have a basis $\omega_1, \ldots, \omega_{i_0}, \ldots, \omega_n$ with $\deg(\omega_i) = -e_i$, $-e_1 \leq \ldots \leq -e_n$ for $\Omega^1(X \cup \{H\})$ such that $\omega_1, \ldots, \omega_{i_0}$ is a generator sequence for $\Omega^1(X)$, which by Lemma 5.3 induces a generator sequence $\omega_1\Big|_H, \ldots, \omega_{i_0}\Big|_H$ for $\Omega(X\Big|_H)$. Since we can again assume that $-e_{i_0} < -e_{i_0+1}$ or $i_0 = n$, this is a contradiction to *(3)*.

(1)&(2) $\Longrightarrow$ *(3)*: Let $\omega_1, \ldots, \omega_n$ be a homogeneous basis for $\Omega^1(X)$, then the argument of Proposition 5.2 allows to assume that $\omega_1, \ldots, \frac{1}{l_H}\omega_{i_0}, \ldots, \omega_n$ is a homogeneous basis for $\Omega^1(X \cup \{H\})$.

Now $\omega_i \in \Omega^1(X)$, thus $H \notin \mathrm{supp}(\omega_i)$ and with this Theorem 4.4 implies $\omega_i\Big|_H \in \Omega^1(X\Big|_H)$ for $1 \leq i \leq n$. But with the Determinant Criterion 3.11 we get that $\omega_1\Big|_H, \ldots, \widehat{\omega_{i_0}\Big|_H}, \ldots, \omega_n\Big|_H$ is a basis of $\Omega(X\Big|_H)$. $\square$

6. Supersolvable Arrangements and Triangular Bases

The following results are all direct consequences of the Strong Preparation Lemma, which even follow without recurrence on the Addition-Deletion Theorem. They lead up to a simple proof of the freeness of supersolvable arrangements and a description of the corresponding bases of $\Omega^1(X)$.

We feel that in the long run Teraos's Theorem 3.20, derived from Theorem 3.21, should get a similarly elementary and transparent proof, using the Strong Preparation Lemma to get a hand on the generators of the modules $\Omega^1(X)$, its syzygies, and the associated degree sequences.

Lemma 6.1:

If X is an arrangement, $H \in X$, and $\left|X\Big|_H\right| = \left|X\right| - 1$ (i. e., no two different hyperplanes in X have the same intersection with H), then $\Omega^1(X)$ is generated (over S) by $\dfrac{\mathrm{d}l_H}{l_H}$ together with $\Omega^1(X - \{H\})$:

$$\Omega^1(X) = S\frac{\mathrm{d}l_H}{l_H} + \Omega^1(X - \{H\}).$$

Proof.

This is a direct corollary of the Strong Preparation Lemma 5.1, for $X' = X$, $\overline{X} = \emptyset$ and $\overline{Q} = 1$. □

Lemma 6.2:

Let X be an arrangement in V, and $H \in X$. If H is a factor of X, (i. e., $r(X - \{H\}) < r(X)$; H is a bridge in $M(X)$), then the restriction map of Corollary 4.5

$$i^* : \Omega^1(X) \longrightarrow \Omega^1(X\Big|_H)$$

is surjective.

Proof.

After change of coordinates we can assume $l_H = x_n$ and $\frac{\partial l_K}{\partial x_n} = 0$ for $K \neq H$. Then the obvious projection map $\pi : V \longrightarrow H$ induces a map of arrangements $\Pi : X\Big|_H \longrightarrow X$. Now $\pi \circ i = id_H$ implies $i^* \circ \pi^* = id$, hence i^* is surjective. □

Lemma 6.3:

Let X be an essential arrangement in V, ℓ a modular coatom of L. Let $X_0 = \{H \in X : \ell \subset H\}$ and $X_1 = X - X_0 = \{H_1, \ldots, H_k\}$, with defining equation $Q_1 = l_1 ... l_k$. Then there is a logarithmic 1-form $\omega_k \in \Omega^1(X)$ of the form

$$\omega_k = \frac{1}{l_k ... l_2}\frac{\mathrm{d}l_1}{l_1} + \omega^1$$

with $\omega^1 \in \dfrac{l_1}{Q}\Omega^1_{alg}(V)$.

Proof.

To construct ω, we proceed by induction on k, the case $k = 1$ being trivial. Let ω_k be given, and $H_{k+1} \notin X$ such that the line ℓ in V is again modular for $X \cup \{H_{k+1}\}$. Then

$$\omega_k\Big|_{H_{k+1}} \in \Omega_1(X\Big|_{H_{k+1}}) = \Omega_1(X_0\Big|_{H_{k+1}})$$

since $X\Big|_{H_{k+1}} = X_0\Big|_{H_{k+1}}$ by modularity of ℓ.

But H_{k+1} is a factor in $X_0 \cup \{H_{k+1}\}$, hence by Lemma 6.2 there is an $\omega_k^0 \in \Omega^1(X_0)$ such that

$$\omega_k\Big|_{H_{k+1}} = \omega_k^0\Big|_{H_{k+1}},$$

hence we can use

$$\omega_{k+1} = \frac{1}{l_{k+1}}(\omega_k - \omega_k^0).$$

□

This Lemma implies the existence of "triangular bases" for supersolvable arrangements (Theorem 6.6), and its proof in fact provides a reasonably simple inductive method to construct such bases.

Definition 6.4:

Let X be a free arrangement of hyperplanes in $V \cong \mathbf{k}^n$. An ordered basis $\{\omega_1, \ldots, \omega_n\}$ of $\Omega^1(X)$ is triangular with respect to coordinates $x_1, \ldots, x_n$ of V if $\omega_i = \frac{1}{Q}\sum_{j=1}^n q_{ij}\mathrm{d}x_j)$ with $q_{ij} = 0$ for $i < j$, that is, if the coefficient matrix (q_{ij}) is lower triangular.

Proposition 6.5:

If $\Omega^1(X)$ has a triangular basis, then we can choose this basis homogeneous, with $e_i = \deg(q_{ii})$, (but not assuming $e_1 \leq e_2 \leq \ldots \leq e_n$!).

For such a basis we have $Q = \det(q_{ij}) = \prod_{i=1}^n q_{ii}$, hence we can factor $q_{ii} = \prod_{j=1}^{e_i} l_{ij}$, where the l_{ij} $(1 \leq i \leq n,\ 1 \leq j \leq e_i)$ are defining equations for the hyperplanes in X.

Proof.

Lemma 3.14, Lemma 3.18 and Proposition 3.11. □

Note that triangular bases for $\Omega^1(X)$ correspond to (analogously defined) triangular bases for $Der(X)$, via Lemma 3.16(ii).

Theorem 6.6:

An arrangement X has a triangular basis in coordinates $x_1, \ldots, x_n$ if and only if

$$V = \mathbf{k}^n \supseteq \{x_1 = 0\} \supseteq \{x_1 = x_2 = 0\} \supseteq \ldots \supseteq \{x_1 = \ldots = x_{d-1} = 0\} \supseteq \{0\}$$

is an M-chain for X.

Consequently, X has a triangular basis for some choice of coordinates if and only if X is supersolvable.

Proof.

Lemma 6.3, Theorem 2.10 and induction on $\Big|X\Big|$. □

Corollary 6.7: [St3] [JT]
Supersolvable arrangements are free.

Note that this includes the case of 2-dimensional arrangements which are always supersolvable. In this case a triangular basis can easily be constructed (see [T1, p. 296]).

Corollary 6.8:
Freeness with triangular basis is a 3-combinatorial property.

We illustrate Theorem 6.6 and its corollaries by some explicit computations:

Example 6.9:
Let $X_0 = D_3^1$ be the three-dimensional arrangement defined over an arbitrary field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) \neq 2$ by $Q = x(x-y)(x+y)(x-z)(x+z)(y-z)(y+z)$, as in Figure 6.1. This is the smallest non-supersolvable, free arrangement. A homogeneous basis for $\Omega^1(X_0)$ is given by

$$\begin{aligned}
\omega_1 =& \frac{\mathrm{d}x}{x} \\
\omega_2 =& \frac{1}{(y-z)}\frac{1}{x}\left\{\frac{d(x-y)}{x-y} + \frac{d(x-z)}{x-z} - \frac{d(x+y)}{x+y} - \frac{d(x+z)}{x+z}\right\}, \\
\omega_3 =& \frac{1}{(y+z)}\frac{1}{x}\left\{\frac{d(x+y)}{x+y} + \frac{d(x-z)}{x-z} - \frac{d(x-y)}{x-y} - \frac{d(x+z)}{x+z}\right\},
\end{aligned}$$

with exponents $\exp(X_0) = [1,3,3]$.
Adding the hyperplane $H = \{y=0\}$, (dashed in Figure 6.1) we get a supersolvable arrangement $X_0 \cup \{H\}$ with exponents $\exp(X \cup \{H\}) = [1,3,4]$, and a triangular basis given by $\omega_1' = \omega_1$, $\omega_2' = \frac{1}{y}(\omega_2 + \omega_3)$, and

$$\begin{aligned}
\omega_2' =& \frac{1}{2y}\{(y-z)\omega_2 - (y+z)\omega_3\} \\
=& \frac{1}{xy}\left\{\frac{d(x-y)}{x-y} - \frac{d(x+y)}{x+y}\right\} \\
=& \frac{2}{(x^2-y^2)}\frac{\mathrm{d}x}{x} - \frac{2}{(x^2-y^2)}\frac{dy}{y}.
\end{aligned}$$

Now adding the hyperplane $\{z=0\}$, we get the Coxeter arrangement B_3 in standard coordinates, free with exponents [1, 3, 5], and a triangular basis $\omega_1'' = \omega_1$, $\omega_2'' = \omega_2'$ and $\omega_3'' = \frac{1}{z}\omega_3'$.

The study of freeness for hyperplane arrangements has produced some remarkable and remarkably resistant conjectures, which ask how well the algebraic structure of hyperplane arrangements is controlled by the combinatorial structure. The most prominent among them, supported by Theorem 3.20 and Corollary 6.8, is:

Terao's Conjecture 6.10: [T1, p. 293] [T4, p. 565] [St3] [St4, p. 167]

Freeness is combinatorial.

Here "combinatorial" means "determined by the intersection lattice" (Definition 2.5). In fact, Corollary 6.8 as well as results in the following sections suggest that even the following stronger conjecture might be true.

Conjecture 6.11:

Freeness is 3-combinatorial.

We will study an approach to these conjectures in Section 8. The ultimate goal would be to give a "combinatorial" construction of a a basis of $\Omega^1(X)$, say, in the case that it is free.

Note that for Conjecture 6.10, it would by Corollary 3.19(ii) be sufficient to show that the *length* of the degree sequence of $\Omega^1(X)$ is combinatorial. However, in Section 8 we will see that this is not true.

Figure 6.1

7. Generic Arrangements and Trivial Generators

We have briefly discussed generic and 3-generic arrangements in Section 2. Here we will be both more precise and more specific, explain the concept of 3-generic planes, and explore their relevance for the understanding of the structure of $\Omega^1(X)$.

The case of generic arrangements (more precisely, 3-generic arrangements) and their algebraic structure is interesting also because it describes the "other extreme case" as compared to the free arrangements, which are "very non-generic". To avoid technicalities, the arrangements considered are always assumed to be essential.

The main results of this section have straightforward generalizations to *multiarrangements* as studied in [Z2]. In particular, Theorem 7.4 and Corollary 7.6 generalize. However, for simplicity and convenience we avoid this extra level of generality here and refer to [Z1, Chapter 6] for a details.

Definition 7.1:

Let X be an arrangement and $H \in X$. The plane H is 3-generic in X if for every 3-circuit $\{K_1, K_2, K_3\} \subseteq X\Big|_H$ there is a 3-circuit $\{H_1, H_2, H_3\} \subseteq X$ such that $K_i = H_i \cap H$ for $i = 1, 2, 3$.

We are in particular interested in the case of 3-generic arrangements. Note that for a 3-generic arrangement, the restriction to a 3-generic hyperplane is again 3-generic.

Proposition 7.2:

If the restriction map $i^* : \Omega^1(X) \longrightarrow \Omega^1(X\Big|_H)$ is surjective, then H is 3-generic in $X \cup \{H\}$.

Proof.

For a 3-circuit $\{K_1, K_2, K_3\} \subseteq X\Big|_H$, consider the form

$$\omega = \frac{1}{l_3}\left(\frac{\mathrm{d}l_1}{l_1} - \frac{\mathrm{d}l_2}{l_2}\right) \in \Omega^1(X\Big|_H),$$

where l_i is a defining equation for K_i $(i = 1, 2, 3)$. If this ω has a preimage in $\Omega^1(X)$, then this preimage can be chosen of the same form, that is, $\{K_1, K_2, K_3\}$ is the restriction of a 3-circuit in X. $\square$

In general, the converse of Proposition 7.2 is not true, as we will see in Example 8.7(iii). However, it seems plausible that the converse of Proposition 7.2 holds for free arrangements – this seems to be necessary if Terao's Conjecture 6.10 is supposed to be true.

More specifically, we propose:

Conjecture 7.3:

If X is a free arrangement in $V = \mathbf{k}^n$ with $n > 3$ and W is a generic subspace of dimension at least 3 in V, then

$$i^* : \Omega^1(X) \longrightarrow \Omega^1(X\Big|_W)$$

is surjective, and $\dim_{\mathbf{k}} \mathbf{k} \otimes \Omega^1(X) = \dim_{\mathbf{k}} \mathbf{k} \otimes \Omega^1(X\Big|_W)$.

Note that if i^* is surjective, then $\dim_{\mathbf{k}} \mathbf{k} \otimes \Omega^1(X) \geq \dim_{\mathbf{k}} \mathbf{k} \otimes \Omega^1(X\Big|_W)$ is clear.

This Conjecture has powerful implications. In particular it would mean that the generic restriction of an arrangement to a plane of dimension at least 3 is never free. Combinatorially, it means that if the intersection lattice of an essential arrangement X in dimension $n > 2$ is a truncation, then X cannot be free.

About the algebraic structure of k-generic arrangements, we have the following main result.

Theorem 7.4:

Let X be a simple, essential arrangement in $V = \mathbf{k}^n$. Then the following are equivalent, for $0 \leq k \leq n-1$:

(1) X is $(k+2)$-generic

(2) $\Omega^k(X) = \Lambda^k \langle \frac{dl_H}{l_H} : H \in X \rangle$

(3) $\Omega^j(X) = \Lambda^j \langle \frac{dl_H}{l_H} : H \in X \rangle \quad$ for $\quad 0 \leq j \leq k$.

Here $\langle M \rangle$ denotes the S-module generated by the elements of M, and $\Lambda^j \langle M \rangle$ denotes the S-module generated by j-fold wedges of elements of M, $\Lambda^0 \langle M \rangle = S$.

Corollary 7.5: $[k = 1]$

A simple essential arrangement X is 3-generic (that is, $L = L_X$ does not have any circuits of size 3) if and only if $\Omega^1(X)$ is generated by $\left\{ \frac{dl_H}{l_H} : H \in X \right\}$.

Proof.

(3) $\Longrightarrow$ *(2)* is trivial. For *(2)* $\Longrightarrow$ *(1)*, let $C \subseteq X$ be a circuit of size $|C| = j+2 \leq k+2$, $C = \{H_0, H_1, \ldots, H_{j+1}\}$, say. We denote l_{H_i} by l_i, for all i. Let $\mathrm{d}x_C = \mathrm{d}l_1 \wedge \ldots \wedge \mathrm{d}l_{j+1}$ and $Q_C = l_1 l_2 ... l_{j+1}$, corresponding to the broken circuit $C_0 = C - \{H_0\}$. If $\{H_1, \ldots, H_n\}$ is a basis of L (set of coordinate hyperplanes) containing C_0, then we can write the Euler vector field $\theta_0 = \sum_i x_i \frac{\partial}{\partial x_i}$ as

$$\theta_0 = \sum_{i=1}^{n} l_i \frac{\partial}{\partial l_i}.$$

Now we get

$$\frac{dx_C}{l_0 Q_C} \in \Omega^{j+1,-(j+2)}(X)$$

and thus by Theorem 3.7(b)

$$\omega_C = i_{\theta_0}(\frac{\mathrm{d}x_C}{l_0 Q_C}) \in \Omega^{j,-(j+1)}(X)$$

which contradicts (3). To see that (2) also cannot hold, we consider

$$\omega_C \wedge \frac{\mathrm{d}l_{j+2}}{l_{j+2}} \wedge \ldots \wedge \frac{\mathrm{d}l_{k+1}}{l_{k+1}} \in \Omega^{k,-(k+1)}(X),$$

which does not vanish.

(1) $\Longrightarrow$ *(2)* For the case $k = 1$ (cf. Corollary 7.5), this follows from the Corollary 6.1 of the Strong Preparation Lemma. Repeated application of the same argument (perhaps best formalized as a "Strong Preparation Lemma for k-forms") proves the general case, too. $\square$

Corollary 7.6:

For every essential 3-generic arrangement X,

$$\dim_{\mathbf{k}} \mathbf{k} \otimes \Omega^1(X) = |X|.$$

Thus 3-generic arrangements with $|X| > n$ are never free. In particular, generic arrangements with $|X| > n \geq 3$ are never free.

Observe that for the situation of Theorem 7.4, circuits of size larger than three are irrelevant for (the generators of) $\Omega^1(X)$. This is consistent with our observations for Conjecture 6.11. Theorem 7.4 allows some nontrivial computations. In particular, we get not only the degree sequence for $\Omega^1(X)$ in the 3-generic case, but also for its minimal free resolution.

In homological algebra terms, this amounts to computation of the homology of $\Omega^1(X)$.

Corollary 7.7:

Let X be 3-generic and essential.

(1) $\Omega^1(X)$ has a minimal free resolution of the form

$$0 \longrightarrow F^1(X) \xrightarrow{r_1} F^0(X) \xrightarrow{r_0} \Omega^1(X) \longrightarrow 0,$$

where $F^0(X) \cong S^m$ has a basis $\{\varphi_i : 1 \leq i \leq m\}$ that by r_0 is mapped to $\{\frac{\mathrm{d}l_i}{l_i} : 1 \leq i \leq m\}$, and $F^1 \cong S^{m-n}$ has a basis $\{C_j : n < j \leq m\}$ corresponding to certain circuits in L.

(2) The degree sequence of $\Omega^1(X)$ and $F^0(X)$ is $[-1, \ldots, -1]$ of length m; in particular,

$$\beta_0^S(\Omega^1(X)) = m.$$

The degree sequence of $F^1(X)$ is $[0, ..., 0]$ of length $m - n$, that is, $\beta_1^S(\Omega^1(X)) = m - n$. Also, $\beta_i^S(\Omega^1(X)) = 0$ for $i > 1$.

8. Critical Subarrangements and Combinatorial Generators

The Strong Preparation Lemma 5.1 and the results about 3-combinatorial invariants lead to the following approach to understand the structure of $\Omega^1(X)$ in terms of combinatorial and 3-combinatorial invariants.

Definition 8.1:

Let X be an arrangement in V.

(i) For $H \in X$, the excess of H in X is

$$e_X(H) = \left|X\right| - \left|X\big|_H\right|.$$

(ii) The excess of X is

$$e(X) = \min_{H \in X} e_X(H).$$

Lemma 8.2:

Let X be an arrangement in V, and $\omega \in \Omega^{1,-k}(X)$ a homogeneous logarithmic 1-form, with $Y := \mathrm{supp}(\omega) \subseteq X$. Then

$$k = -\deg(\omega) \leq e(\mathrm{supp}(\omega)) = e(Y).$$

Proof.

This follows directly from the Strong Preparation Lemma 5.1. ∎

Note that the excess is not a monotone function: we cannot conclude that $k \leq e(X)$.

Lemma 8.2 suggests to consider those minimal subsets $Y \subseteq X$ which support a form of degree $k < 0$.

Definition 8.3:

Let X be an arrangement in $V = \mathbf{k}^n$.

(i) $Y \subseteq X$ is k-supporting (for some $k \geq 0$) if

$$\Omega^{1,-k}(Y) \neq 0,$$

that is,

$$\dim_{\mathbf{k}} \Omega^{1,-k}(Y) > 0.$$

(ii) $Y \subseteq X$ is k-critical (for some $k \geq 0$) if Y is k-supporting and no proper subset of Y is k-supporting. Equivalently, the k-critical sets are the minimal subsets Y of X (under inclusion) such that

$$Y = \mathrm{supp}(\omega) \quad \textit{for some} \quad \omega \in \Omega^{1,-k}(X).$$

Observe that, by Lemma 8.2, $e(Y) \geq k$ for every k-supporting and hence in particular for every k-critical subarrangement $Y \subseteq X$.

Examples 8.4:

(i) The only 0-critical subset of X is $\emptyset$, by definition. The 1-critical sets are exactly the sets $\{H\}$ for $H \in X$. The 2-critical sets are the 3-circuits in L: for every 3-circuit C, we get

$$\Omega^{1,-2}(X) = \mathbf{k}\omega_C,$$

where ω_C is the form constructed in the proof of Theorem 7.4.

(ii) The arrangements in Figure 8.1 are both 3-critical. The corresponding 1-forms of degree -3 are of the form

$$\omega_{(a)} = \frac{1}{l_4 l_3}\left(\frac{\mathrm{d}l_1}{l_1} - \frac{\mathrm{d}l_2}{l_2}\right)$$

and

$$\omega_{(b)} = \frac{1}{l_6 l_5}\left(\frac{\mathrm{d}l_1}{l_1} + \frac{\mathrm{d}l_2}{l_2} - \frac{\mathrm{d}l_3}{l_3} - \frac{\mathrm{d}l_4}{l_4}\right).$$

Figure 8.2(a) depicts an arrangement X of 9 planes in $\mathbf{R}^3$ such that $e(X) = 3$, but $\Omega^{1,-3}(X) = 0$ (we will do the computations for this in Example 8.7). This shows that the k-critical sets do *not* in general coincide with the minimal subsets of X with excess k.

We now derive the following strengthening of Corollary 7.5.

Proposition 8.5:

Let X be an essential arrangement such that $e(Y) \leq 2$ for all subarrangements $Y \subseteq X$. Then $\Omega^1(X)$ is generated by

$$\left\{\frac{\mathrm{d}l_H}{l_H}\right\} \cup \{\omega_C : C \text{ is a 3-circuit in } X\}.$$

Proof.

Induction on $|X|$. Let $\omega \in \Omega^1(X)$. We can assume that $X = \operatorname{supp}(X)$. Now choose $H \in X$ such that $e_X(H) = e(X)$.

If $e_X(H) = 1$, then we are done by Lemma 6.1.

For $e_X(H) = 2$, let $C = \{H, H_1, H_2\}$ be the 3-circuit in X containing H, and (by Strong Preparation Lemma 5.1) write ω as

$$\omega = \frac{p}{l_1}\frac{\mathrm{d}l_H}{l_H} + \omega^0.$$

where ω^0 does not have a pole at H. Now we have

$$\omega_C = \frac{1}{l_1}\left(\frac{\mathrm{d}l_H}{l_H} - \frac{\mathrm{d}l_2}{l_2}\right)$$

in our set of generators. But

$$\omega - p\omega_C = \omega^0 + \frac{1}{l_1}\frac{\mathrm{d}l_2}{l_2} \in \Omega^1(X - \{H\}),$$

and we are done by induction. $\square$

Now Example 8.4(ii) leads to

Figure 8.1(a)

Figure 8.1(b)

Figure 8.2(a)

Figure 8.2(b)

Conjecture 8.6:
For every k-critical arrangement X, $e(X) = k$.

Together with the Strong Preparation Lemma 5.1 and the argument of Proposition 8.5 this Conjecture 8.6 would imply that
(i) $\dim_{\mathbf{k}} \Omega^{1,-k}(X) = 1$ for every k-critical X;
(ii) $\Omega^{1,-k}(X)$ is generated by $\{\Omega^{1,-k}(Y) : k \geq 1; Y \subseteq X \ k\text{-critical}\}$.
Note that for $k \leq 2$, Conjecture 8.6 and its implications (i) and (ii) are true by Example 8.4(i).

We believe that Conjecture 8.6 is sufficiently sharp and concrete to be somewhat promising. Its proof would certainly give some new ideas about the algebraic structure of generators for $\Omega^1(X)$. However, it is not clear how this would help to understand the *combinatorial* structure of generators. In fact, being k-critical is *not* a combinatorial property for $k = 3$. This will follow from the following final example.

Example 8.7:
Let X_1 be the arrangement of nine hyperplanes in $\mathbf{R}^3$ sketched in Figure 8.2(a), which we mentioned before in 8.5. This arrangement has excess 3, where every proper subarrangement has excess at most 2.
Let ω be a homogeneous logarithmic form of degree -3 at X, $\omega \in \Omega^{1,-3}(X)$. Now let $H = H_9$ and $l_H = l_9$, then $l_H\omega \in \Omega^{1,-2}(X)$ can by Proposition 8.5 be written as

$$l_H\omega = \frac{c_1}{l_3}\left(\frac{\mathrm{d}l_1}{l_1} - \frac{\mathrm{d}l_2}{l_2}\right) + \frac{c_2}{l_7}\left(\frac{\mathrm{d}l_1}{l_1} - \frac{\mathrm{d}l_5}{l_5}\right) + \frac{c_3}{l_3}\left(\frac{\mathrm{d}l_6}{l_6} - \frac{\mathrm{d}l_8}{l_8}\right) + \frac{c_4}{l_7}\left(\frac{\mathrm{d}l_6}{l_6} - \frac{\mathrm{d}l_4}{l_4}\right)$$

Here $l_H\omega\Big|_H = 0$ is equivalent to the following four conditions:

$$\begin{aligned} c_1 l_7 + c_2 l_3 &\in \langle l_9, l_1\rangle \\ c_1 l_7 + c_4 l_3 &\in \langle l_9, l_2\rangle \\ c_3 l_7 + c_4 l_3 &\in \langle l_9, l_5\rangle \\ c_3 l_7 + c_2 l_3 &\in \langle l_9, l_6\rangle. \end{aligned}$$

These four conditions translate into four linear equations for $\mathbf{c} = (c_1, c_2, c_3, c_4)$.
(i) *Generically*, they do not have a nonzero solution in $\mathbf{c}$: they require that the intersection lines $H_5 \cap H$ and $H_6 \cap H$ are conjugate in H with respect to $H_1 \cap H$ and $H_2 \cap H$. In other words, it requires a particular symmetry of $\{H_1, H_2, H_5, H_6\}\Big|_H$.
Thus generically

$$\dim_{\mathbf{R}} \Omega^{1,-3}(X_1) = 0.$$

In this case, we compute the extended degree sequence of $\Omega^1(X_1)$ (listing first the degree sequence of the module and then of its relations) as $[-2, -2, -2, -2, -2, -2, -1| -1, -1, -1, -1]$. The extended degree sequence of $Der(X_1)$ is $[1, 6, 6, 6, 6, 6, 6|7, 7, 7, 7]$ in this case.

(ii) Now consider the *"special position"* arrangement X_2 in Figure 8.2(b), given by

$$\begin{array}{rcccc} l_1 = & x & +y & -z & \\ l_2 = & x & -y & +z & \\ l_3 = & x & & & \\ l_4 = & 2x & -2y & +z & \\ l_5 = & 2x & -y & -2z & \\ l_6 = & 2x & +y & +z & \\ l_7 = & & y & & \\ l_8 = & 2x & -y & -z & \\ l_9 = & & & z & \end{array}$$

which is combinatorially equivalent to X_1. However, X_2 has the extra symmetry required! In fact, the above conditions are easily verified to have the solution $\mathbf{c} = (1, 1, -1, -\frac{1}{2})$, such that

$$\dim_{\mathbf{R}} \Omega^{1,-3}(X_2) = 1.$$

The extended degree sequence of $\Omega^1(X_2)$ is $[-3, -2, -2, -2, -2, -1| -1, -1, -1]$. The extended degree sequence for $Der(X_2)$ turns out to be $[1, 5, 6, 6, 6|7, 8]$.

(iii) Now let $\widehat{X_2}$ be the arrangement in $\mathbf{R}^4$ defined by

$$\begin{array}{rcccc} \widehat{l_1} = & x & +y & -z & \\ \widehat{l_2} = & x & -y & +z & \\ \widehat{l_3} = & x & & & \\ \widehat{l_4} = & 2x & -2y & +z & \\ \widehat{l_5} = & 2x & -y & -2z & \\ \widehat{l_6} = & 2x & +y & +z & +w \\ \widehat{l_7} = & & y & & +w \\ \widehat{l_8} = & 2x & -y & -z & -w \\ \widehat{l_9} = & & & z & -w \end{array}$$

The hyperplane $K = \{w = 0\}$ is 3-generic in $\widehat{X_2}$, and $\widehat{X_2}\Big|_K = X_2$. However, there is no nonzero 1-form of degree -3 at $\widehat{X_2}$. (In fact it is easy to see that if the four equations for $\mathbf{c}$ have a nonzero solution, then the corresponding arrangement does not have a nontrivial extension, i.e., it is not the 3-generic restriction of an essential 4-dimensional arrangement.)

With this we can compute the extended degree sequence for $\Omega^1(\widehat{X_2})$ to be $[-2, -2, -2, -2, -2, -2, -1| -1, -1, -1]$. The extended degree sequence for $Der(\widehat{X_2})$ turns out to be $[1, 3, 3, 3, 3|4]$. In particular, we observe that with $\Omega^{1,-3}(\widehat{X_2}) = 0$, the restriction map

$$i^* : \Omega^1(\widehat{X_2}) \longrightarrow \Omega^1(X_2)$$

cannot be surjective, that is, the converse of Proposition 7.2 is false.

(Whereas the generators, relations and degree sequences for the modules of differential forms are quite easily determined by hand (using Proposition 8.5), we have used the Computer Algebra System `Macaulay` for the computation of resolutions and degree sequences for $Der(X_i)$.)

Corollary 8.8

The degree sequences for $\Omega^1(X)$ and $Der(X)$ and their lengths are not combinatorial. The Solomon-Terao polynomial $\Psi(X; x, t)$ is not combinatorial.

In [T1], Terao also studies a second sequence of integers associated with every arrangement, the *structure sequence*, defined as the sequence of minimal degrees for a maximal set of S-independent elements in the module considered. This structure sequence is a subsequence of length n of the degree sequence. We have not discussed it because it seems to carry less information about the arrangement than the structure sequence. In any case, Example 8.7 also implies that the structure sequences of $Der(X)$ and $\Omega^1(X)$ are not combinatorial, either.

This raises questions about Terao's Conjecture 6.10 and Conjecture 6.11.

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