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ARRANGEMENTS OF HYPERPLANES WITH A LATTICE OF REGIONS

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I: INTRODUCTION

Let $\mathcal{H}$ be a finite set of hyperplanes through the origin in a real vector space $\mathbb{R}^d$. We study the combinatorial structure of the set $\mathcal{R}$ of regions, that is connected components of the complement $\mathbb{R}^d \setminus \bigcup_{H\in\mathcal{H}} H$ of the arrangement $\mathcal{H}$.

In particular, the adjacency graph of $\mathcal{H}$ has $\mathcal{R}$ as its set of vertices and connects two regions by an edge if they are adjacent (separated by exactly one hyperplane). If one of the regions is chosen as a distinguished base region B, the adjacency graph, directed away from B, gives rise to a finite, graded poset $P(\mathcal{H};B)$, the poset of regions of $\mathcal{H}$ with respect to the base B [Ed1].

In this paper we discuss under which conditions $P(\mathcal{H};B)$ is a lattice. For this to hold, B necessarily has to be a simplicial cone. We show that this condition is also sufficient for $d \le 3$ but not for $d > 3$.

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However, for several highly structured situations more can be said.

If $\mathcal{H}$ is <u>simplicial</u> (every region of $\mathcal{H}$ is a simplicial cone), then $P(\mathcal{H};B)$ is a lattice for every region B. This generalizes results by Björner [Bj3] on the weak ordering of finite COXETER groups.

An arrangement $\mathcal{H}$ is called <u>supersolvable</u> if the lattice $L(\mathcal{H})$ of intersections of hyperplanes in $\mathcal{H}$ (ordered by reverse inclusion) is a supersolvable lattice as defined by Stanley [St2]. We give a geometric description of supersolvable arrangements. For such arrangements there is a canonical choice of base regions B such that $P(\mathcal{H};B)$ is a lattice. Furthermore, for such choice of a base region the rank generating function factorizes in a way which is similar to the expression known for the case of COXETER arrangements. This suggests a generalization to arrangements that are <u>free</u> in the sense of [Te1].

The results of this section IV were announced (without proofs) in [Bj1].

Much of out discussion of the lattice property of the bounded posets $P(\mathcal{H};B)$ depends on a "local" criterion for the lattice property in bounded posets of finite length (Lemma 2.1) that appears to be new.

We proceed to describe the lattice property and lattice operations geometrically in terms of an (abstract) closure operator defined on the set $\mathcal{R}$ of regions. With this closure operator, the set $S(R)$ of hyperplanes in $\mathcal{H}$ separating a region R from the base region is both closed and coclosed. If $P(\mathcal{H}, B)$ is a lattice, the converse also holds and the lattice operations can be expressed in terms of the closure operator.

(more explicitly?)

In our final sections we show how our results generalize to oriented matroids, which can be thought of as combinatorial abstractions of hyperplane arrangements. The adjacency graph of a hyperplane arrangement determines the the corresponding oriented matroid. Thus the lattice $L(\mathcal{H})$ of intersections can be constructed from the poset of regions $P(\mathcal{H}; B)$ for any B.

Our approach to hyperplane arrangements in this paper is entirely combinatorial. We refer to [EW], [De] and [OS1] for topological aspects (which become more profound when working in complex vector spaces), to [Za] and [Ca] for enumerative aspects and a study of the lattice of intersections, and [Te1], [Te2] for algebraic ramifications.

§4: SUPERSOLVABLE ARRANGEMENTS

Now we proceed to study "supersolvable" arrangements, a large class of examples defined by a combinatorial condition on their lattices of intersections:

DEFINITION 4.1: Let L be a finite geometric lattice of rank d. An element V of L is modular if $rk(V \vee W) + rk(V \wedge W) = rk V + rk W$ for every $W \in L$. L is supersolvable if it has a maximal chain $\hat{1} = V_0 > V_1 > \ldots > V_d = \hat{0}$ of modular elements [St2].

Stanley has shown [St1] that an element V of L is modular if its complements form an antichain. This is the case exactly if all complements of V have the same rank $d - rk V$.

From now on, let L be a supersolvable geometric lattice, and $\hat{0} = V_0 < V_1 < \ldots < V_d = \hat{1}$ a fixed maximal chain of modular elements in L.

For $1 \le i \le d$, let e_i be the number of atoms of L that lie below V_i, but not below V_{i-1}, i.e. the number of atoms in $[\hat{0}; V_i] \setminus [\hat{0}; V_{i-1}]$. Trivially we get $e_1 = 1$, and $\sum_{i=1}^{d} e_i$ is the number of atoms in L.

A main result of [St2] states that the characteristic polynomial of L factors as $\chi_L(t) = \prod_{i=1}^{d}(t - e_i)$. In particular this shows that the multiset $\{e_1, \ldots, e_d\}$ does not depend on the maximal chain of modular elements chosen in L.

DEFINITION 4.2.: An arrangement $\mathcal{H}$ is supersolvable if its lattice $L(\mathcal{H})$ of intersections is a supersolvable lattice. The integers $e_1, \ldots, e_d$ associated with $L(\mathcal{H})$ are called the exponents of $\mathcal{H}$.

(compare §2)

Supersolvable arrangements have first been considered by [St3] and [JT] in the context of free arrangements of hyperplanes as defined by Terao [Te].

The following result, generalizing Theorem 5.4 of [JT] to arbitrary dimension d, describes the geometric structure and construction of supersolvable arrangements inductively:

THEOREM 4.3: Every arrangement $\mathcal{H}$ of rank at most 2 is supersolvable. An arrangement $\mathcal{H}$ of rank $d \geq 3$ is supersolvable if and only if it can be written as $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1$ (disjoint union, $\mathcal{H}_1 \neq \emptyset$), where $\mathcal{H}_0$ is a supersolvable arrangement of rank $d-1$, and for any $H', H'' \in \mathcal{H}_1$ $(H' \neq H'')$ there is an $H \in \mathcal{H}_0$ such that $H' \cap H'' \subseteq H$.

PROOF: Every geometric lattice of rank 2 is supersolvable, which proves the first statement.

If $\mathcal{H}$ is supersolvable, let V_{d-1} be the coatom (line) in a maximal modular chain ("flag") of $L(\mathcal{H})$ as above, and define $\mathcal{H}_0 = \{H \in \mathcal{H} \mid H \supseteq V_{d-1}\}$, $\mathcal{H}_1 = \mathcal{H} \setminus \mathcal{H}_0$.

Then $\mathcal{H}_0$ is obviously supersolvable of rank $d-1$, and for $H', H'' \in \mathcal{H}$, $H' \neq H''$ we find that H' and H'' are both complements of V_{d-1} in $L(\mathcal{H})$. Thus $V_{d-1} \vee (H' \vee H'') = \hat{1}$, hence by modularity $H := V_{d-1} \wedge (H' \vee H'')$ is an atom in $L(\mathcal{H})$, that is a hyperplane in $\mathcal{H}$. But $H \leq V_{d-1}$ means $H \in \mathcal{H}_0$, and $H \leq H' \vee H''$ means $H \supseteq H' \cap H''$.

To prove the converse, let $\mathcal{H}_0$ and $\mathcal{H}_1$ be given as above. Any maximal chain of modular elements in $L(\mathcal{H}_0)$ together with $\hat{1}$ defines a maximal chain in $L(\mathcal{H})$. By the argument in [St1, p.217] it is sufficient to show that V_{d-1} (the maximal element of $L(\mathcal{H}_0)$) is modular in $L(\mathcal{H})$. For this let $Y \in L(\mathcal{H})$. For $Y \in L(\mathcal{H}_0)$, there is nothing to show. For $Y \in L(\mathcal{H}_0)$, write Y as a join of $r(Y)$ atoms such that the number of atoms (hyperplanes) from $\mathcal{H}_1$ in this expression is minimal. But for $H', H'' \in \mathcal{H}_1$, $H' \neq H''$, there is an $H \in \mathcal{H}_0$ such that $H \vee H' = H' \vee H''$. Thus Y can be written as $Y = Y_0 \vee H'$, where $Y_0 \in L(\mathcal{H}_0)$, $H' \in \mathcal{H}_1$, and $\operatorname{rk} Y_0 = \operatorname{rk} Y - 1$ from semimodularity. But this shows

$$\operatorname{rk}(V_{d-1} \vee Y) + \operatorname{rk}(V_{d-1} \wedge Y) = \operatorname{rk} \hat{1} + \operatorname{rk} Y_0 =$$
$$(\operatorname{rk} V_1 + 1) + (\operatorname{rk} Y - 1) = \operatorname{rk} V_1 + \operatorname{rk} Y \ \#.$$

We remark that ~~especially~~ the exponents $\{e_1, \ldots, e_d\}$ of $\mathcal{H}$ are recursively given by Theorem 4.3 as the multiset $\{e_1, \ldots, e_{d-1}\}$ of exponents of $\mathcal{H}_0$ together with $e_d = |\mathcal{H}_1|$.

To study the poset of regions of a supersolvable arrangement $\mathcal{H}$, we observe that Theorem 4.3 describes a canonical, orderpreserving, surjective map

$$\pi : P(\mathcal{H}; B) \longrightarrow P(\mathcal{H}_0; \pi(B))$$

which is inclusion of the regions of $\mathcal{H}$ into the larger regions of $\mathcal{H}_0$, where $\mathcal{H}_0$ is a supersolvable arrangement of lower rank. This allows constructions and proofs by induction.

Inductively, define a base region $\mathcal{B}$ for $\mathcal{H}$ to be <u>canonical</u> if it is chosen such that $\pi(\mathcal{B})$ is canonical for $\mathcal{H}_0$, and that $\pi^{-1}(\pi(\mathcal{B}))$ is linearly ordered in $P(\mathcal{H},\mathcal{B})$. For $n \leq 2$, every base region is canonical.

Note that, given any base region $\mathcal{B}_0$ for $\mathcal{H}_0$, the regions of the arrangement $\mathcal{H}$ contained in $\mathcal{B}_0$ are linearly ordered by adjacency, such that for every canonical $\mathcal{B}_0$ for $\mathcal{H}_0$ there are exactly two canonical $\mathcal{B}$ for $\mathcal{H}$ such that $\pi(\mathcal{B}) = \mathcal{B}_0$.

Thus every supersolvable arrangement of rank d has at least 2^d canonical base regions.

<u>THEOREM 4.4:</u> Let $\mathcal{H}$ be a supersolvable arrangement, $\mathcal{B}$ a canonical base region for $\mathcal{H}$, then the rank generating function for $P(\mathcal{H},\mathcal{B})$ is

$$\prod_{i=1}^{d}(1+q+q^2+\ldots+q^{e_i}),$$

where the e_i are the exponents of $L(\mathcal{H})$.

<u>PROOF:</u> If $\mathcal{B}$ is canonical, then all the fibres of $P(\mathcal{H},\mathcal{B})$ under the mapping π are chains of length $|\mathcal{H}_1| = e_d$.
Recall from [Ed] that $P(\mathcal{H},\mathcal{B})$ is graded with rank being given by $\mathrm{rk}\,R = |S(R)|$. Thus, if R is a region and $h(R)$ denotes its rank in the fibre $\pi^{-1}(\pi(R))$, then $\mathrm{rk}\,R = \mathrm{rk}(\pi(R)) + h(R)$.
Hence we can compute the rank generating

function by induction as the product of the rank generating functions $\prod_{i=1}^{d-1}(1+q+\ldots+q^{e_i})$ for $\mathcal{H}_0$ and $1+q+\ldots+q^{e_d}$ for the fibres ∎

<u>COROLLARY 4.5:</u> ([Te, JT])

The number of regions of a supersolvable arrangement is $\prod_{i=1}^{d}(1+e_i)$

<u>THEOREM 4.6:</u> Let $\mathcal{H}$ be a supersolvable arrangement, B a canonical base region. Then $P(\mathcal{H},B)$ is a lattice.

<u>PROOF</u>: For every region R of the arrangement $\mathcal{H}$, let $F(R)$ be its fibre under π, that is $F(R)=\pi^{-1}(\pi(R))$. $F(R)$ is a chain of length $|\mathcal{H}_1|$; it is an interval of $P(\mathcal{H},B)$.

Let R_1 and R_2 be two regions of $\mathcal{H}$, and let $T\in\pi^{-1}(\pi(R_1)\vee\pi(R_2))$ be minimal in $F(\pi(R_1)\vee\pi(R_2))$ such that $T\geq R_1$ and $T\geq R_2$. This is well-defined, because we can assume that $P(\mathcal{H}_0,\pi(B))$ is a lattice by induction on the rank, and the maximal element of $F(\pi(R_1)\vee\pi(R_2))$ is an upper bound for R_1 and R_2. Note that T is a minimal upper bound for R_1 and R_2 by construction.

Let T' be a diferent minimal upper bound. This implies that $T\not\leq T'$ and $\pi(T)<\pi(T')$.

Now let H be the hyperplane in $\mathcal{H}_1$ that separates T from the region it covers in $\mathcal{F}(T)$. Choose $H' \in S(T) \setminus S(T')$: this is possible because $T \nleq T'$. Note that $H' \in \mathcal{H}_1$ because $\pi(T) \leq \pi(T')$

We will now use that for every fibre $\mathcal{F}$ (i.e. for every region of $\mathcal{H}_0$) there is a unique linear order on $\mathcal{H}_1$ defined by $H_1 \leq H_2$ if and only if $H_2 \in S(R)$ implies $H_1 \in S(R)$ for $R \in \mathcal{F}$, that is "H_2 is higher than H_1" in the corresponding region of $\mathcal{H}_0$.

By construction we have $\underline{H' < H \text{ in } \mathcal{F}(T)}$.

The construction of T implies $H \in S(R_1) \cup S(R_2)$; we can assume $H \in S(R_1)$. On the other hand, $H' \notin S(T')$, thus $H' \notin S(R_1) \cup S(R_2)$ and $H' \notin S(R_1)$, and therefore $\underline{H < H' \text{ in } \mathcal{F}(R_1)}$.

Finally $H \in S(T')$, but $H' \notin S(T')$, hence $\underline{H < H' \text{ in } \mathcal{F}(T')}$.

Now if H_0 is the (unique) hyperplane in $\mathcal{H}_0$ that contains $H \cap H'$, then these data implie that $H_0 \in S(T_1) \,\Delta\, S(T)$ and $H_0 \in S(T) \,\Delta\, S(T')$, which contradicts $\pi(C_1) \leq \pi(T) < \pi(T')$. Thus T as constructed above is the unique minimal upper bound for R_1 and R_2 ∎

Since the conclusions of the Theorems 4.4 and 4.6 are known to hold also if $\mathcal{H}$ is a COXETER arrangement ([So], respectively §3), it is natural to ask whether they generalize to the case where $\mathcal{H}$ is a "free" arrangement in the sense of Terao [Te1]. In this general case (which encompasses the case of supersolvable and of COXETER arrangements) Corollary 4.5 is known to hold by [Te2], with respect to the exponents defined for a free arrangement in a unified way as described by Terao, see [Te1] or [St3].

It has been conjectured by Terao and Wagreich [Wa, p.137] that Theorem 4.4. holds for free arrangements, that is for suitable choice of a base chamber B, the rank generating function of $P(\mathcal{H}, B)$ factors as $\prod_{i=1}^{d}(1+q+\ldots+q^{e_i})$ for the (generalized) exponents e_i. However, H. Terao

?picture? has found [Te4] that the arrangement "$A_4(17)$" from Grünbaums list [Gr2] of simplicial arrangements forms a counterexample to this conjecture. Using the symmetry of this arrangement, it is not hard to check that $A_4(17)$ is free with generalized exponents 1, 7, 9 [Te1, p.308], but for none of the regions as basis $P(\mathcal{H}, B)$ has the "correct" rank generating function

$$(1+q)(1+q+\ldots+q^7)(1+q+\ldots+q^9)$$
$$= 1+3q+5q^2+7q^3+9q^4+11q^5+\ldots$$

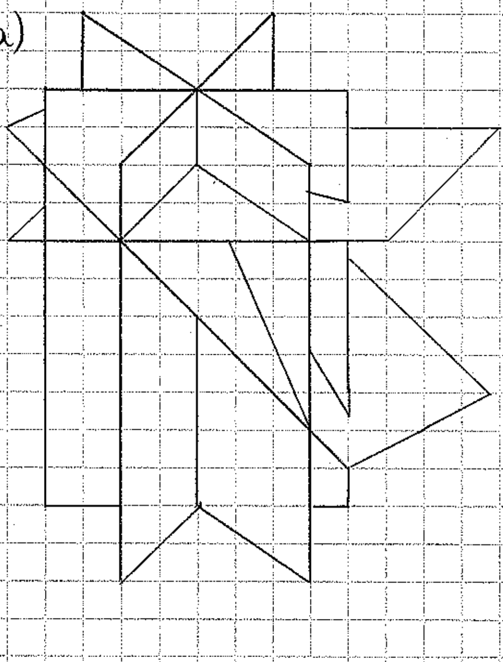
The generalization of Theorem 4.6 to free arrangements is true for $d \le 3$ by Theorem 3.2 and the existence of simplicial base regions [Ca]. For large enough dimension it is probably false, too.

If $\mathcal{H}$ is a COXETER arrangement, then the posets of regions $P(\mathcal{H})$ obtained for different bases are canonically isomorphic via the transitive action of the corresponding COXETER group on the regions. However, an analogous statement for supersolvable arrangements is false.

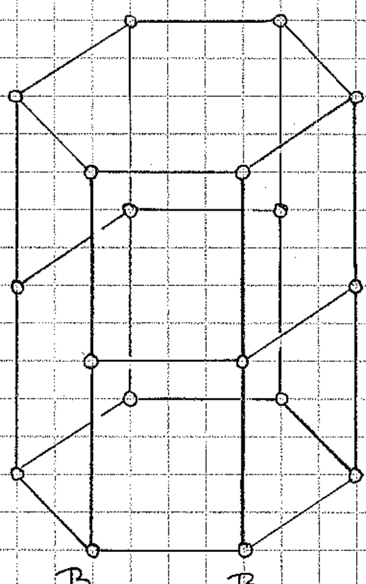
Example

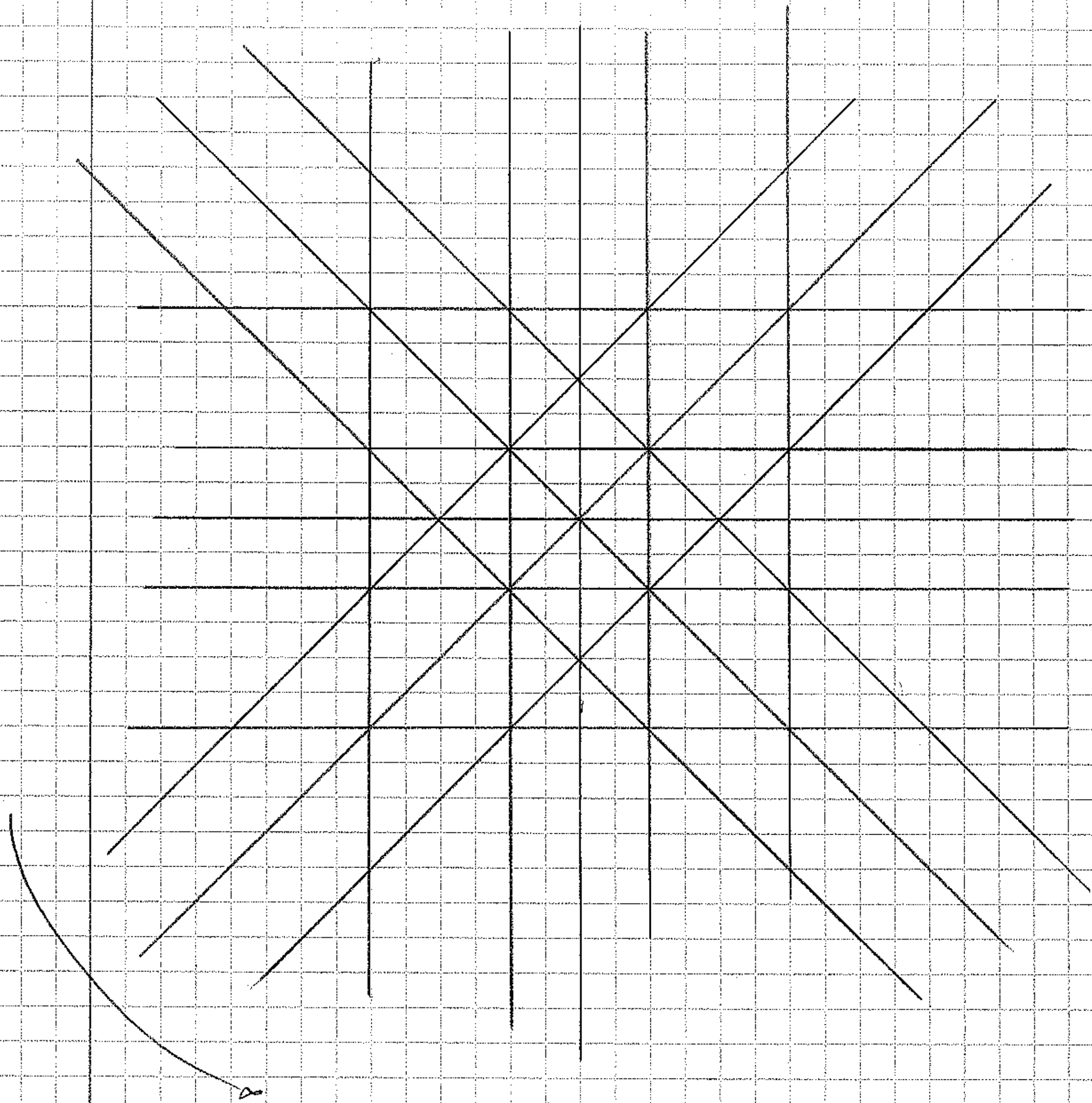
Let $\mathcal{H}$ be the supersolvable arrangement in $\mathbb{R}^3$ with exponents 1, 2, 2 (Figure (a)). Its adjacency graph is given by Figure (b).

(a)



(b)





A_4 (17)

The base regions B_1 and B_2 are both canonical, but the posets $P(\mathcal{H}, B_1)$ and $P(\mathcal{H}, B_2)$ (obtained by directing the adjacency graph away from the respective base) are not isomorphic, though they share the lattice property and the same rank generating function $(1+q)(1+q+q^2)^2$.

In connection with Theorem 3.2., it seems interesting that the regions of supersolvable arrangements, though not simplicial in general, are of a very restricted combinatorial type. In particular, the regions of a supersolvable arrangement in $\mathbb{R}^d$ have at most $2d$ bounding hyperplanes. In fact, they are cones over $(d-1)$-dimensional polytopes that can be described inductively via the construction of Theorem 4.3. We omit the details.