FREIE UNIVERSITÄT BERLIN
FORSCHUNGSSEMINAR SS 14

Pro-étale cohomology

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INTRODUCTION

The Étale cohomology theory which was initially suggested by Grothendieck in 1960s plays a very important role in Modern algebraic geometry. In particular, when X is an algebraic variety over an algebraically closed field k of characteristic $p \neq l$, the l-adic cohomology which is derived from the general étale cohomology theory provides a nice Weil-cohomology theory. However, since the higher étale cohomology groups of a locally constant sheaf over a normal variety are always torsion, in order to get a Weil-cohomology theory with coefficients in a field of characteristic 0, one has to define the l-adic cohomology groups

$$H^i(X_{\text{ét}}, \mathbb{Q}_l) := (\varprojlim_{n \in \mathbb{N}^+} H^i(X_{\text{ét}}, \mathbb{Z}/l^n\mathbb{Z})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

in an "indirect" manner, i.e. one has to define $H^i(X_{\text{ét}}, \mathbb{Q}_l)$ by a projective limit of $H^i(X_{\text{ét}}, \mathbb{Z}/l^n\mathbb{Z})$ (and then tensor with $\mathbb{Q}_l$) instead of "directly" taking cohomology for the constant étale sheaf $\mathbb{Q}_l$. If one wants to stick to the "direct" manner of defining $H^i(X_{\text{ét}}, \mathbb{Q}_l)$, then one would get

$$H^i(X_{\text{ét}}, \mathbb{Q}_l) = \begin{cases} \mathbb{Q}_l & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

assuming X to be a smooth projective connected curve over an algebraically closed field. This certainly can't fit into a Weil-cohomology theory with coefficients in a field of characteristic 0.

For some reasons (mainly for the study of perverse sheaves), one has to work with complexes of l-adic constructible sheaves and do operations at the level of derived categories instead of working with the single l-adic sheaf $\mathbb{Q}_l$ and considering only cohomology groups. Again these notions are defined in an "indirect" manner, i.e. one has to deal with projective systems instead of single objects. For example, a constructible $\mathbb{Z}_l$-sheaf is not defined as a single étale sheaf but as a projective system $\{\mathscr{F}_n\}_{n\in\mathbb{N}^+}$ of étale abelian sheaves satisfying

(i) $\mathscr{F}_n$ is annihilated by l^n and is a constructible $\mathbb{Z}/l^n\mathbb{Z}$-sheaf;

(ii) if $n \geq m$ then the transition map $\mathscr{F}_n \to \mathscr{F}_m$ induces an isomorphism $\mathscr{F}_n \otimes_{\mathbb{Z}/l^n\mathbb{Z}} \mathbb{Z}/l^m\mathbb{Z} \to \mathscr{F}_m$ in the category of $\mathbb{Z}/l^m\mathbb{Z}$-sheaves.

In this way we obtain the category of constructible $\mathbb{Z}_l$-sheaves (with obvious morphisms). To get the category of constructible $\mathbb{Q}_l$-sheaves from that of constructible $\mathbb{Z}_l$-sheaves we just keep the objects and apply $-\bigotimes_{\mathbb{Z}_l}\mathbb{Q}_l$ to the group of morphisms. The above operations still work if one replaces $\mathbb{Q}_l$ by a finite sub-extension inside $\mathbb{Q}_l \subset \bar{\mathbb{Q}}_l$ and $\mathbb{Z}_l$ by the corresponding ring of integers. Now we can define the category of constructible $\bar{\mathbb{Q}}_l$-sheaves by taking the 2-inductive limit of constructible E-sheaves along all finite subextensions E inside $\mathbb{Q}_l \subset \bar{\mathbb{Q}}_l$.
Similarly, the derived category of bounded complexes of constructible sheaves $D^b_c(X,\mathbb{Z}_l)$ is defined to be the category of projective systems of complexes of abelian sheaves $\{K_n\}_{n\in\mathbb{N}^+}$ satisfying

(i) $K_n \in D^b(X,\mathbb{Z}/l^n\mathbb{Z})$ has constructible cohomology sheaves;

(ii) if $n \geq m$ then the transition map $K_n \to K_m$ induces an isomorphism $K_n \bigotimes^L_{\mathbb{Z}/l^n\mathbb{Z}} \mathbb{Z}/l^m\mathbb{Z} \to K_m$ in the category $D^b(X,\mathbb{Z}/l^m\mathbb{Z})$.

Then one uses the same strategy to pass from $D^b_c(X,\mathbb{Z}_l)$ to $D^b_c(X,\mathbb{Q}_l)$ and $D^b_c(X,\bar{\mathbb{Q}}_l)$.
These notions, although a bit "indirect", work very well: $H^i(X_{\text{ét}},\mathbb{Q}_l)$ is a Weil-cohomology theory, the category of l-adic lisse sheaves (which is the full subcategory of the category of constructible l-adic sheaves consisting of objects whose n-components are locally constant for each n) is equivalent to the category of continous l-adic representations of $\pi_1^{\text{ét}}(X,x)$, $D^b_c(X,\mathbb{Q}_l) \subseteq D^b(X,\mathbb{Q}_l)$ is a sub-triangulated category and is preserved by Grothendieck's six operations (under suitable finiteness conditions). However, it is a natural question to ask whether there is a suitable way to define a cohomology which is more "direct" and more intuitional while it still keeps the nice properties.
The goal of this seminar is to understand the new "direct" approach to the étale cohomology theory which is developed by Bhatt and Scholze in a recent paper [BS]. There they have the following definition.

Definition 1. *Let E be an algebraic extension of $\mathbb{Q}_l$. Let X be a scheme whose topological space is Noetherian.*

(1) A sheaf of E-modules on $X_{\text{proét}}$ is lisse if it is locally free of finite rank. (Warning: here E is treated as a topological ring rather than a discrete ring.)

(2) A sheaf of E-modules on $X_{\text{proét}}$ is constructible if there is a finite stratification $\{X_i \hookrightarrow X\}$ into locally closed subsets $X_i \subseteq X$ such that the restriction of the sheaf to each X_i is lisse.

(3) An object $K \in D(X_{\text{proét}},E)$ is constructible if it is bounded, and all cohomology sheaves are constructible. Let $D_{\text{cons}}(X_{\text{proét}},E) \subseteq D(X_{\text{proét}},E)$ be the corresponding full triangulated subcategory.

Then they can prove the following results.

(1) $H^i(X_{\text{ét}}, \mathbb{Q}_l) = H^i(X_{\text{proét}}, \mathbb{Q}_l)$ where $H^i(X_{\text{proét}}, \mathbb{Q}_l)$ is obtained by the "direct" cohomology sheaf.

(2) $D_{\text{cons}}(X_{\text{proét}}, \mathbb{Q}_l) \subseteq D(X_{\text{proét}}, \mathbb{Q}_l)$ is a full triangulated subcategory stable under Grothendieck's six operations, and there is a natural equivalence between $D_{\text{cons}}(X_{\text{proét}}, \mathbb{Q}_l)$ and $D^b_c(X, \mathbb{Q}_l)$.

(3) There is an equivalence between the category of finite dimensional continous representations of $\pi_1^{\text{proét}}(X, x)$ and that of lisse $\mathbb{Q}_l$-sheaves (in the sense of Definition 1.) on a locally topologically Noetherian connected scheme X.

1 OVERVIEW (24/04/2014)

Aim of the talk: EXPLAIN THE *Introduction* PART OF [BS, BS].

2 THE PRO-ÉTALE TOPOLOGY (24/04/2014)

Aim of the talk: INTRODUCE THE PRO-ÉTALE SITE.

(1) Recall the definition of a Grothendieck topology and pretopology[Gi, Chapter 0, Définition 1.2., 1.5., 1.6.] or [Vi, Definition 2.24, pp. 27], and give examples[Gi, Exemple 2.5.3.]. Be short here and avoid the set-theoretic issues.

(2) Introduce the notion of weakly étale and ind-étale morphisms [BS, Definition 2.2.1. (5), 2.3.1.] and their basic properties [BS, Proposition 2.3.2.]. Mention [BS, Theorem 2.3.4.] but don't prove it.

(3) Introduce the pro-étale site [BS, Definition 4.1.1] and then explain [BS, Lemma 4.16.-4.18.].

(4) Explain [BS, Example 4.1.4., 4.1.9., 4.1.10.], if time permits then also [BS, Example 4.1.13.].

3 THE PRO-ÉTALE TOPOS (15/05/2014)

Aim of the talk: INTRODUCE THE PRO-ÉTALE TOPOS.

(1) Briefly recall the definition of sheaves [Gi, Chapter 0, Définition 2.1., Lemme 2.3.] and the notion of topos, [Gi, Chapter 0, Théorème 2.5., 2.6.].

(2) Define the subcategory $X^{\text{aff}}_{\text{proét}} \subseteq X_{\text{proét}}$ [BS, Definition 4.2.1.] and explain [BS, Lemma 4.2.2., 4.2.4., 4.2.6., 4.2.7.]. Mention [BS, Lemma 4.2.8.] but don't prove it.

(3) Explain in detail [BS, Lemma 4.2.12.] as it will be used again and again. If time permits then also explain [BS, Lemma 4.2.13.].

(4) Explain [BS, Example 4.2.10., 4.2.11.].

4 Detour to w-local rings (15/05/2014)

Aim of the talk: PROVE [BS, THEOREM 1.7.].

(1) Briefly "recall" the notion and basic properties of spectural spaces from [Hoc] (at least the *introduction* part). Introduce the notion of w-local spectral spaces in [BS, Definition 2.1.1.].

(2) Explain [BS, Example 2.1.2] and [BS, Lemma 2.1.3., 2.1.4.].

(3) Define w-local ring, w-strictly local, etc. [BS, Definition 2.2.1.].

(4) Then take all your efforts to sketch the proof of [BS, Lemma 2.2.4.]. Along the way you will deal with [BS, Lemma 2.1.10., Remark 2.1.11.].

5 The w-contractiblity of the pro-étale topos (22/05/2014)

Aim of the talk: PROVE THAT THE PRO-ÉTALE TOPOS IS LOCALLY W-CONTRACTIBLE.

(1) Introduce the notion of w-contractible rings [BS, Definition 2.4.1.].

(2) Explain whatever you think is interesting (e.g. Lemma 2.4.10.) in [BS, §2.4.], but at least you have to include a sketch of the proof of [BS, Lemma 2.4.9.] in the talk. Along the way you have to explain some lemmas in [BS, §2.2.] and [BS, Lemma 2.4.8.].

(3) Introduce the notion of (locally) w-contractible topos [BS, Definition 3.2.1.] and conclude that the pro-étale topos is locally w-contractible [BS, Proposition 4.2.8.].

(4) Mention [BS, Theorem 1.8.]. If time permits then conclude it from [BS, Lemma 2.4.2., 2.2.9., 2.4.8.].

6 Comparison between étale and pre-étale topology (05/06/2014)

Aim of the talk: STUDY THE PULL-BACK AND PUSH-FORWARD FUNCTORS BETWEEN THE PROÉTALE TOPOS AND THE ÉTALE TOPOS.

Fix a scheme X. Since an étale map is also a weakly étale map, there is a continous functor between sites $X_{\text{ét}} \subseteq X_{\text{proét}}$ defined by the natural inclusion. Thus we get a natural push-foward functor

$$\nu_* : \text{Shv}(X_{\text{proét}}) \to \text{Shv}(X_{\text{ét}}),$$

and its left adjoint

$$\nu^* : \text{Shv}(X_{\text{ét}}) \to \text{Shv}(X_{\text{proét}}).$$

Similarly we have these pull-back and push-forward for the corresponding derived categories. The aim of this talk is to study the two functors, especially at the derived level.

(1) Introduce the notion of continous functors between sites and related notions and properties [Gi, Définition 3.1., 3.3., Proposition 3.2., 3.3.2].

(2) Please at least include [BS, Lemma 5.1.2., 5.1.4., 5.2.3., Proposition 5.2.6., Corollary 5.1.5., 5.1.6., Definition 5.2.1.] in your talk.

(3) If time permits then also discuss the unbounded case [BS, Proposition 5.2.9.].

7 THE REPLETE TOPOS AND THE LEFT-COMPLETION (05/06/2014)

Aim of the talk: INTRODUCE THE REPLETE TOPOS AND DESCRIBE THE LEFT COMPLETION OF $D(X_{\text{ét}})$ IN TERMS OF $D(X_{\text{proét}})$.

(1) Introduce the notion of replete topos [BS, §3.1.], the notion of left-completion [BS, Definition 3.3.1.], explain [BS, Proposition 3.2.3., 3.3.2., 3.3.3.].

(2) Explain [BS, §5.3.].

(3) Please at least include [BS, Lemma 5.4.1., 5.4.2., 5.4.3.] in your talk.

8 FUNCTORIALITY FOR LOCALLY CLOSED IMMERSIONS (12/06/2014)

Aim of the talk: STUDY THE BASIC FUNCTORIALITY OF PUSH-FORWARD AND PULL-BACK FUNCTORS.

(1) First consider the functoriality for closed immersions [BS, §6.1.]. Put emphasis on [BS, Corollary 6.1.4., Example 6.1.5.] where it is showed that the proétale site has the advantage that a closed immersion induces a continous morphism between the sites while the usual étale site does not. [BS, Lemma 6.1.11.] will be used in the 11th talk for the construction of $f_!$, if you don't explain it the speaker of the 11th talk may kill you.

(2) Explain the locally closed case [BS, §6.2.].

9 CONSTRUCTIBLE COMPLEXES IN THE ÉTALE TOPOLOGY AND DERIVED COMPLETIONS (19/06/2014)

Aim of the talk: (1) STUDY THE CONSTRUCTIBLE COMPLEXES IN ÉTALE AND PROÉTALE TOPOLOGY; (2) STUDY THE DERIVED COMPLETIONS IN A REPLETE TOPOS

(1) Introduce constructible complexes in the étale topology [BS, §6.3.]. Please don't omit [BS, Lemma 6.3.13.] as it will be used later. There are some very important notions such as *perfect complex, compact objects* in a derived category which can be found in Stack Project or nLab, please explain them clearly.

(2) First study the derived completions in the general case (for a replete ringed topos) [BS, §3.4.]. Then study the derived completions in the case when when the ring object of the ringed topos is a constant sheaf of a Noetherian ring R with a fixed maximal ideal $\mathfrak{m}$ [BS, §3.5.].

10 Constructible complexes in the pro-étale topology (26/06/2014)

Aim of the talk: (1) study the constructible complexes in étale and proétale topology; (2) prove that the notion of a constructible complex on $X_{\text{proét}}$ coincides with the more intuitive one if X is Noetherian.

(1) Explain [BS, §6.5.] as much as you can, especially [BS, Definition 6.5.1., Lemma 6.5.3., 6.5.4., 6.5.5., 6.5.6., 6.5.9.]. In my opinion, this part illustrates the state of the art: the proétale site translates the projective systems of constructible sheaves on the étale site into constructible sheaves on the proétale site. It looks even better after step (2). So please put more emphasis on this part.

(2) Then take all your efforts to prove [BS, Proposition 6.6.11.].

(3) If you have time then you could also explain [BS, Remark 6.6.13.].

11 Grothendieck's six operations and l-adic pro-étale sheaves (03/07/2014)

Aim of the talk: (1) construct $f_!$ and $f^!$ for $D_{\text{cons}}(X_{\text{proét}}, \hat{R})$ and show that $D_{\text{cons}}(X_{\text{proét}}, \hat{R})$ is stable under Grothendieck's six operations. (2) construct constructible complexes for coefficient rings which are not complete, for example $\bar{\mathbb{Z}}_l$ and $\bar{\mathbb{Q}}_l$.

(1) The pull-back f^* has been included in the 9th talk [BS, Lemma 6.5.9.]. The push-forward f_* is [BS, Lemma 6.7.2.]; the $f_!$ is [BS, Lemma 6.7.7.]; the $f^!$ is [BS, Lemma 6.7.19.]; $\otimes$-products is [BS, Lemma 6.5.5.]; the Internal Hom is [BS, Lemma 6.7.13.].

(2) Explain smooth base change theorem [BS, Lemma 6.7.4.], base change for proper morphisms (f_* version) [BS, Lemma 6.7.5.], base change for separated finitely presented morphisms ($f_!$ version) [BS, Lemma 6.7.5.].

(3) In fact all the material covered in [BS, §6.8.] are extremely interesting. It shows how should one pass from the derived category with coefficients in the ring of integers to the derived category with coefficients in the algebraic extension of the quotient field. Hope the speaker could cover as much as possible. At any rate, please finish [BS, Proposition 6.8.4., 6.8.11., 6.8.14.]

12 Infinite Galois theory (10/07/2014)

Aim of the talk: INTRODUCE THE INFINITE GALOIS THEORY.

(1) Introduce the notion of Noohi groups follow [BS, §7.1.].

(2) Briefly recall the classical construction of Galois categories [SGA1, Exposé V, §5].

(3) Introduce the infinite Galois category in [BS, §7.2.].

13 The proétale fundamental group (17/07/2014)

Aim of the talk: CONSTRUCT THE PROÉTALE FUNDAMENTAL GROUP AND DISCUSS ITS RELATIONS WITH $\pi_1^{\text{ét}}(X,x)$.

(1) Construct the proétale fundamental group follow [BS, §7.3., Lemma 7.4.1.].

(2) Introduce the fundamental group $\pi_1^{\text{SGA3}}(X,x)$ following [SGA3, Exposé X, §6].

(3) Explain the relation among $\pi_1^{\text{proét}}(X,x)$, $\pi_1^{\text{SGA3}}(X,x)$ and $\pi_1^{\text{ét}}(X,x)$. More precisely, you should prove [BS, Theorem 1.10.].

References

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