p-adic Non-abelian Hodge Theory

Here is the outline for our research seminar on p-adic non-Abelian Hodge theory:

1 A mini-course given by a colleague of Michael on the classical non-Abelian Hodge theory (over complex numbers).
2 A brief introduction to [OV07].
3 For the main part we concentrate on [LSZ13].
4 Introduce the open problems or continue to study Faltings' approach.

We aim to get an understanding of the p-adic Simpson correspondence and the work of Lan-Sheng-Zuo [LSZ13]. Suggestions and comments are welcome.

In below we take a brief review on the history of Non-Abelian Hodge Theory and a short introduction to the references.

1. History and background

The theory of non-abelian Hodge theory starts from Hitchin. Hitchin in [Hit87] studied the self-dual Yang-Mills equations and obtained the famous Hitchin equations as follows.

Let $X/\mathbb{C}$ be a smooth projective curve over the complex numbers. E is a C^∞-complex vector bundle of rank 2 on X together with a C^∞-Hermitian metric. ∇ is a C^∞-connection on E, and $\Phi \in End(E) \otimes \mathcal{A}^{(1.0)}$ is an $End(E)$-valued (1.0) form satisfying

$$\begin{cases} F(\nabla) + [\Phi, \Phi^*] &= 0 \\ \nabla''(\Phi) &= 0 \end{cases}$$

where $F(\nabla)$ is the curvature of ∇, Φ^* is the complex conjugation of Φ, and ∇'' is the $(0,1)$-component of the connection ∇.

Theorem [Kob14, Chapter I, Proposition 1.3.7] tells us that on a Riemann surface, a C^∞-complex vector bundle with a C^∞-connection has a unique holomorphic structure determined by the $(0,1)$-part of the connection. So if (E, ∇, Φ) is a solution of the Hitchin equations, ∇'' will induce a holomorphic structure $E^{\nabla''}$ on E. From the second equation, we see that Φ is holomorphic with respect $E^{\nabla''}$. In particular, the pair $(E^{\nabla''}, \Phi)$ forms a Higgs bundle. By calculation, the connection $\widetilde{\nabla} := \nabla + \Phi + \Phi^*$ is flat and induces another holomorphic structure on E (and the Chern classes of E vanish).

Hitchin shows in [Hit87] that the solutions of the Hitchin equations modulo gauge equivalent are in one-to-one correspondence with poly-stable Higgs bundles with vanishing Chern classes via

$$(E, \nabla, \Phi) \mapsto (E^{\nabla''}, \Phi).$$

In [Don87] after [Hit87], Donaldson showed that every irreducible flat connection is gauge equivalent to a connection of the form $\nabla + \Phi + \Phi^*$ where (E, ∇, Φ) is a solution of the Hitchin equations. Therefore we get a bijection between poly-stable Higgs bundles with vanishing Chern classes and semi-simple local systems on X as follows:

$$\begin{matrix}\text{semi-simple} \\ \pi_1(X)\text{-representations}\end{matrix} \leftrightarrow \begin{matrix}\text{bundles with} \\ \text{flat connections}\end{matrix} \leftrightarrow \begin{matrix}\text{solutions of} \\ \text{Hitchin equations}\end{matrix} \leftrightarrow \begin{matrix}\text{polystable Higgs} \\ \text{-bundles}\end{matrix}.$$

For $\Phi = 0$, we recover the famous result by Narasimhan and Seshadri [NS65].

Simpson in [Sim88, Sim92, Sim94, Sim97] generalized this correspondence to the general case where X is a smooth projective variety of arbitrary dimension, E is a C^∞-vector bundle of rank r, and he refined the correspondence with additional information given by variation of Hodge structures. He showed the corresponding moduli spaces $\mathbf{M}_{dR}(X/\mathbb{C})$, $\mathbf{M}_B(X/\mathbb{C})$, $\mathbf{M}_{Dol}(X/\mathbb{C})$, which we mean rank r integrable connections, local systems, semi-stable Higgs bundles with vanishing Chern class respectively, are real analytically isomorphic. Furthermore, he give a comparison of Dolbeault cohomology and de Rham cohomology and show that the correspondence between semi-stable Higgs bundles and flat connections are compatible with extensions. Simpson called this correspondence **non-abelian Hodge theory**, and some other mathematicians called this **Simpson correspondence**.

2. *p*-adic non-abelian Hodge Theory

The first step of p-adic analogue of Simpson correspondence is given by Faltings in [Fal05]. For **curves** or **small affine space** over p-adic field, Faltings constructed an equivalence between the category of Higgs bundles (the Chern classes may not vanish) and "generalised representations" using p-adic Hodge theory and almost étale coverings (the local computation looks similar to the calculation in [Ols05, Chapter 3,4]). This kind of representations include usual representations as a subcategory.

Faltings' construction appears to be satisfactory only for curves and, even in this case, many fundamental questions remain open.

The **equivalence depends on the choice of an exponential function** for the multiplicative group.

If we restrict the inverse functor as a functor from usual representations of geometric π_1 to Higgs bundles. It is difficult to characterise its image. All Higgs bundles in the image are semi-stable of slops zero, but people don't know whether the image contains all those semi-stable Higgs bundles of slop zero. People also don't know what kind of Higgs bundles comes from "genuine representations".

After [Fal05], Abbes , Gros and Tsuji in [AGT16] explicitly explained the theory of Faltings.

Liu Ruochuan and Zhu Xinwen in [LZ17] formalized the Simpson correspondence functor on a rigid analytic space via pulling back to pro-étale site [Sch12, Sch13]. (One can show that this is implied by the results of Faltings et al., but this is difficult).

They used Scholze's machinery to construct a Riemann-Hilbert correspondence for p-adic local systems on rigid analytic varieties using this Simpson correspondence functor. As a consequence, they obtained rigidity theorems for p-adic local systems on a connected rigid analytic variety. Finally, they gave an application of their results to Shimura varieties.

On the other hand, the p-adic Simpson correspondence can also be built from lifting a characteristic p correspondence. Ogus and Vologodsky in [OV07] established the nonabelian Hodge theorem in positive characteristic. They constructed a functor, which they called "inverse Cartier transform" from a category of certain nilpotent Higgs modules to a category of certain nilpotent flat modules on a $W_2(k)$-liftable smooth variety. (Here k is some perfect field of positive characteristic)

Lan Guitang, Sheng Mao and Zuo Kang in [LSZ13] lifted the inverse Cartier transform to the truncated Witt ring $W_n(\overline{\mathbb{F}}_q)$ and introduced the notion of Higgs-de Rham flows. They showed that there is a Higgs correspondence from crystalline representations to periodic Higgs-de Rham flows by passing form $W_n(\overline{\mathbb{F}}_q)$ to $W(\overline{\mathbb{F}}_q)$.

The theory developed in [LSZ13] also turns out to be useful in the study of Higgs bundles with nontrivial Chern classes. This has been demonstrated in the recent work [Lan15] of A. Langer on an algebraic proof of the Bogomolov's inequality for Higgs sheaves on varieties in positive characteristic p that can be lifted modulo p^2.

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