Forschungsseminar: Langlands correspondence for function fields

April 18, 2017

The aim of this seminar is to understand the rudiments of the Langlands Programme in the function field case, with a particular focus on the geometric nature of the constructions. We want to start discussing Tate's thesis, subsequently we introduce the automorphic side of the Langlands' correspondence to arrive to the precise statement of it. After it we will see some aspects of Drinfeld's proof of the direction from Automorphic to Galois in the case of GL_2. Hence we will define Shtukas, their moduli space and its cohomology. In last talks we move to Geometric Langlands Programme. We present the geometric reformulation of Langlands' correspondence, introducing all the characters we need and we will see Deligne's proof of Geometric Class Field Theory. We will also present the general ideas of Vincent Lafforgue's proof of the Langlands' correspondence for general reductive groups, using the Geometric Satake equivalence, as well as the moduli space of G-Shtukas. The last talks will focus on Drinfeld modules and their analytic moduli space. We will see that the cohomology of this moduli space gives, in some particular cases, another way to define a map from Automorphic to Galois for GL_2. The computations in this cases are simpler and they can help to handle the correspondence.

Some good references to handle the subject are:

- An introduction to classical and geometric Langlands programme [F09];
- Tate's Thesis [Kud];
- A gentle and short introduction to the Langlands programme for GL_n [Laf];
- Notes of Chin's seminar on the Langlands programme for GL_n [Car];
- Outline of Drinfeld's proof for GL_2 [Dr];
- Final proof of "Automorphic to Galois"[Dr2].

Notation

The objects that are called "Elliptic modules" in Drinfeld's papers are commonly called nowadays "Drinfeld modules" and we will use this last notation. At the same time we will call "Shtuka" Drinfeld's "F - sheaf".

Outline

1. Overview: Langlands/function fields (Michael, 20/04)
2. Tate's thesis for function fields I (Yun, 27/04)
   - review of local fields and adèles ([Kud, p. 1-4])
   - connection between idèles and line bundles [RV13, 7.2], [F09, p. 36]
   - recall definition and basic properties of the Haar measure [RV13, 1.2]
   - local zeta integrals and meromorphic continuation [Kud, p. 10]
   - interpretation of local zeta integrals eigendistributions [Kud, p. 7ff]

3. Tate's thesis for function fields II (Yun, 04/05)
   - recall definition and basic properties of Pontryagin duality and Fourier transform [RV13, 3.3 & 3.4]
   - Poisson summation formula (sketch proof if possible) [RV13, 7.2]
   - Riemann-Roch and Serre duality for algebraic curves via harmonic analysis [RV13, 7.2]
   - global zeta integrals
   - deduce functional equation [RV13, 7.3] or [Kud, section 4]
4. Automorphic side I (Simon, 11/05)
   - Hecke algebras [Gr]
   - Satake isomorphism (focus on example: SL_2 and PGL_2) [Gr]
   - L-functions [Gr]
5. Automorphic side II (Marco, 18/05)
   - Automorphic representations
   - Statement of Langlands correspondence for function fields
6. The stack of bundles: Bun_G (Lei, 01/06)
   - algebraicity (overview) [Av09, G09]
   - basic properties: smooth, not quasi-compact, locally of finite presentation [Av09]
7. Shtukas for GL_2, I (Fabio, 08/06)
   - Define Hecke stacks [Ngo, Section 1].
   - Define Shtukas and prove Theorem 2.2 in [Ngo].
   - Explain Proposition 2.5 in [Ngo].
8. Shtukas for GL_2, II (Tanya, 15/06)
   - Define Harder-Narasimhan polygon and state Proposition 3.2 [Ngo, Section 3].
   - Define degenerated and pre-iterated Shtukas and state [Ngo, Proposition 5.1].
   - State Theorem 5.4 and sketch its proof [Ngo, Section 6].
9. The cohomology of the stack of Shtukas (22/06)
   - Explain the construction of the Automorphic and Galois representation on the cohomology ([Dr2, 5]).
   - Explain the Basic Theorem at page 29 of [Dr2] and how to deduce its corollary.
10. From the classical Langlands programme to its geometric counterpart (29/06)
    - connection between adèles and bundles [F09, p. 36]
    - natural measures on $G(\mathbb{A})$ [ADK08, p. 13-15]
    - connection to counting bundles, and formulae [ADK08, p. 13-15]
    - function-sheaf dictionary [F09, p. 50]
    - character sheaf / Hecke eigenproperty [G, 277 - 280]
11. Geometric Class Field Theory (Shane, 06/07)

- character sheaf (Hecke eigensheaf) construction using symmetric powers [F09, p. 48 - 53], [Toth]
- the functional equation revisited (optional, ask Michael for notes)

12. Shtukas for reductive groups and general ideas of V. Lafforgue proof (Michael, 06/07 afternoon/other day?)

    - quick summary of first half of [Str]
    - follow second half of [Str, p.18ff]

13. Drinfeld Modules (Marcin, 13/07)

    - Definition of Drinfeld Module, torsion points. Remark on "Drinfeld level structure" [DH, 1.1,1.2,1.3]
    - Describe algebraic moduli [DH, 1.6]
    - Analytic theory (i.e. analogy with elliptic curves.) [DH, 2.1,2.2]

14. Analytic moduli space of Drinfeld's Modules and their cohomology (Raju, 20/07)

    - follow [DH]

References

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[Dr2] Drinfeld, V. *Cohomology of compactified manifolds of modules of* F *- sheaves of rank 2* `http://link.springer.com/article/10.1007/BF01099348` (document)

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[G] Gaitsgory, D. *Informal introduction to geometric Langlands.* An introduction to the Langlands program. Birkhuser Boston, 2004. 269-281. (document), 2, 10, 11

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[Goss] Goss, D. *What is ... a Shtuka*`http://www.ams.org/notices/200301/what-is.pdf` 10

[Gr] Gross, B., *On the Satake isomorphism.* `http://www.math.harvard.edu/~gross/preprints/sat.pdf` 6

[Kud] Kudla, S., *Tate's thesis.* `http://u.cs.biu.ac.il/~reznikov/courses/kudla-1.pdf`

[Laf] Lafforgue, L, *Chtoucas de Drinfeld, formule des traces d'Arthur-Selberg et correspondance de Langlands* `https://arxiv.org/abs/math/0212399` 4

[Mum] Mumford, D. *An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg deVries equation and related non-linear equations* (document), 2, 3

[Ngo] Ngô Dac Tuan *Introduction to the stack of Shtukas.* `https://www.impan.pl/swiat-matematyki/notatki-z-wyklado~/ngo.pdf` (document)

[RV13] Ramakrishnan, Dinakar, and Robert J. Valenza. *Fourier analysis on number fields.* Vol. 186. Springer Science & Business Media, 2013.

[Str] Stroh, B., *La paramétrisation de Langlands globale (d'après Vincent Lafforgue).* `http://www.math.univ-paris13.fr/~stroh/bourbaki.pdf` 7, 8

[Toth] Toth, P. *Geometric abelian class field theory.* 2, 3

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