Geometric Langlands Correspondance

X = sm proj, geo conn Curve $/\mathbb{F}_q$. $F = K(X)$

§ Classical langlands

$\mathcal{G}_n = \left\{ \begin{array}{l} n\text{-dim'l Galois reps} \\ G(\bar{F}/F) \to GL_n(\bar{\mathbb{Q}}_\ell) \\ \bullet \text{ irr} \\ \bullet \text{ almost everywhere unramified} \end{array} \right\} \longleftrightarrow \mathcal{A}_n := \left\{ \begin{array}{l} \text{cusp. aut reps} \\ \bullet \text{ irr} \\ \bullet \text{ almost everywhere unramified (spherical)} \end{array} \right\}$

automatic (for irr)

Recall a bit of aut reps :

$\pi : GL_n(\mathbb{A}) \longrightarrow \mathrm{Aut}_{\mathbb{C}}\big(\underbrace{\mathcal{C}_{cusp}}_{\text{smooth, cuspidal functions}}(GL_n(F)\backslash GL_n(\mathbb{A}))\big)$

smooth $\uparrow$ loc. const ; cuspidal $\uparrow$ $\int_{N_{n_1,n_2}(F)\backslash N_{n_1,n_2}(\mathbb{A})} f(ug)\,du = 0$

multiplicity 1 thm $\Rightarrow$ it decompose into a direct sum of irr reps, each of them appear once, and each of them being 1-dim'l.

These irr, 1-dim'l reps are called irr cusp. aut reps.

Unramified (spherical) rep : unlike cuspidal, unramified is a local concept:

$$\pi = \bigotimes_{x \in X}{}' \pi_x$$

with each $\pi_x : GL_n(F_x) \longrightarrow$ ~~$\mathrm{Aut}_{\mathbb{C}}(\mathcal{C}_{cusp}(GL_n(F)\backslash GL_n(\mathbb{A})))$~~

unramified at x means $\exists f_x \in \pi_x (\subset \pi)$ s.t. $f_x \in \pi_x^{GL_n(\mathcal{O}_x)}$

Prop tells us that f_x is unique up to scalar.

in other words, π_x is 1-dim'l.

if π irr & π unr at x, then

To build the correspondance :

for unramified x,

$\mathcal{H}_x := \mathcal{C}_c(GL_n(F_x))$ that is bi-invariant under $GL_n(\mathcal{O}_x)$-action.

$\hookrightarrow$ Local Hecke algebra.

Let $\overbrace{H_{1,x}, \cdots, H_{n,x}}^{\boxed{\text{Hecke operators}}}$ be the characteristic function of $\mathbb{1}_{S_1}, \cdots, \mathbb{1}_{S_n}$. where

$$S_i = GL_n(\mathcal{O}_x) \begin{array}{c} \left[\begin{array}{c|c} \begin{matrix} \varpi_x & & \\ & \ddots & \\ & & \varpi_x \end{matrix} & \\ \hline & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right] \\ \underbrace{\hspace{3em}}_{i} \end{array} \Big]\, i \quad GL_n(\mathcal{O}_x)$$

$\mathcal{H}_x$ has an action on π_x

$$H_{i,x} * f_x := \int_{GL_n(F_x)} H_{i,x}(g)(g \cdot f_x)\, dg$$

$$h \mapsto \int_{S_i} f_x(hg)\, dg$$

$$= \int_{hS_i} f_x(g)\, dg \rightsquigarrow \text{pass to } GL_n(k_x) \text{ doesn't affect measure and integral.}$$

Cheat a bit

Since π_x 1-dim $\Rightarrow H_{i,x} * f_x = c_{i,x} \cdot f_x$

反解 : $\begin{cases} S_1(z_1, \cdots, z_n) = c_{1,x} \in \mathbb{C}^* \\ \vdots \\ S_n(z_1, \cdots, z_n) = c_{n,x} \in \mathbb{C}^\times \quad \in \mathbb{C} \end{cases}$

有 $n!$ 组解 $\xrightarrow[\text{i.e: } S_n \text{ 作用下的轨道数为 } 1]{\text{unordered.}}$ $(z_1(\pi_x), \cdots, z_n(\pi_x))$

Hecke eigen value

(实际是你妹的eigenvalue啊!)

$\xleftrightarrow[\text{equal up to scalar}]{\text{Langlands correspondance}}$ $(z_1(F_x), \cdots, z_n(F_x))$

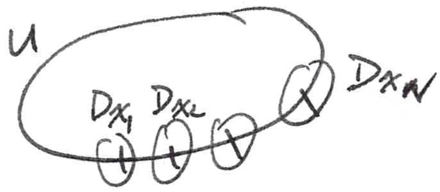
§ Geometric realization

$\mathcal{O} := \prod_{x \in X} \mathcal{O}_x$

First observation: "Lemma 2"

$$GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O})$$
$$\cong$$
set of isom classes of rk n vb. on X.

Proof.



Any vb is trivialized at some dense open U

$$\rightsquigarrow \underbrace{GL(X \setminus \{x_1, \dots, x_N\})}_{\text{different trivializations on } U} \Big\backslash \underbrace{\prod^N GL_n(D_{x_i}^\times)}_{\text{transition map}} \Big/ \underbrace{\prod_{i=1}^N GL_n(D_{x_i})}_{\text{different trivializations around } x.}$$

pass to formal disc $\rightsquigarrow$ $$GL_n(F) \Big\backslash \prod_{x \in X} GL_n(F_x) \Big/ \prod_{x \in X} GL_n(\mathcal{O}_x)$$ $\square$

This identification tells us that, in the geometric picture, ~~we are only~~ on the automorphic side, we should restrict ourselves to functions on the double quotient ($\Leftrightarrow$ functions on Bun_n), which equivalently means automorphic functions that are EVERYWHERE unramified:

$$f = \bigotimes_{x \in X} f_x \in \bigotimes_{x \in X}{}' \pi_x = \pi, \quad f_x \in \pi_x^{GL_n(\mathcal{O}_x)}$$
$$\Rightarrow f \in \pi^{GL_n(\mathcal{O})} \text{ (by def)}$$

4.

So up to now, what we are expecting should be

$$\mathcal{G}_n \longleftrightarrow \mathcal{A}_n$$

$$\mathcal{G}_n = \boxed{\left\{ G(\bar{F}/F) \to GL_n(\bar{\mathbb{Q}}_\ell) \right\} + \text{irr \& unramified}}$$

$\left(\text{i.e. } Gal(F^{ur}/F) \cong \pi_1(X) \to GL_n(\bar{\mathbb{Q}}_\ell) + \text{irr} \right)$.

$$\mathcal{A}_n = \left\{ GL_n(\mathbb{A}) \to Aut\left(\mathcal{C}_{cusp}\left(GL_n(F) \backslash GL_n(\mathbb{A}) \Big/ \prod_{x \in X} GL_n(\mathcal{O}_x) \right) \right) + \text{irr} \right.$$

unramified

One more thing to be translated to the geo. language:

Translation of Cuspidality:

for this we need a correspondance. Let

$$Fl^n_{n_1,n_2} = \left\{ \begin{array}{l} \text{the set of s.e.s of vb's on } X \\ 0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0 \\ \text{with } rk(\mathcal{M}) = n, \ rk(\mathcal{M}_i) = n_i \end{array} \right\}$$

equivalence:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M}_1 \arrow[r] \arrow[d, equal] & \mathcal{M} \arrow[r] \arrow[d, "\cong"] & \mathcal{M}_2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{M}_1 \arrow[r] & \mathcal{M}' \arrow[r] & \mathcal{M}_2 \arrow[r] & {} \end{tikzcd}$$

Correspondance

$$\begin{tikzcd} & Fl^n_{n_1 n_2} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Bun_n & & Bun_{n_1} \times Bun_{n_2} \\ \mathcal{M} & & (\mathcal{M}_1, \mathcal{M}_2) \end{tikzcd}$$

~~Prop.~~ Define

$\boxed{\text{Constant term functor}}$

$$r^n_{n_1,n_2} : \mathcal{C}(Bun_n) \to \mathcal{C}(Bun_{n_1} \times Bun_{n_2})$$
$$f \longmapsto q_!(p^* f).$$

Prop $f \in \mathcal{C}(\mathrm{Bun}_n) = \mathcal{C}\left(GL_n(F)\backslash GL_n(\mathbb{A}) / \prod GL_n(\mathcal{O}_x)\right)$ is cuspidal (i.e. $\int_{U_{n_1,n_2}(F)\backslash U_{n_1,n_2}(\mathbb{A})} f(gh)\,dg = 0 \ \forall h \in G(\mathbb{A})$)

$\Longleftrightarrow$ $\quad \mathcal{E}_!(p^*f) = 0$ for any partition $n = n_1 + n_2$.

Proof For fixed n_1, n_2, $h \in \mathcal{C}(\mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2}) = \mathcal{C}\left(\backslash \mathrm{Levi}_{n_1,n_2}(\mathbb{A}) / \right)$

$$\int_{U(F)\backslash U(\mathbb{A})} f(gh)\,dg$$

$$\| $$

$$\int_{U(F)\backslash U(\mathbb{A})\cdot h} f(g)\,dg \overset{\substack{\text{def of the measure} \\ + f \text{ smooth (i.e. loc. const)}}}{\underset{!}{=}} \sum_{p(g) \in (\mathcal{E}^{-1}(h))} f(p(g))$$

$$\| \text{ def}$$

$$\mathcal{E}_!(p^*f)(h) \qquad \square$$

i.e. looking back on the geo correspondance on P4,

$$\mathcal{A}_n = \left\{ \begin{array}{l} GL_n(\mathbb{A}) \to \mathrm{Aut}_{\mathbb{C}}(\mathcal{C}(\mathrm{Bun}_n)) \\ + \text{ irr } \& \ \underline{\text{cuspidal}} \\ \qquad \text{defined as } r^n_{n_1,n_2}(f) = 0 \end{array} \right\}.$$

To translate the "equal eigenvalue" property, we need a translation for the local Hecke operators/eigenvalues.

$\boxed{5.5}$ translation of

<u>local Hecke operator</u>:

$$\mathcal{H}_x^i := \{ \mathcal{M}' \overset{\beta}{\hookrightarrow} \mathcal{M} \mid \mathcal{M}, \mathcal{M}' \in \mathrm{Bun}_{n,X} \text{ and } \mathcal{M}/\mathcal{M}' = k_x^{\oplus i} \}.$$

$$\begin{tikzcd} & \mathcal{H}_x^i \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow}"] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \end{tikzcd}$$

$\mathcal{M}'$ $\qquad\qquad$ $\mathcal{M}$

This gives rise to

$\boxed{\text{local}}\boxed{\text{Hecke operator}}$ $\qquad H_{i,x} : \mathcal{C}(\mathrm{Bun}_n) \to \mathcal{C}(\mathrm{Bun}_n)$

$$f \longmapsto h^{\rightarrow}_! (h^{\leftarrow *}(f)).$$

This is exactly the same local Hecke operator:

$\forall\, h \in \mathrm{Bun}_n \cong GL_n(F) \backslash GL_n(\mathbb{A}) / \prod_{x \in X} GL_n(\mathcal{O}_x)$, $\quad f \in \pi_X$,

$$h^{\rightarrow}_! (h^{\leftarrow *}(f))(h) = \sum_{h^{\leftarrow}(h') \in (h^{\rightarrow})^{-1}(h)} f(h') = \sum_{h' \in \underbrace{Gr(\mathcal{M}_x \otimes k_x,\, n-i)}_{\overset{\text{dual}}{=} Gr(\mathcal{M}_x \otimes k_x,\, i)}} f(h')$$

$\downarrow P_2$ (cheat a bit)

$$= \int_{h' \in h S_i} f(h')\, dh' \qquad \square$$

§ From functions to sheaves

Note that Bun_n is the $\mathbb{F}_q$-pt of the moduli stack $\mathcal{B}un_n$. thus we can translate one step further:

$\boxed{\text{Grothendieck sheaf – function correspondance}}$ [Laumon]

	$D^b_c(X,\overline{\mathbb{Q}}_\ell)$ ↙ we can define for complexes, but actually only the coh sheaf matters.	functions $X(\mathbb{F}_q)\to\overline{\mathbb{Q}}_\ell$
Def:	K	$f^K: x\mapsto \sum(-1)^\nu \mathrm{Tr}(F, \mathcal{H}^\nu(K)_x)$
Prop	$K\to M\to L\to K[1]$ $\rightsquigarrow$	$f^M=f^K+f^L$
	$K\otimes L$	$f^{K\otimes L}=f^K\cdot f^L$
	$R\varphi_! K$	$y\mapsto \sum_{x\in\varphi^{-1}(y)} f^K(x)$ i.e. $\varphi_! f^K$
	$\varphi^* K$	$x\mapsto f^K(\varphi(x))$ i.e. $\varphi^* f^K$
	$K[d]$	$(-1)^d f^K$ $\boxed{\because w(K[d])=w(K)+d}$
	$K(n)$	$q^{-n} f^K$ $\boxed{\because w(K(n))=w(K)-2n}$

- Examples: Artin–Schreier: in the perverse $f^{\mathcal{L}(\psi)}=\psi\circ\mathrm{Tr}$; $f^{\varphi_{0*}\overline{\mathbb{Q}}_\ell}=\sum_{t\in(\mathbb{F}_q,+)}\psi_t$
- Similar proposition holds for functions on $\mathbb{F}_{q^m}$-pts: $f^K_m: X(\mathbb{F}_{q^m})\to\overline{\mathbb{Q}}_\ell$, $x\mapsto\sum(-1)^\nu\mathrm{Tr}$
- Proof of ↑: Grothendieck-Lefschetz trace formula

$$f^{R\varphi_!(K)}(y)\overset{def}{=}\mathrm{Tr}(F,(R\varphi_!K)_y)$$

$$= \mathrm{Tr}(F, R\Gamma_c(X_y, K|_{X_y})) \overset{\text{Trace formula}}{=\!=} \sum_{x\in \underbrace{X_y(\mathbb{F}_q)}_{=\varphi^{-1}(y)}} \mathrm{Tr}(F,K_x) \overset{def}{=}\varphi_! f^K(y)$$

∎

7.

Grothendieck gp of this triangulated cat

- $$f^\bullet : K(D_c^b(X,\bar{\mathbb{Q}}_\ell)) \longrightarrow \prod_{m\geq 1} \mathcal{C}(X(\mathbb{F}_{q^m}), \bar{\mathbb{Q}}_\ell)$$
$$K \longmapsto (f_m^K)_{m\geq 1}$$
is injective.

<u>Proof</u>: This is a consequence of <u>Čebotarev thm</u>:

K', K'' semisimple perverse. Then
$$f^{K'} = f^{K''} \Longrightarrow K' \cong K''$$

+ the fact that in $K(D_c^b(X,\bar{\mathbb{Q}}_\ell))$
$$[K] = \sum_v (-1)^v [\mathcal{H}^v(K)] = \sum_v (-1)^v [{}^p\mathcal{H}(K)]$$
□

<u>Note</u>: <u>For lisse sheaves</u>:

$$f^{\mathcal{F}} = f^{\mathcal{G}} \Longrightarrow Tr(F^r; \mathcal{F}_x) = Tr(F^r, \mathcal{G}_x), \forall r$$

$\Longrightarrow$ eigenvalues are the same

(∵ are the only solution of symmetric polys up to [redacted] change of orders)

philosophically $\Longrightarrow$ we know Frob action at every geo pt

$\xRightarrow{\text{Čebotarev density thm}}$ know $\pi_1(X)$-action somehow.

□

$\bigcup_{x\in X(\mathbb{F})} Im(Gal(\bar{k(x)}/k(x)) \to \pi_1(X))$ is dense in $\pi_1(X)$

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From the prop in P7 we know that the info in functions on the geo pts of X, can be faithfully reflected via "sheaves" in $D_c^b(X, \overline{\mathbb{Q}}_\ell)$.

But there is a even smaller category to think about:

$\boxed{\text{Perverse sheaves}}$ [KW]

(• t-str
• ab cat
• motivation (P39 Frankel)
  - Experience tells us interesting functions ~~always~~ usually comes from perverse sheaves
  - more handy because it's a ab cat
  - behave well wrt ~~6 functors~~ & Verdier duality

△ sm case: $\{\text{lisse sheaves}\} \hookrightarrow \{\text{perverse sheaves } \mathcal{F}[d]\}$

☆ middle extension prop. (Cor 5.4 in P149 [KW]).

• Example!

Another ~~eb.~~ Appendix:

$\boxed{\text{Hecke eigensheaves}}$

Exaple! Deligne vanishing thm

§ From functions to sheaves (continue)

Expected correspondance

$$\left\{\begin{array}{l} \overset{\mathcal{G}_n}{\pi_1(X) \to GL_n(\overline{\mathbb{Q}}_\ell)} \\ + \text{irr } \left(\text{i.e. irr lisse sheaves}\right) \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \overset{\mathcal{A}_n}{GL_n(\mathbb{A}) \to \mathrm{Aut}_{\mathbb{C}}(\mathcal{C}(\mathrm{Bun}_n))} \\ + \text{irr \& cuspidal} \end{array}\right\}$$

$$\left(z_1(F_x), \dots, z_n(F_x)\right) \overset{\text{up to scalar}}{=} \left(z_1(\pi_x), \dots, z_n(\pi_x)\right)$$

with $S_1(z_1(\pi_x), \dots, z_n(\pi_x))$
$\vdots$
$S_n(z_1(\pi_x), \dots, z_n(\pi_x))$
are eigen values of the Hecke operator at x

$$H_{i,x} : f \mapsto \overrightarrow{h}_!\left(\overleftarrow{h}^*(f)\right)$$

Still, two translations to be done are Hecke operators and cuspidality.

Hecke : $\mathcal{H}_x^i$: stack s.t. $\mathrm{Hom}(S, \mathcal{H}_x^i) = \left\{ \beta : \mathcal{M}' \hookrightarrow \mathcal{M} \;\middle|\; \begin{array}{l} \mathcal{M}, \mathcal{M}' \in \mathrm{Bun}_{n, X\times S} \\ \mathcal{M}/\mathcal{M}' \text{ support on } x \times S \subset X \times S \text{ and} \\ \text{is a rk } i \text{ vb.} \end{array} \right\}$

Local Hecke correspondance

$$\begin{tikzcd} & \mathcal{H}_x^i \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \underset{\mathcal{M}}{\mathrm{Bun}_n} & & \underset{\mathcal{M}'}{\mathrm{Bun}_n} \end{tikzcd}$$

Local Hecke operator : $H_{i,x} : D(\mathrm{Bun}_n) \to D(\mathrm{Bun}_n)$

$$\mathcal{K} \longmapsto \overrightarrow{h}_!\left(\overleftarrow{h}^*(\mathcal{K})\right)[i(n-i)]$$

Globalize $\rightsquigarrow$ (下一页)

$\mathcal{H}^i$ stack s.t. $\mathrm{Hom}(S, \mathcal{H}^i)$

$$= \left\{ (\underset{X(S)}{\phi \in}, \mathcal{M}' \overset{\beta}{\hookrightarrow} \mathcal{M}) \;\middle|\; \begin{array}{l} \mathcal{M}, \mathcal{M}' \in \mathrm{Bun}_{n, X \times S} \\ \mathrm{Supp}(\mathcal{M}/\mathcal{M}') = \Gamma_\phi \subset X \times S \\ \text{and is a rk } i \text{ vb on } \Gamma_\phi. \end{array} \right\}$$

<u>Global Hecke correspondance</u>

$$\begin{tikzcd} & \mathcal{H}^i \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "s \times h^{\rightarrow}"] & \\ \mathrm{Bun}_n & & X \times \mathrm{Bun}_n \end{tikzcd}$$

$\mathcal{M}'$ $\qquad\qquad\qquad$ $\phi, \mathcal{M}$

<u>Global Hecke operator</u> $\quad H_i : D(\mathrm{Bun}_n) \to D(X \times \mathrm{Bun}_n)$

$$K \mapsto (s \times h^{\rightarrow})_! (h^{\leftarrow *}(K)) [i(n-i)]$$

<u>Note $s^{-1}(x) = \mathcal{H}_x^i$ so all the local info</u> is contained in this global operator.

We are expecting sheaves that satisfy the eigen property:

<u>Hecke eigensheaf</u>: An object $\mathrm{Aut}_E \in D_c^b(\mathrm{Bun}_n, \overline{\mathbb{Q}_\ell})$ is called a Hecke eigensheaf wrt local system E, if

$$\begin{tikzcd} \mathrm{Aut}_E \in D(\mathrm{Bun}_n) \arrow[r, "H_i"] & D^b(X \times \mathrm{Bun}_n) \arrow[dl] \arrow[d] \\ E \in D(X) & D(\mathrm{Bun}_n) \ni \mathrm{Aut}_E \end{tikzcd}$$

$$H_i(\mathrm{Aut}_E) \cong \Lambda^i E \boxtimes \mathrm{Aut}_E, \quad \forall\ 1 \le i \le n$$

"<u>locally</u>": $\quad H_{i,x}(\mathrm{Aut}_E) = \Lambda^i E_x \boxtimes \mathrm{Aut}_{E,x}$

sheaf function dict.

eigenvalues $c_1, \dots, c_n$ $\qquad\qquad$ $z_1, \dots, z_n$

$$c_1 \overset{= \mathrm{Tr}(F, \Lambda^1 E)}{\underset{!}{=}} S_1(z_1, \dots, z_n),$$

$$c_n \overset{= \mathrm{Tr}(F, \Lambda^n E)}{=} S_n(z_1, \dots, z_n).$$

i.e. get the eigen equality in the classical case.

<u>Rmk</u> Unlike the classical case, now we have an explicit corresp: $E \mapsto \mathrm{Aut}_E$. But Aut_E is hard to construct.

In history, Laumon gave his first try, and FGV finished

Cuspidality

with no suprise, cuspidality is translated using the same correspondance as in function case

$\boxed{\text{Constant term functor}}$ $r^{n}_{n_1,n_2} : D(\mathrm{Bun}_n) \to D(\mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2})$

$$\mathcal{F} \mapsto q_!(p^*\mathcal{F})$$

$\mathcal{K}$ $\mathcal{K}$

Cuspidal $\iff$ $q_!(p^*\mathcal{F}) = 0$, $\forall$ partition n_1, n_2.

But surprisingly:

Thm Any Hecke eigensheaf Aut_E is cuspidal.

So we need no restriction in the formulation of the correspondance

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To sum up:

$$\begin{array}{ccc} \mathcal{G}_n & & \mathcal{A}_n \\ \left\{ \begin{array}{l} \pi_1(X) \to GL_n(\overline{\mathbb{Q}}_\ell) \\ + \text{irr} \end{array} \right\} & \longleftrightarrow & \left\{ \text{Hecke eigensheaf } \mathrm{Aut}_E \right\} \\ E & \longmapsto & \mathrm{Aut}_E \end{array}$$

Rmk. Must require E irr,

C-ex: $\underbrace{E = \bigoplus^{n} \text{rk } 1 \text{ loc. systems}}$ $\rightsquigarrow$ $\mathrm{Aut}_E = \bigoplus^{\text{inf}}$ irr perverse sheaves on Bun_n.

labeled by $\mathbb{Z}^n$

↳ "geometric Eisenstein series"

It can happen that there are several non-isomorphic Aut_E corresponding to the same E.

$\rightsquigarrow$ category $\mathcal{A}ut_E$ . . .

2-1 Second Lecture on geo-Langlands —

Proof of the geometric Langlands conjecture [FGV]

X = sm proj, geo conn curve$/\mathbb{F}_q$

Recall the statement:

$$\left\{\begin{array}{l}\boxed{\text{Galois side}}\\ \text{irr } n\text{-dim'l local system}\\ \pi_1(X) \to GL_n(\overline{\mathbb{Q}}_\ell)\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\boxed{\text{Automorphic side}}\\ \text{irr Hecke eigensheaf}\\ \text{i.e. } \mathcal{K} \in D(\mathrm{Bun}_n) \text{ satisfying}\\ H^i_n(\mathcal{K}) \cong \bigwedge^i E \boxtimes \mathcal{K},\ \forall 1 \le i \le n.\\ \text{wrt some rk } n \text{ loc. system } E.\end{array}\right\}$$

$$E \longmapsto \mathrm{Aut}_E.$$

Hecke operator

$$\mathcal{H}^i_n = \{\underset{\substack{\Downarrow \\ X(S)}}{\phi},\ \mathcal{M}' \overset{\beta}{\hookrightarrow} \mathcal{M}\}$$

$$\begin{tikzcd} & \mathcal{H}^i_n \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "s \times h^{\rightarrow}"] & \\ \mathrm{Bun}_n & & X \times \mathrm{Bun}_n \end{tikzcd}$$

$\mathcal{K}$, Bun_n

$H^i_n : \mathcal{K} \longmapsto (s \times h^{\rightarrow})_! h^{\leftarrow *}(\mathcal{K})$

Two remarks:

- irreducibility is essential
- cuspidality is automatic on ~~the~~ RHS

Aim of today's talk: sketch ~~the~~ proof following [FGV]

More precisely, the geometric Langlands Conj says:

1.3 Conjecture $\forall$ irr rk n loc. system E on X,

$\exists\ \mathrm{Aut}_E \in \underline{\mathrm{Perv}}(\mathrm{Bun}_{n,X})$ s.t.

- $\mathrm{Aut}_E|_{\mathrm{Bun}^d_n}$ is $\underline{\text{irr}}$, $\forall d \in \mathbb{Z}$
- Aut_E is a $\underline{\text{Hecke eigen sheaf}}$ wrt E.

① whether two def of Hecke eigensheaf equal

§1. Vanishing conjecture

Before everything, we need to ~~introduce a few notations~~ fix some symbols.

- $\mathcal{C}oh_n$ = the stack classifying coherent sheaves on X of generic rk n.
  i.e. $\mathrm{Hom}(S, \mathcal{C}oh_n)$ = groupoid with
  obj: coh sheaves $\mathcal{M}$ on $X \times S$, flat over S, and $\forall s \in S$ geometric point, $\mathcal{M}_s / X \times_{\mathbb{F}_q} s$ of generic rk n.
- $\mathcal{C}oh_n^d \subseteq \mathcal{C}oh_n$: coherent sheaves of generic rk n and degree d.
- $\mathcal{C}oh_0^{rss} \hookrightarrow \mathcal{C}oh_0$: regular semisimple torsion sheaves.
  i.e. "geometrically": a geo pt of $\mathcal{C}oh_0^{rss} \subseteq \mathcal{C}oh_0$ are exactly those coherent sheaves, that are the direct sum of skyscraper sheaves of length 1 supported at distinct pts

$$i_{x_1 *}k(x_1) \oplus i_{x_2 *}k(x_2) \oplus \cdots \oplus i_{x_n *}k(x_n)$$

- $\mathcal{C}oh_0^{rss,d} = \mathcal{C}oh_0^{rss} \cap \mathcal{C}oh^d$: degree d part.
  Have a natural smooth map
- $X^{(d)} \setminus \Delta \xrightarrow{\text{nat}} \mathcal{C}oh_0^{rss,d}$

$$(x_1, \cdots, x_d) \longmapsto i_{x_1 *}k(x_1) \oplus \cdots \oplus i_{x_d *}k(x_d) \quad \text{(at geo pts)}.$$

  really pairwise distinct! not $X^d \setminus \Delta$

- $E^{(d)} = \left(\mathrm{sym}_!\left(E^{\boxtimes d}\right)\right)^{S_d}$ where $\mathrm{sym} : X^d \to X^{(d)}$
  $\hookrightarrow$ perverse sheaf on $X^{(d)}$.
- $E^{(d)}|_{X^{(d)} \setminus \Delta}$ is a local system $\Big\}$ descend $\boxed{\mathrm{nat}^*\left(\mathring{\mathcal{L}}_E^d\right) = E^{(d)}|_{X^{(d)} \setminus \Delta}}$

gap • $\mathring{\mathcal{L}}^d_E$ loc. system on $\mathcal{C}oh_0^{rss,d}$

suffices to show irr
But I can't sho

2-3

- $\mathcal{L}_E^d := j^d_{0!*}(\overset{\circ}{\mathcal{L}}{}^d_E)$ where $\mathrm{Coh}_0^{\mathrm{rss},d} \overset{j_0^d}{\hookrightarrow} \mathrm{Coh}_0^d$

  $\hookrightarrow$ perverse sheaf on Coh_0^d, $\boxed{\text{irr.}}$
- $\mathcal{L}_E :=$ perverse sheaf on Coh_0, with $\mathcal{L}_E|_{\mathrm{Coh}_0^d} = \mathcal{L}_E^d$, $\forall d \in \mathbb{Z}$.

  $\hookrightarrow$ $\boxed{\text{Laumon sheaf}}$.

  $\hookrightarrow$ (it will play a role similar to Artin–Schreier in Deligne–Fourier transform)
- $\mathrm{Mod}_k^d :=$ stack classifying

$$\left\{ (\mathcal{M}' \overset{\beta}{\hookrightarrow} \mathcal{M}) \;\middle|\; \begin{array}{l} \mathcal{M}, \mathcal{M}' \in \mathrm{Bun}_k \\ \mathcal{M}/\mathcal{M}' \text{ torsion sheaf of length } d \end{array} \right\}$$

$$\begin{tikzcd} & \mathrm{Mod}_k^d \arrow[dl, "\overleftarrow{h}"'] \arrow[d, "\overrightarrow{h}"] \arrow[dr, "\pi"] & \\ \mathrm{Bun}_k & \mathrm{Bun}_k & \mathrm{Coh}_0^d \\ \mathcal{M}' \ \text{小} & \mathcal{M} \ \text{大} & \mathcal{M}/\mathcal{M}' \end{tikzcd}$$

After introducing all those notations, we turn to

$\boxed{\text{averaging functor}}$ $\quad H^d_{k,E} : D(\mathrm{Bun}_k) \longrightarrow D(\mathrm{Bun}_k)$

$$\mathcal{X} \longmapsto \overrightarrow{h}_! \big(\underbrace{\overleftarrow{h}^* \mathcal{X} \otimes \pi^* \mathcal{L}_E^d}_{\text{main part}} \big)$$

(Little Rmk: Almost all functors, equalities, etc, will be only written up to Tate twists & cohomological shift.)

2.3 Vanishing Conjecture

X = sm proj geo conn curve / field $\mathbb{k}$.

E = irr loc. system of rk n. Then

$$H^d_{k,E} = 0, \quad \forall\, 1 \le k \le n-1, \ \forall\, d > kn(2g-2)$$

~~Rmk~~ Rmk This will be ~~the~~ used in all big thms afterwords:

cleanness, descent of $\mathcal{F}_{E,k}$, cuspidality.

Rmk • Formulated in [FGV] and hinted by Laurent, Lafforgue.

• $\mathbb{k}=1$ is a direct consequence of Deligne's vanishing thm (point 2.6 in [FGV])

• If vanishing conjecture holds for any irr rk n local system E, then geo Langlands Conj 1.3 holds. (Proof given in §10 of [FGV]).

In particular, this is the case when $\mathbb{k}=\mathbb{F}_q$

(Optional)

— reformulation of the vanishing Conj (§2.4, §2.5)

— $\mathbb{k}=1$: Deligne vanishing thm $\Rightarrow$ vanishing conjecture.

2-5 附 what is $\mathcal{C}_k$?

intuitively : the not so "very unstable" part of $\mathcal{C}oh_k^d$ for large d.

Prop [Fix $\mathcal{L}^{est}$ of degree $>(2n+2)(g-1)$] $\exists$ integer $c_{g,n}$ s.t. for any $d \geqslant c_{g,n}$ and $\mathcal{M} \in Bun_n^d(\overline{\mathbb{F}_q})$ with $Hom(\mathcal{M}, \mathcal{L}^{est}) \neq 0$, $\Rightarrow \mathcal{M}$ is very unstable

$\boxed{\begin{array}{l}\mathcal{M} \cong \mathcal{M}_1 \oplus \mathcal{M}_2 \ (\mathcal{M}_i \neq 0) \\ Ext^1(\mathcal{M}_1, \mathcal{M}_2) = 0.\end{array}}$

For $d \geqslant c_{g,n}$

$$\mathcal{C}_k^d := \{\mathcal{M} \in \mathcal{C}oh_k^d \mid Hom(\mathcal{M}, \mathcal{L}^{est}) = 0\}.$$

$$\hookrightarrow \mathcal{C}oh_k^d$$

§2. Construction of Aut_E (via Fourier-transform)

$n = \mathrm{rk}\, E$

Fix notations:

Coh'_n = stack classifying $\{(\Omega^{n-1} \overset{s}{\hookrightarrow} \mathcal{M}) \mid \mathcal{M} \in \mathrm{Coh}_n\}$

$\mathrm{Bun}'_n = \varrho_n^{-1}(\mathrm{Bun}_n)$ where $\varrho_n : \mathrm{Coh}'_n \to \mathrm{Coh}_n$ is the forgetful map.

$\mathcal{C}^d_k \overset{\circ}{\hookrightarrow} \mathrm{Coh}^d_k$ open substack that appears only for technical reason: (Optional)

$\mathcal{C}_k = \bigcup_{d \geq c_{g,n}} \mathcal{C}^d_k \hookrightarrow \mathrm{Coh}_k$

$\boxed{\text{For } k=1 \quad \mathcal{C}_0 \cong \mathrm{Coh}_0}$

Step 1 Fundamental diagram and $\mathcal{F}_{E,n} / \mathcal{E}^\circ_n$

(irr. perverse (up to cohom. shift) on each conn. comp of $\mathcal{E}^\circ_n$)

$\mathcal{E}_k$ = stack classifying $\{(\mathcal{M}_k, s_k) \mid \mathcal{M}_k \in \mathcal{C}_k \subseteq \mathrm{Coh}_k,\ s_k \in \mathrm{Hom}(\Omega^{k-1}, \mathcal{M}_k)\}$

$\mathcal{E}_k \xrightarrow[\text{nat proj}]{p_k} \mathcal{C}_k (\subseteq \mathrm{Coh}_k)$ $\boxed{\text{This is a vb on } \mathcal{C}_k}$

$\mathcal{E}^\vee_k$ = stack classifying $\{0 \to \Omega^k \to \mathcal{M}_{k+1} \to \mathcal{M}_k \to 0 \mid \text{the last term } \mathcal{M}_k \in \mathcal{C}_k\}$

$\mathcal{E}^\vee_k \xrightarrow{p^\vee_k} \mathcal{C}_k (\subseteq \mathrm{Coh}_k)$

$\boxed{\text{Fact: dual bundle on } \mathcal{C}_k \text{ (due to the def of } \mathcal{C}_k)}$

$\mathcal{E}^\circ_k = \{(\mathcal{M}_k, s_k) \in \mathcal{E}_k \mid s_k \text{ inj}\}$ $\overset{j_k}{\hookrightarrow} \mathcal{E}_k$

$\mathcal{E}^{\vee\circ}_k = \{(0 \to \Omega^k \to \mathcal{M}_{k+1} \to \mathcal{M}_k \to 0) \in \mathcal{E}^\vee_k \mid \mathcal{M}_{k+1} \in \mathcal{C}_{k+1},\ (\mathcal{M}_k \in \mathcal{C}_k)\} \subseteq \mathcal{E}^\vee_k$ $\hookrightarrow \mathrm{Coh}'_n$

$\boxed{\mathcal{E}^\circ_k \cong \mathcal{E}^{\vee\circ}_{k-1}}$ ← Easy

$\boxed{\text{2-6 前附}}$

~~vert~~

(partial) <u>Fourier transform</u> of $\mathbb{A}^n = \mathbb{A}^r \times \mathbb{A}^s$ wrt $\mathbb{A}^r$

$$\begin{tikzcd}
 & \mathbb{A}^r \times \mathbb{A}^r \times \mathbb{A}^s \arrow[dl, "p_{13}"'] \arrow[dr, "p_{23}"] \arrow[rr, "m_{12}"] & & \mathbb{A}^1 \quad \mathcal{L}_0(\psi) \\
\mathbb{A}^n & & \mathbb{A}^n &
\end{tikzcd}$$

$\boxed{\psi : (\mathbb{F}_q, +) \to \overline{\mathbb{Q}}_\ell^\times \quad \text{Fix once and for all}}$

<u>relative</u>

$$\begin{tikzcd}
 & \mathbb{A}^r \times \mathbb{A}^r \times \mathbb{A}^s \times Y_0 \arrow[dl] \arrow[dr] \arrow[rr, "m_{12}"] & & \mathbb{A}^1 \\
K \quad \mathbb{A}^n \times Y_0 & & \mathbb{A}^n \times Y_0 &
\end{tikzcd}$$

$$T_\psi(K) = Rp_{23!}\left(p_{13}^* K \otimes m_{12}^* \mathcal{L}_0(\psi)\right)[r]$$

<u>Vecter bundle</u>

do locally + glue

<u>Properties</u> preserves ι-mixedness, perversity, irr.

2–6

$\boxed{\text{Foundamental diagram}}$

$$\begin{tikzcd}[column sep=small]
 & \mathcal{F}_{E,n} & & & & \mathcal{F}_{E,n-1} & \\
\mathcal{E}_n \arrow[d, "p_n"'] & \mathcal{E}_n^{o} \arrow[l, hook', "j_n"'] \arrow[r, phantom, "\cong"] & \mathcal{E}_{n-1}^{\vee o} \arrow[r, hook] & \mathcal{E}_{n-1}^{\vee} \arrow[dr, "p_{n-1}^{\vee}"] & \mathcal{E}_{n-1} \arrow[d, "p_{n-1}"] & \mathcal{E}_{n-1}^{o} \arrow[l, hook', "j_{n-1}"'] & \cdots \\
\mathcal{C}_n & & & & \mathcal{C}_{n-1} & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & j_{2!}\mathcal{F}_{E,2} & \mathcal{F}_{E,2} & & & j_{1!}\mathcal{F}_{E,1} & \mathcal{F}_{E,1} & & & \\
\cdots & \mathcal{E}_2^{\vee} \arrow[d, "p_2^{\vee}"'] & \mathcal{E}_2 \arrow[dl, "p_2"] & \mathcal{E}_2^{o} \arrow[l, hook', "j_2"'] \arrow[r, phantom, "\cong"] & \mathcal{E}_1^{\vee o} \arrow[r, hook] & \mathcal{E}_1^{\vee} \arrow[dr, "p_1^{\vee}"] & \mathcal{E}_1 \arrow[d, "p_1"] & \mathcal{E}_1^{o} \arrow[l, hook', "j_1"'] \arrow[r, phantom, "\cong"] & \mathcal{E}_0^{\vee o} \arrow[r, hook] & \mathcal{E}_0^{\vee} \arrow[d, "p_0^{\vee}"] & \\
 & \mathcal{C}_2 & & & & & \mathcal{C}_1 & & & \mathcal{C}_0 \cong \mathcal{C}oh_0 & \\
 & & & & & & & & & \mathcal{L}_E &
\end{tikzcd}$$

We start from $\mathcal{L}_E$ on $\mathcal{C}oh_0$ and define inductively

$$\mathcal{F}_{E,1} = \left(\mathcal{E}_1^{o} \cong \mathcal{E}_0^{\vee o} \hookrightarrow \mathcal{E}_0^{\vee} \xrightarrow{p_0^{\vee}} \mathcal{C}_0 \cong \mathcal{C}oh_0\right)^{*}(\mathcal{L}_E)$$

$$\mathcal{F}_{E,2} = \mathrm{Four}\left(j_{1!}\mathcal{F}_{E,1}\right)\Big|_{\mathcal{E}_2^{o}}$$

$$\vdots$$

$$\mathcal{F}_{E,n} = \mathrm{Four}\left(j_{n-1\,!}\,\mathcal{F}_{E,n-1}\right)\Big|_{\mathcal{E}_n^{o}}$$

§5 3.7

Thm (Cleaness property of $\mathcal{F}_{E,n}$, Conjectured by Laumon)

$$j_{k!}(\mathcal{F}_{E,k}) \xrightarrow{\sim} j_{k*}(\mathcal{F}_{E,k}), \quad \forall\ 1 \leq k \leq n-1.$$

- Rely on Vanishing conj.

Rmk
- $\mathcal{K} \in D(Y)$ is called <u>clean</u> wrt an embedding $Y \overset{j}{\hookrightarrow} \overline{Y}$ if $j_!(\mathcal{K}) \xrightarrow{\sim} j_*(\mathcal{K})$ $\left(\Leftrightarrow\ j_*(\mathcal{K})\big|_{\overline{Y}-Y} = 0\right)$
- When $\mathcal{K}$ perverse, cleaness $\Rightarrow$ $j_!(\mathcal{K}) \cong j_{!*}(\mathcal{K}) \cong j_*(\mathcal{K})$

- Laumon's sheaf $\mathcal{L}_E^d$ on $\mathcal{C}oh_0^d$ is perverse & $\boxed{\text{irr}}$

$\boxed{\text{pulling back preserves perversity}}$ (& $\mathcal{E}_1 \xleftarrow{\text{loc. closed}} \square$, $\mathcal{E}_1^0 \overset{\text{o}}{\subseteq} \mathcal{E}_1$, $\downarrow vb \Rightarrow \downarrow vb$, $C_1 \hookleftarrow Z$; irr lisse is stable under pullback to vb)

$\Longrightarrow \mathcal{F}_{E,1}$ is perverse + irr

$\boxed{\text{Fourier transform preserves perversity \& irr}}$

$\Longrightarrow \mathcal{F}_{E,2}$ is perverse + irr

$\vdots$

$\boxed{\text{Cor. } 3.8}$ $\Longrightarrow \mathcal{F}_{E,n}$ is perverse + irr, when restricted to each conn comp of $\mathcal{E}_n^0$ (up to cohomological shift).

As noted before, we have natural map

$$\mathcal{E}_n^0 \ (\subseteq \mathcal{C}oh_n') \xrightarrow{\rho_n} \mathcal{C}_n := \bigcup_{d \geq c_{g,n}} \mathcal{C}_n^d \quad (\Omega^{k-1} \hookrightarrow \mu_k) \mapsto \mu_k$$

Thm 3.9 (Descent property, also conjectured by Laumon)

use the technical stack $\mathcal{C}_k$

$$\mathcal{F}_{E,n} \cong \rho_n^*(\delta_E^0) \qquad \text{(up to coh shift)}$$

where δ_E^0 on $\mathcal{C}_n$ is perverse, and the restriction to each conn comp of $\mathcal{C}_n$ is nonzero irr perverse.

$$\mathcal{C}_n := \bigcup_{d \geq c_{g,n}} \mathcal{C}_n^d \overset{j}{\hookrightarrow} \bigcup_{d \geq c_{g,n}} \mathcal{C}oh_n^d$$

$$\delta_E^0 \rightsquigarrow \underset{\delta_E}{\cancel{}} := j_{!*}\, \delta_E^0$$

$$\mathrm{Aut}_E := (j_{!*}\delta_E^0)\Big|_{\bigcup_{d \geq c_{g,n}} \mathrm{Bun}_n^d}$$

(perverse)

Excursion
- $\mathcal{C}oh_k^d \subseteq \mathcal{C}oh_k$ open & closed, but possibly can be a fin union of conn comp.
- $\mathcal{C}oh_0^d \subseteq \mathcal{C}oh_0$ are precisely the conn comp.
- $\mathrm{Bun}_k^d \subseteq \mathrm{Bun}_k$ are precisely ~~th~~ conn comp

2-8前

Hecke-Laumon property / eigensheaf.

$\mathcal{HL}_n^d$ = Heck-Laumon stack, classifying $\left\{ 0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{J} \to 0 \right\}$, $\mathcal{M}' \in \mathcal{C}oh_n$, $\mathcal{J} \in \mathcal{C}oh_0^d$

$$\begin{tikzcd} & \mathcal{HL}_n^d \arrow[dl, "\overleftarrow{h_\ell}"'] \arrow[dr, "\overrightarrow{h_\ell}"] & \\ \mathcal{C}oh_n & & \mathcal{C}oh_0^d \times \mathcal{C}oh_n \end{tikzcd}$$

$\mathcal{M}$ $\qquad$ $(\mathcal{M}', \mathcal{J})$

$\boxed{\text{Recall for } n=0,\ \mathcal{HL}_n^d = Fl_d^{d', d''}}$

$\boxed{\text{Hecke-Laumon Functor}}$ Feels like "肢解"

$$HL_n^d : D(\mathcal{C}oh_n) \to D(\mathcal{C}oh_0^d \times \mathcal{C}oh_n)$$
$$\mathcal{K} \mapsto \overrightarrow{h_\ell}_!\left(\overleftarrow{h_\ell}^*(\mathcal{K})\right)$$

(Only up to Tate twists & coh shift.
there's Tate twists & coh shift for $n \geq 1$

$\boxed{\text{Hecke-Laumon eigensheaf (wrt } E)}$ $\mathcal{K} \in D(\mathcal{C}oh_n)$ s.t. $\forall d \in \mathbb{Z}$

$$HL_n^d(\mathcal{K}) \cong \mathcal{L}_E^d \boxtimes \mathcal{K}$$

Laumon proved The perverse sheaf $\mathcal{S}_E^0$ on $\mathcal{C}_n$ is a Hecke-Laumon eigensheaf wrt E.

2-8 **Thm 7.7** The perverse sheaf $\mathrm{Aut}_E|_{\mathcal{E}_n \cap \mathrm{Bun}_n}$ is a Hecke eigensheaf wrt E.

Thm 7.1 The perverse sheaf Aut_E can be uniquely extended to the entire stack $\mathrm{Bun}_n \left(= \bigcup_{d \in \mathbb{Z}} \mathrm{Bun}_n^d\right)$ such that it becomes a Hecke eigensheaf wrt E.

Proof of Thm 7.7

Prop 8.4 Let δ be a perverse sheaf on Coh_n (or $\mathcal{E}_n$), $\mathbb{D}(\delta)$ its Verdier dual. If both δ & $\mathbb{D}(\delta)$ satisfy ~~the~~ Hecke-Laumon property wrt $\underbrace{E \text{ and } E^*}_{\text{local systems}}$ (should be $E^\vee$)

Then $\underbrace{\delta|_{\mathrm{Bun}_n}}_{\left(\text{or } \delta|_{\mathrm{Bun}_n \cap \mathcal{E}_n}\right)}$ is a Hecke eigensheaf.

Prop 8.4 ⇒ 7.7 Apply to $\delta = \delta_E^\circ$ on $\mathcal{E}_n$

$\Rightarrow$ $\delta_E^\circ|_{\mathcal{E}_n \cap \mathrm{Bun}_n} = \mathrm{Aut}_E|_{\mathcal{E}_n \cap \mathrm{Bun}_n}$ is a Hecke eigensheaf □

Why δ_E°, $\mathbb{D}(\delta_E^\circ) = \delta_{E^*}^\circ$ satisfy the Hecke-Laumon property? (wrt E & E^* resp).

8-3 ⇐ 8.2

$\boxed{\text{See 2-8 前 } \underbrace{}_{\text{Page}}}$

7.7 ⇒ 7.1

Sketch $\mathcal{M} \in \mathrm{Bun}_n \setminus \mathcal{E}_n$

Pick $x \in |X(\mathbb{F}_q)|$ and fix it.

Then we can twist $\mathcal{M}$ into $\mathcal{E}_n \cap \mathrm{Bun}_n$: $\exists d''_{(\mathcal{M})} \in \mathbb{Z}$ s.t

$\mathrm{mult}_x : \mathrm{Bun}_n \to \mathrm{Bun}_n$, $\mathcal{M} \mapsto \underbrace{\mathcal{M}(d'' \cdot x)}$

Set Aut_E around $\mathcal{M}$ to be $\mathrm{mult}_{x,d''}(\mathrm{Aut}_E) \otimes (\wedge^n E_x)^{\otimes -d''}$ ~~no long~~ degree $> C_{g,n}$ & not very unstable

Infact, can have d'' defined uniformly on a small open around $\mathcal{M}$ Hecke eigenproperti □