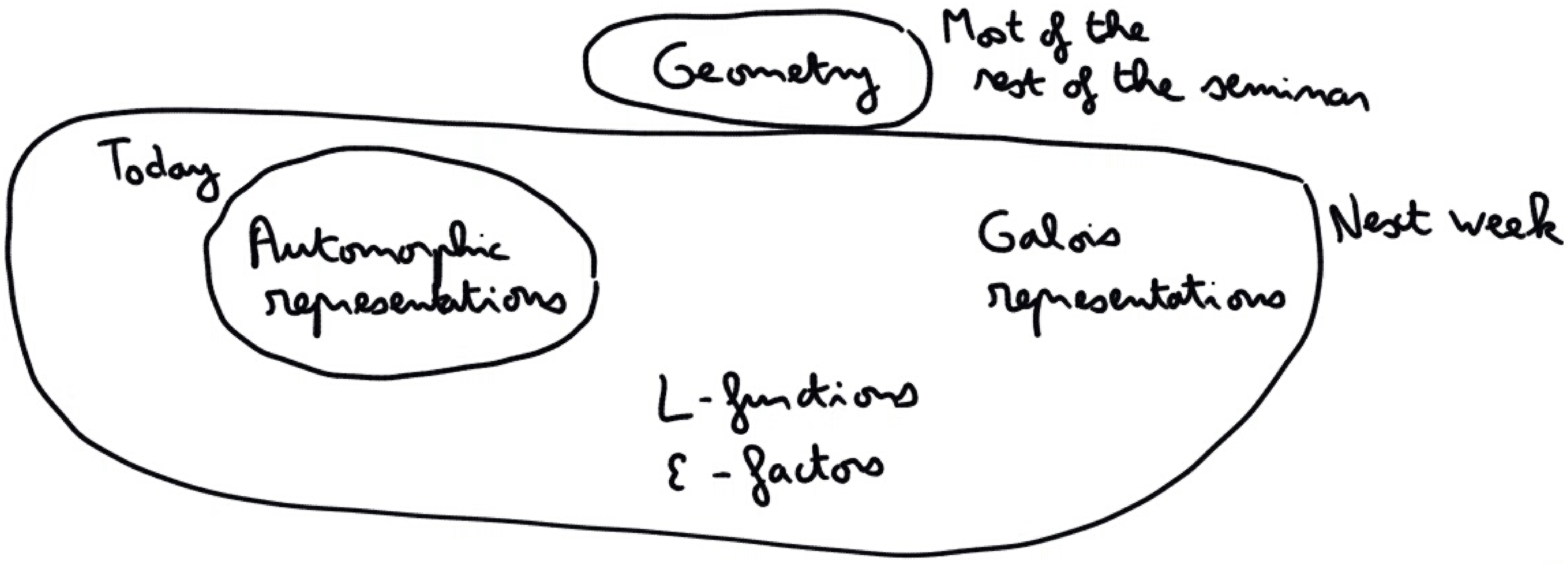
Automorphic side I :



Starting point : k global field, G reductive group over k.

1) Reductive groups

- G reductive algebraic group over k := smooth affine alg. group with no normal unipotent subgroup.

ex : · G torus : $G_{\bar{F}} \simeq (\mathbb{G}_m)^n$; $\{x^2+y^2=1\}$

· $G = \underline{\underline{GL_n}}, SL_n, PGL_n, \underset{\text{herm. form}}{U(h)}, \underset{\text{central simple algebra}}{GL_n(A)}, \dots$ (type A)

· $G = \underset{\text{quad. form}}{O(q)}, SO(q), Spin(q), \dots$ (types B, D)

· $G = Sp_{2n}, GSp_{2n}, \dots$ (type C)

· G "=" G_2, F_4, E_6, E_7, E_8 (exceptional types)

- Write Z = center of G ; $Z \simeq \overset{\text{torus}}{Z^0} \times \overset{\text{finite}}{\pi_0(Z)}$ (Z finite $\overset{\text{def}}{\iff}$ G semisimple)

- **def:** | G is split if G admits a maximal torus for inclusion which is split, i.e $\simeq (\mathbb{G}_m)^n$. ($\Rightarrow Z^0$ split torus)

ex : · GL_n, SL_n, PSL_n (using diagonal matrices)

· Sp_{2n}

- **Fact:** | · $\exists F'/F$ finite separable, $G_{F'}$ is split.
  · G split $\Rightarrow \exists ! \mathcal{G}$ reductive group scheme over $\mathbb{Z}$, $G \simeq \mathcal{G}_F$.

- We will focus on G split, and even on $G = GL_n$. The general case is useful:
  - because of Langlands' functoriality conjecture, one can (sometimes) transfer between groups.
  - in the number field case, because GL_n has no Shimura variety for $n \geq 3$.
  - for the case G anisotropic ($\mathbb{G}_m \not\hookrightarrow G$) because then the analytic aspects are simplified by $G(F)\backslash G(\mathbb{A})$ compact.

2) Adelic groups and local groups:

- $\mathbb{A}$ (resp k_v) is a top. k-algebra $\Rightarrow$ can form $G(\mathbb{A})$ (resp. $G(k_v)$) top. group by using any <u>closed</u> embedding $G \hookrightarrow \mathbb{A}^n_k$ (<u>ex</u>: $GL_n \hookrightarrow M_{n+1}$, $g \longmapsto \left(\begin{array}{c|c} g & 0 \\ \hline 0 & \det(g)^{-1} \end{array}\right)$)

<u>Lemma</u>: $G(k) \subseteq G(\mathbb{A})$ is discrete.

- $G(\mathbb{A})$ (resp. $G(k_v)$) is an LC-group $\Rightarrow \exists$ left Haar measures.

<u>Lemma</u>: Let G be a linear algebraic group over k (resp. k_v) with unipotent rad. $R_u(G)$.
Let μ be a left Haar measure on $G(\mathbb{A})$ (resp. $G(k_v)$).
Put $\delta_G(g) := |\det(Ad(g)|\ \mathrm{Lie}\ R_u(G))| > 0$ (<u>module of G</u>)
Then $\delta_G \cdot \mu$ is a right Haar measure.

- In particular for G reductive, $G(\mathbb{A})$ and $G(k_v)$ are <u>unimodular</u> ($\delta = 1$)

- The groups $G(\mathbb{A}_f)$ and $G(k_v)$ for v non-archimedean are examples of the following notion (which thus covers everything in the fct° field case):

<u>def</u>: A topological group G is an <u>LCTD-group</u> if $1 \in G$ has a basis of neighbourhoods consisting of compact open subgroups, and if for K such a subgroup, G/K is countable.

<u>ex for G split/k fct° field</u>: $G(\mathcal{O}) := \mathcal{G}(\mathcal{O}) \subseteq G(\mathbb{A})$, $G(\mathcal{O}_v) := \mathcal{G}(\mathcal{O}_v) \subseteq G(k_v)$ are compact open subgroups.

- Then $G(\mathbb{A}) = \prod_v{}' G(k_v) = \left\{ (g_v) \,\middle|\, \begin{array}{l} \text{for almost all } v, \\ g_v \in G(\mathcal{O}_v) \end{array} \right\}$

- $k = \mathbb{F}_q(X)$. $N = \sum n_v \cdot [v]$ effective divisor on X.

$$K_N := \prod_v K_{n_v} := \left\{ (g_v) \in G(\mathcal{O}) \,\middle|\, g_v \equiv 1 \mod m_v^{n_v} \right\} \subseteq G(\mathcal{O})$$

(principal level N subgroup)

- Every compact open subgroup of $G(\mathbb{A})$ (resp. of $G(k_v)$) is contained in a K_N (resp. a K_{n_v}).

def Let π be a representation of an LCTD group G on a $\mathbb{C}$-vs V (no topology on V, $\dim(V)$ can and will be ∞)

i) π is <u>smooth</u> if $V = \bigcup V^K$ with K ranging through the compact open subgroups of G

ii) π is <u>admissible</u> if it is smooth and if for all K compact open:

$$\dim(V^K) < \infty .$$

<u>rmk</u>: · The other natural class of representations are unitary ones. The advantage of admissibility is that it is purely algebraic.

· For $G(\mathbb{R})$ or $G(\mathbb{C})$, the right notion is that of "admissible $(\mathfrak{g}\ K_\infty)$-modules".

· <u>def</u> Let G be an LCTD group and K compact open subgroup.

Choose an Haar measure μ on G (assumed unimodular). The <u>Hecke algebra for (G,K)</u> is

$$\mathcal{H}(G,K) := \mathcal{C}_c(K\backslash G/K) = \left\{ \begin{array}{l} f : G \longrightarrow \mathbb{C} \\ \text{loc. constant} \end{array} \middle| \begin{array}{l} \cdot\ f \text{ has compact support.} \\ \cdot\ f \text{ is } K\text{-biinvariant}: \\ \forall g \in G\ \ \forall k, k' \in K,\ f(kgk') = f(g) \end{array} \right\}$$

equipped with the convolution product

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)\, \mu(x)$$

$$\left[\begin{array}{l} (f_1 * f_2)(gk) = (f_1 * f_2)(g) \checkmark \\ (f_1 * f_2)(kg) = \int f_1(x) f_2(x^{-1}kg) \\ = \int f_1(ky) f_2(y^{-1}g) = (f_1 * f_2)(g) \checkmark \\ \mathrm{Supp}(f_1 * f_2) \subseteq \mathrm{Supp}(f_1) \cdot \mathrm{Supp}(f_2) \end{array} \right.$$

The <u>Hecke algebra of G</u> is

$$\mathcal{H}(G) := \operatorname*{colim}_{\{K\} \text{ basis of } 1} \mathcal{H}(G,K) = \mathcal{C}_c(G)$$

⚠ $\mathcal{H}(G,K)$ has a unit e_K (char. funct° of K)

$\mathcal{H}(G)$ has <u>no</u> unit.

<u>PROP</u>: The construction $V \mapsto V^K$ induces:

· an equivalence of categories

$$\left\{ \begin{array}{c} \text{smooth representations} \\ \text{of } G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{smooth } \mathcal{H}(G)\text{-modules} \\ (\text{i.e. } \exists K,\ e_K \cdot M = M) \end{array} \right\}$$

· a bijection (for fixed K):

$$\left\{ \begin{array}{c} \text{irreducible smooth rep. } V \\ \text{with } V^K = \{0\} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{simple} \\ \mathcal{H}(G,K)\text{-modules} \end{array} \right\}$$

thm | (Bernstein) F local, G reductive. Any irred. smooth rep of $G(F)$ is admissible.

3) Automorphic representations:

- Our model is the definition of idèle-class characters for $G = GL_1$.

- $G(k)\backslash G(\mathbb{A})$ inherits an measure. It is not finite volume in general.

thm | · (Borel-Harish-Chandra) Let $Z_s \subseteq Z$ be the maximal split subtorus.
Then $\mu\left(Z_s(\mathbb{A})\, G(k)\backslash G(\mathbb{A})\right) < \infty$

· G anisotropic $\Leftrightarrow$ $G(k)\backslash G(\mathbb{A})$ cocompact

- Hyp: k function field

- Fix a central character $\omega : Z(k)\backslash Z(\mathbb{A}) \longrightarrow \mathbb{C}^\times$ (a.k.a an automorphic rep. for Z)

(for $G = GL_n$, ω = idèle-class character)

- Let $\mathcal{A}^{cusp}_{G,\omega} := \left\{ f : G(k)\backslash G(\mathbb{A}) \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} \cdot\ \exists K' \text{ compact open subgroup s.th } \forall k \in K', \forall g \in G(\mathbb{A}), \\ \quad f(g\cdot k) = f(g) \\ \cdot\ \forall \underset{\in G(\mathbb{A})}{z \in Z(\mathbb{A})},\ f(z\cdot g) = \omega(z) f(g) \\ \cdot\ f \text{ cuspidal (see Marco's talk)} \end{array} \right\}$

smooth $\circlearrowleft$ $G(\mathbb{A})$

- For $g \in G(\mathbb{A})$, $f \in \mathcal{A}^{cusp}_{G,\omega}$, $(g\cdot f)(x) := f(x\cdot g)$ (right action)

def : | A cuspidal automorphic representation (with central char. ω) is an irred. representation of $G(\mathbb{A})$ which occurs as a subquotient of $\mathcal{A}^{cusp}_{G,\omega}$.

thm | (Gelfand, Piatetski-Shapiro, Shalika)

1) We have (non-canonically)

$$\mathcal{A}^{cusp}_{G,\omega} \simeq \bigoplus_{\pi \text{ aut. cusp.}} \pi^{\oplus m(\pi)} \quad \text{with } 0 < m(\pi) < \infty \quad \left(G = GL_n \Rightarrow m(\pi) = 1\right)$$

In particular $\left\{ \begin{array}{l} \pi \text{ aut. cusp} \\ \text{with } \omega \text{ central char} \end{array} \right\}$ is countable.

2) π aut. cusp $\Rightarrow$ π admissible

3) Given $K \subseteq G(\mathbb{A})$ compact open

$\left\{ \begin{array}{l} \pi \text{ aut cusp} \\ \text{with } \omega \text{ central char} \end{array} \;\middle|\; \pi^K \neq 0 \right\}$ is finite.

· thm: (Flath) Let π be an irr. admissible representation of $G(\mathbb{A})$.

For every place v, there exists a unique admissible irr. representation π_v of $G(k_v)$ (the local component of π at v) such that

- for almost all v, π_v is unramified with spherical vector $s_v \in \pi_v$.

- π is the restricted tensor product of the π_v's:

$$\pi = \bigotimes_v{}' \pi_v := \left\langle (x_v)_v \;\middle|\; x_v \in \pi_v,\ x_v = s_v \text{ for almost all } v \right\rangle$$

(better: colim over finite sets of places)

rmk: Given a collection $(\pi_v)_v$ satisfying the conditions above, deciding when $\bigotimes_v{}' \pi_v$ is automorphic is very difficult!

· The rest of this talk is devoted to explaining the underlined terms, and to classify unramified representations.

rmk: What about the other π_v's, including rep of $G(\mathbb{R})/G(\mathbb{C})$? For F local, rep. of $G(F)$ are the object of the (conjectural) local Langlands correspondence which relates them to rep of G_F (thm for $G = GL_n$ and all k / for all G and $k = \mathbb{R}, \mathbb{C}$)

4) Unramified representations

· def: F local field, $G/_F$ reductive

G is unramified if
- G is quasi-split, i.e admits a Borel subgroup def $/_F$.
- $\exists E/_F$ unramified, G_E is split over E.

· G split $\Rightarrow$ G unramified $\Leftrightarrow$ G has a reductive group scheme model $/\mathcal{O}$.

· For G unramified, $G(F)$ admits so-called hyperspecial compact subgroups, which are compact open subgroups of the form $\mathcal{G}(\mathcal{O})$ for reductive models $\mathcal{G}/\mathcal{O}$; for G split, $G(\mathcal{O})$ hyperspecial.

· prop: | G reductive over k global. Then for almost all v, G_{k_v} is unramified. (not true for "split", e.g. unitary groups)

⌈ rmk: In "$\pi = \bigotimes_v' \pi_v$", the restricted tensor product is taken wrt a compatible system of hyperspecial compact subgroups arising from a reductive model $\mathcal{G}/_U$ with $U \subseteq X$ open. ⌋

· For the rest of the talk, fix F local field, $G/_F$ split reductive, $K \subseteq G(F)$ hyperspecial. (the unramified case is similar)

def: | An irreducible admissible representation π of $G(F)$ is unramified (or spherical) if $\pi^K \neq \{0\}$.

· By the general result on Hecke algebras:

$$\left\{ \begin{array}{l} \text{unramified rep} \\ \text{of } G(F) \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{l} \text{simple } \mathcal{H}(G(F), K)\text{-} \\ \text{modules} \end{array} \right\}$$

ex: Let $T = (\mathbb{G}_m)^n$ be a split torus.

Then irr. admissible rep of $T(F)$ are 1-dimensional because $T(F)$ is abelian. Such a rep. is a product of characters of $\mathbb{G}_m$:

$$\chi : T(F) \longrightarrow \mathbb{C}^\times$$

$$(t_1, \dots, t_n) \longmapsto \chi_1(t_1) \cdots \chi_n(t_n)$$

χ unramified $\iff \chi_{|T(\mathcal{O})} = 1 \iff \forall\, 1 \leq i \leq n,\ \chi_i(t) = |t|^{s_i}$ for some $s_i \in \mathbb{C}$

· More canonically, write $\begin{cases} X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T) \text{ for the } \underline{\text{cocharacter lattice}} \\ X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m) \text{ for the } \underline{\text{character lattice}} \end{cases}$

· $X_*(T)$, $X^*(T)$ are dual lattices ($\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$).

· $0 \longrightarrow T(\mathcal{O}) \longrightarrow T(k) \longrightarrow X_*(T) \longrightarrow 0$ (uniformizer)

Then we find:

prop: | · The set of unramified characters of T is $\simeq \mathrm{Hom}(X_*(T), \mathbb{C}^\times)$
· We have an isomorphism $\mathcal{H}(T(F), T(\mathcal{O})) \simeq \mathbb{C}[X_*(T)]$

5) Reductive groups II

- Need to recall some structure theory for G and $G(F)$.

Algebraic structure

G split $\Rightarrow$ can pick $T \subset \underset{\overset{\cap}{N}}{B} \subset G$ with $\begin{cases} T \text{ split maximal torus} \\ B \text{ Borel subgroup, } N \text{ unipotent radical} \end{cases}$

Write $X^* = X^*(T)$, $X_* = X_*(T)$ dual lattices

- Structure theory of reductive groups $\Rightarrow$

$$\underset{\substack{\uparrow \\ \text{positive roots}}}{\Phi_+} \subseteq \underset{\substack{\uparrow \\ \text{roots}}}{\Phi} \subseteq X^* \quad \text{and} \quad \underset{\substack{\uparrow \\ \text{positive coroots}}}{\Phi^\vee_+} \subseteq \underset{\substack{\uparrow \\ \text{coroots}}}{\Phi^\vee} \subseteq X_*$$

ex: $G = GL_2$: $T = \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}$, $B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$, $N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$.

$\Phi = \left\{ \pm \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto a b^{-1} \right) \right\}$, $\Phi^\vee = \left\{ \pm \left(t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \right\}$

- Weil group $W := N_G(T) / Z_G(T)$ finite group, "acts on everything"

ex: $G = GL_n \Rightarrow W \simeq S_n$

$$\begin{cases} X^*_+ := \{ \lambda \in X^* \mid \forall \alpha^\vee \in \Phi^\vee_+,\ \langle \lambda, \alpha^\vee \rangle \geq 0 \} & \underline{\text{dominant characters}} \\ X^+_* := \{ \mu \in X_* \mid \forall \alpha \in \Phi_+,\ \langle \alpha, \mu \rangle \geq 0 \} & \underline{\text{dominant cocharacters.}} \end{cases}$$

We have $W \cdot X^*_+ \simeq X^*$, $W \cdot X^+_* \simeq X_*$.

- $\{$ iso. classes of alg. irrep of $G \} \longleftrightarrow X^*_+$ (highest weight theory)

$\Rightarrow \mathrm{Rep}_F(G) := K_0(\text{cat of alg reps of } G) \xrightarrow[\sim]{\text{character}} F[X^*]^W$.

- Isomorphism class of $G \longleftrightarrow$ Root datum $(X^*, X_*, \Phi, \check{\Phi})$
- The root datum $(X_*, X^*, \check{\Phi}, \Phi)$ corr. to another split reductive group (over $\mathbb{C}$), the Langlands dual group $G^\vee$. Rmk "$W = W^\vee$"

$$\boxed{\mathrm{Rep}_{\mathbb{C}}(G^\vee) \simeq \mathbb{C}[X_*]^W.}$$

ex:

G	GL_n	SL_n	SO_{2n+1}	$SO(2n)$	ooo
$G^\vee$	GL_n	PGL_n	Sp_{2n}	$SO(2n)$	ooo

Local structure :

thm: (Iwasawa decomposition) $G(F) = B(F)\cdot K$

thm (Cartan decomposition)

$$G(F) = \coprod_{\mu \in X_*^+} \underbrace{K\,\mu(\pi)\,K}_{\text{indpt of choice of } \pi}$$

· $GL_n(F)$ is $\displaystyle\coprod_{m_1 \geq \dots \geq m_n \geq 0} K \begin{pmatrix} \pi^{m_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{m_n} \end{pmatrix} K$

rmk for $G = GL_n$, elementary proofs using row and column operations.

· $B(F)$ is <u>not</u> unimodular, its modular function is determined by its values on $T(F)$ ($B(F) = N(F)T(F)$, $\delta_B(nt) = \delta_B(t)$) and we have for $\mu \in X_*$:

$$\delta_B(\mu(\pi)) = q^{\langle 2\rho, \mu\rangle} \quad \text{with} \quad 2\rho = \sum_{\alpha \in \Phi^+} \alpha$$

i.e. $\delta_B(t) = \prod_{\alpha \in \Phi^+} |\alpha(t)|$

$G = GL_2$: $\delta_B\begin{pmatrix} t_1 & 0 \\ 0 & t_2\end{pmatrix} = \left|\frac{t_1}{t_2}\right|$

6) Unramified principal series

· Let $\chi : T(F)/T(\mathcal{O}) \longrightarrow \mathbb{C}^\times$ unramified character. $\leftrightsquigarrow$ closed pt of $\mathrm{Spec}(\mathbb{C}[X_*(T)])$

χ extends uniquely to $\chi : B(F)/B(\mathcal{O}) \longrightarrow \mathbb{C}^\times$ (N unipotent)

def The <u>parabolic induction</u> of χ from B to G is

$$i_B^G(\chi) = \left\{ f \in \mathcal{C}(G) \;\middle|\; \begin{array}{l} \cdot\ \exists K' \subset G(F),\ \forall g \in G\ \forall k \in K,\ f(gk) = f(g) \\ \cdot\ \forall b \in B(F)\ \forall g \in G(F), \\ \quad f(bg) = \delta_B^{1/2}(b)\,\chi(b)\,f(g) \end{array} \right\}$$

equipped with its natural G-action ($g\cdot f(g') = f(g'g)$).

prop: $i_B^G(\chi)$ is an admissible rep. of $G(F)$, and $\dim\left(i_B^G(\chi)\right)^K = 1$.

proof: Let K' be another compact open. Then $K \cap K'$ is of f. index in K, K'.

By Iwasawa, we get $\left| B(F)\backslash G(F)/K' \right| < \infty$. A function $f \in \left(i_B^G(\chi)\right)^{K'}$ is uniquely determined by its values on these double cosets. $\square$

· **prop**: $i_B^G(\chi)$ has exactly one unramified subquotient $\pi(\chi)$.

proof: follows from – $i_B^G(\chi)$ finite length (non-trivial!)

– $(-)^K$ exact functor.

· **prop**: $\forall w \in W$, $i_B^G(\chi)$ and $i_B^G(w\cdot\chi)$ have the same J-H factors $\Rightarrow \pi(\chi) \simeq \pi(w\cdot\chi)$

proof: Not at all obvious, follows from study of "Jacquet functor", Bruhat decomposition $\square$

ex: $G = GL_2$. $\chi = \chi_1 \chi_2$ unramified.

Then
- $\chi_1 \chi_2^{-1} \neq |\cdot|^{\pm 1} \Rightarrow i_B^G(\chi) \simeq \pi(\chi)$ irreducible
- $\chi_1 \chi_2^{-1} = |\cdot| \Rightarrow 0 \to \underset{1\text{-dim}}{St \otimes \chi'} \to i_B^G(\chi) \to \pi(\chi) \to 0$ $\qquad \chi', \chi''$ explicit
- $\chi_1 \chi_2^{-1} = |\cdot|^{-1} \Rightarrow 0 \to \pi(\chi) \to i_B^G(\chi) \to St \otimes \chi'' \to 0$ (with $\pi(\chi)$ 1-dim)

with $St \simeq \mathcal{C}_c\left(B(F) \backslash GL_2(F) \right) / \text{cst fct}^{\circ}$ <u>Steinberg representation</u>

- We thus get a set $\left\{ \pi(\chi) \mid \chi \in \mathrm{Hom}(X_*, \mathbb{C}^\times)/W \right\}$ of un. rep.

7) Satake isomorphism $\quad \mathcal{H}_K := \mathcal{H}(G(F), K)$

- By general th. of Hecke alg., have action

$$\mathcal{H}_K \times i_B^G(\chi)^K \longrightarrow i_B^G(\chi)^K$$
$$(h, f) \longmapsto (h \cdot f)(g) = \int_{G(F)} h(g') f(g'g) \mu(g')$$

- Since $\dim_{\mathbb{C}} i_B^G(\chi)^K = 1$, get scalar $S(h, \chi) \in \mathbb{C}$. By construction,

$$S(-,-): \mathcal{H}_K \times \mathrm{Hom}(X_*, \mathbb{C}^\times) = \mathcal{H}_K \times \mathrm{Hom}_{alg}(\mathcal{H}(T(F), T(\mathcal{O})), \mathbb{C}) \longrightarrow \mathbb{C}$$

such that for any fixed χ, $S(-,\chi)$ algebra hom.

<u>Lemma</u>: $(*)$ is induced by an alg. homomorphism $S: \mathcal{H}_K \to \mathcal{H}(T(F), T(\mathcal{O}))$ with

$$(Sh)(t) = \delta_B^{1/2}(t) \cdot \int_{N(F)} h(tn)\, dn$$

normalized Haar measure on N.

Proof: - The expression yields a $\mathbb{C}$-linear map $\mathcal{H}_K \to \mathcal{H}(T(F), T(\mathcal{O}))$.
- $f_{\chi,B}(tnk) = (\delta_B^{1/2} \chi)(a)$ is a basis of $i_B^G(\chi_3)^K$. We compute:
- $$S(h, \chi) = \int_{G(F)} h(g) f_{\chi,B}(g)\, dg$$

$$= \int_{B(F)} \int_K h(bk) f_{\chi,B}(bk)\, dk\, d_e b \quad (\text{Iwasawa})$$

$$= \int_{B(F)} h(b) f_{\chi,B}(b)\, d_e b \quad (h, f_{\chi,B} \text{ right } K\text{-inv})$$

$$= \int_{T(F)} \int_{N(F)} h(tn) \delta_B^{1/2}(t) \chi(t)\, dn\, dt \quad (B = NT + \text{def of } f_{\chi,B})$$

$$= \int_{T(F)} \left(\delta_B^{1/2}(t) \cdot \int_{N(F)} h(tn)\, dn \right) \chi(t)\, dt$$

$\square$

thm: S induces an isomorphism
$$S : \mathcal{H}_K \xrightarrow{\sim} \mathbb{C}[X_*]^W \simeq \mathrm{Rep}_{\mathbb{C}}(G^\vee)$$

proof:

- The fact that $\mathrm{Im}(S)$ lands in $\mathbb{C}[X_*]^W$ follows from the fact that $i_B^G(w\cdot\chi)^K \simeq \pi(w\cdot\chi)^K \simeq \pi(\chi)^K \simeq i_B^G(\chi)^K$ $\mathcal{H}_K$-equivariantly.
- By Cartan, $\{R_\mu := \mathbb{1}_{K\mu(\pi)K}\}_{\mu\in X_*^+}$ is a $\mathbb{C}$-basis of $\mathcal{H}_K$.

  By $X_* = W\cdot X_*^+$, the elements $g_\upsilon := \sum_{w\in W} w\cdot\upsilon$ $(\upsilon\in X_*^+)$ form a $\mathbb{C}$-basis of $\mathbb{C}[X_*]^W$.

Write $S R_\mu = \sum c_{\mu,\upsilon}\, g_\upsilon$.

- The proof then consists in showing that

(1) $c_{\mu,\mu} \neq 0$ (easy from integral formula)

(2) $c_{\mu,\upsilon} = 0$ unless $\upsilon \leq \mu \overset{\text{def}}{\iff} \mu - \upsilon = \sum_{\substack{\alpha^\vee \text{ simple} \\ \text{positive coroot}}} \ell_\alpha \cdot \alpha^\vee$ with $\ell_\alpha \geq 0$

(Heart of the proof!)

- From (1), (2) we see that the "matrix of S is triangular" $\Rightarrow$ isomorphism $\square$

- The coefficients $c_{\mu,\upsilon}$ are quite complicated in general.

ex: $G = GL_2$. For $m\geq n\geq 0$, put $R_{m,n} = \mathbb{1}_{K\begin{pmatrix}\pi^m & 0\\ 0 & \pi^n\end{pmatrix}K}$. Then
$$\mathcal{H}_K \underset{\mathbb{C}\text{-ev}}{\simeq} \bigoplus \mathbb{C}\cdot R_{m,n} \underset{\text{thm}}{\simeq} \mathbb{C}[R_{1,0}, R_{1,1}^{\pm 1}].$$
(" T_p ", " S_p " Hecke ops in modular forms)

- We have $R_{k,k}\cdot R_{m,n} = R_{m+k,n+k}$

lemma: $K\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix}K = \begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}K \amalg \coprod_{b \bmod \pi} \begin{pmatrix}\pi & b\\ 0 & 1\end{pmatrix}K$

proof:
- $\supseteq$: $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix} = \begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\pi & b\\ 0 & 1\end{pmatrix}$
- $\subseteq$: Let $g = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in K\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix}K$. Then $\begin{cases}|\det(g)| = |\pi| \\ \text{one of } a,b,c,d \text{ is in } \mathcal{O}^\times.\end{cases}$
- right mult $\leftrightarrow$ column ops + elementary operations

$\square$

- From this one can deduce

$$\forall k \geq 1, \quad R_{1,0} R_{k,0} = R_{k+1,0} + q\, R_{1,1} R_{k-1,0}$$

and this suffices to describe the full structure of $\mathcal{H}_K$.

- One can then compute $\quad S R_{1,0}$.

$$(S R_{1,0})(t) = \delta_B^{1/2}(t) \cdot \int_{N(F)} R_{1,0}(t n)\, dn$$

$$= \delta_B^{1/2}(t) \left(\int_{N(F)} \mathbb{1}_{\left(\begin{smallmatrix}1&0\\0&\pi\end{smallmatrix}\right)K}(t n)\, dn + \sum_{e \in \mathcal{O}/(e)} \int \mathbb{1}_{\left(\begin{smallmatrix}\pi&e\\0&1\end{smallmatrix}\right)}(n t)\, dn \right)$$

- The first term is 0 unless $t \in \begin{pmatrix}1&0\\0&\pi\end{pmatrix} T(\mathcal{O})$, and then the integral is 1.
- The second term is 0 unless $t \in \begin{pmatrix}\pi&0\\0&1\end{pmatrix} T(\mathcal{O})$, and then each int. is 1.

$$\Rightarrow (S R_{1,0})(t) = \delta_B^{1/2}(t) \left(\mathbb{1}_{\left(\begin{smallmatrix}1&0\\0&\pi\end{smallmatrix}\right)T(\mathcal{O})}(t) + q \cdot \mathbb{1}_{\left(\begin{smallmatrix}\pi&0\\0&1\end{smallmatrix}\right)T(\mathcal{O})}(t) \right)$$

$$= q^{+1/2}\, \mathbb{1}_{\left(\begin{smallmatrix}1&0\\0&\pi\end{smallmatrix}\right)T(\mathcal{O})}(t) + \underbrace{q^{-1/2} \cdot q}_{q^{1/2}}\, \mathbb{1}_{\left(\begin{smallmatrix}\pi&0\\0&1\end{smallmatrix}\right)T(\mathcal{O})}(t) \quad \text{W-invariant}$$

- More concisely $S R_{1,0} = q^{1/2}(\mu_1 + \mu_2)$ with μ_1, μ_2 usual basis of X_*.

<u>Cor:</u>
- $\mathcal{H}_K$ is commutative.
- $\{\text{all } \pi \text{ unramified reps of } G(F)\} \simeq \{\pi(\chi),\ \chi \text{ unr. char of } T\}$

$$\simeq \mathrm{Hom}(X_*, \mathbb{C}^\times)/W$$

$$\simeq G^\vee(\mathbb{C})^{ss}/\text{conj}$$

<u>proof:</u> - Simple $\mathcal{H}_K$-modules are 1-dimensional because $\mathcal{H}_K$ is commutative

$$G^\vee(\mathbb{C})^{ss} = \bigcup_{\check{T} \subseteq \check{G} \text{ max}} \check{T}(\mathbb{C}) \Rightarrow \check{G}(\mathbb{C})^{ss}/\text{conj} = \check{T}(\mathbb{C})/\text{conj} = \check{T}(\mathbb{C})/W$$

$$\check{T}(\mathbb{C})/W = \mathrm{Hom}(X^*(\check{T}), \mathbb{C}^\times)/W = \mathrm{Hom}(X_*(T), \mathbb{C}^\times)/W \quad \square$$