Forschungsseminar
Chow groups of zero cycles over p-adic fields

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WS 14/15

Introduction

Let K be a p-adic field, i.e. a finite extension of $\mathbb{Q}_p$ and V a smooth geometrically connected variety over K. Denote by $CH_0(V)$ the Chow group of zero-cycles on V and by $A_0(V) \subset CH_0(V)$ the subgroup of degree zero-cycles. It was conjectured by Colliot-Thélène that the group $A_0(V)$ is the direct sum of a finite group and a p'-divisible group. Here a group D is called p'-divisible if the multiplication map $D \to D$, $d \mapsto nd$ is surjective for any natural number n which is prime to p. In this seminar we want to understand the proof of this conjecture given by Saito-Sato in the case of quasi-semistable reduction:

Theorem 1 ([SS10, Thm 9.7])**.** *Let K be a p-adic field with ring of integers $\mathcal{O}_K$ and residue field k. Let X be a regular connected scheme which is flat and projective over $\mathcal{O}_K$ and assume that $(X \otimes_{\mathcal{O}_K} k)_{\text{red}}$ is a divisor with simple normal crossings.*

Then $A_0(X \otimes_{\mathcal{O}_K} K)$ is the direct sum of a finite group and a p'-divisible group.

A key ingredient in the proof is the following theorem:

Theorem 2 ([SS10, Thm 1.16])**.** *Let R be an excellent henselian DVR with fraction field K and residue field k, which we assume to be either finite or separably closed. Let X be a regular scheme, flat and projective of pure relative dimension d over R and assume that $(X \otimes_R k)_{\text{red}}$ is a divisor with simple normal crossings.*

Then for any natural number $n \geq 1$ which is prime to the characteristic of k the cycle map

$$\rho_X : CH_1(X)/n \xrightarrow{\simeq} H^{2d}_{ét}(X, \mu_n^{\otimes d})$$

is an isomorphism.

Now to prove Theorem 1 one can use Theorem 2 and Bertini arguments to reduce to the case where $V = X \otimes_{\mathcal{O}_K} K$ is a curve and then theorem follows from a classical result of Mattuck on the structure of the group of K-rational points of an abelian variety over K.

Most of the seminar will be devoted to understand the techniques used in the proof of Theorem 2. In fact as remarked by Bloch [Blo13, Rmk A.4] one can use Gabber's refinement of de Jong's alteration theorem to obtain Theorem 2 also in the case without the quasi-semistable assumption on X. In case the residue field k is separably closed Bloch also gives a shorter proof of this theorem. Since we are mainly interested in the case of finite residue field we essentially follow [SS10], but we try to incorporate a simplification of Bloch. As an additional reference, one can mention [CT11].

The Talks

1. Introduction: The Chow group of zero-cycles over various fields (23.10.14).

Aim of the talk: Give a rough picture of what is known about the structure of the Chow group of zero-cycles of a smooth projective and geometrically connected variety over a finite field, a local field, a number fields and the complex numbers, respectively.

Details: Give the relevant definitions and facts from [CT11, p. 1-2]. It would be nice if you could also discuss [Beĭ87, 5.0-5.2], [Blo80, Lecture 1, Appendix, Thm 6, Cor 9] and [AS07, Thm 1.1]. Finish by stating Theorem 1.

Lars

2. The extraordinary inverse image functor in étale cohomology (30.10.14).

Aim of the talk: Introduce the extraordinary inverse image functor $f^! : D^+(Y_{\text{ét}}, \Lambda) \to D^+(X_{\text{ét}}, \Lambda)$, where $f : X \to Y$ is a separated finite type morphism between separated and noetherian schemes and Λ is a torsion ring (e.g. $\mathbb{Z}/\ell^n\mathbb{Z}$ or $\mathbb{Q}_\ell/\mathbb{Z}_\ell$).

Details: Try to explain (without proofs!) [Del73, Thm 3.1.4, Def 3.1.6, Prop 3.1.7, Prop 3.1.8, Thm 3.2.5] and [ILO12, Exp. XVI, Thm 3.1.1, Cor 3.1.2].

NN

3. Étale Homology (06.11.14).

Aim of the talk: Define the homology theory $H_q(X, \Lambda)$ and prove some first properties properties. This homology theory will be an essential tool in the rest of the seminar.

Details: Give [SS10, Def (1.4)] and prove [SS10, Prop 1.5, Lem 1.8]. Note that the canonical isomorphism [SS10, (1.9)] in particular yields an isomorphism $H^{2d-q+2}(X, \Lambda(d)) \cong H_q(X, \Lambda)$, in particular $H^{2d}(X, \Lambda(d)) \cong H_2(X, \Lambda)$.

Gabriel

4. The cycle map (13.11.14).

Aim of the talk: Define the cycle map $\rho_X : CH_1(X) \otimes \Lambda \to H_2(X, \Lambda)$ and prove some first properties. Then state Theorem 2. The aim of the next 8 talks is to explain the proof of this theorem.

Details: Explain [SS10, Cor 1.10, (1.12), Prop 1.13] in detail. Then state [SS10, Thm 1.16].

Challenge:(This is not needed in the following.) In [SS10, Thm 1.16] the theorem is stated only for X quasi-semistable. As remarked by Bloch [Blo13, Rmk A.4] the theorem also holds in the general case where X is regular and flat and projective over B. If you want you can try to explain this. (For this one needs [ILO12, Exp IX, Thm 1.1] and the fact that if $f : Y \to X$ is an alteration between regular schemes of degree prime to ℓ then the cycle map $\rho_X : CH_1(X) \otimes \mathbb{Z}/\ell^n \to H_2(X, \mathbb{Z}/\ell^n)$ is a direct factor of $\rho_Y : CH_1(Y) \otimes \mathbb{Z}/\ell^n \to H_2(Y, \mathbb{Z}/\ell^n)$.)

Elena

5. Kato homology (20.11.14).

Aim of the talk: Introduce the Kato homology of X. This invariant plays a prominent role in the following.

Details: Explain [SS10, Def 2.1]. In particular, explain that the differentials in the complex $KC(X, \Lambda)$ are the negative of the boundary maps between the Galois cohomology groups. For the definition of the latter boundary maps see e.g. [JSS14, 0.7, (I.1) and (II)], for the comparison with the boundary maps described here see [JSS14, Thm 2.1.1]. Then explain and prove [SS10, (2.3)-(2.5), Lem 2.6, Lem 2.7].

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6. The Vanishing Theorem (27.11.14).

Aim of the talk: Explain and prove the Vanishing Theorem [SS10, Thm 3.2.]

Jean-Baptiste

7. The Bertini theorem over a DVR (4.12.14).

Aim of the talk: Explain the Bertini Theorem over a DVR [SS10, Thm 4.2] and its proof. this talk is independent from the previous talks.

Sina

8. Surjectivity of the cycle map (11.12.14).

Aim of the talk: Show that the cycle map $\rho_X : CH_1(X) \otimes \Lambda \to H_2(X, \Lambda)$ for X quasi-semistable over R and $\Lambda \in \{\mathbb{Z}/\ell^n, \mathbb{Q}_\ell/\mathbb{Z}_\ell\}$ is surjective. This involves the identification of the 2nd Kato homology of X with coefficients in $\mathbb{Z}/\ell^n$ with the ℓ^n-torsion in the Brauer group of X.

Details: Explain and prove [SS10, Thm 5.1]. Also explain the definition and the properties of the Brauer group used on the way. Then prove [SS10, Thm 5.3]. Finally combine [SS10, Lem 1.17] with the surjectivity of the cycle map to conclude that [SS10, Thm 1.16] is now proved for $\dim X \leq 2$.

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9. A Moving Lemma (18.12.15).

Aim of the talk: Explain the proof of the Moving Lemma [GLL13, Thm 2.3] under the assumption that S is excellent. This talk is independent from the previous talks. It will be used in **10** and **11** for Bloch's simplification of the proof of Theorem 2.

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10. Proof of Theorem 2, I (08.01.15).

Aim of the talk: In this and the next talk a proof of Theorem 2 should be presented, using the ingredients of the previous talks together with Bloch's simplification. We are mostly interested in the case of a finite residue field. According to [Blo13, Rmk A.4] this case can be proven using the arguments from [Blo13] and [SS10, §8, Step 2]. There are some details to fill in. If time permits Bloch's much shorter proof in the case of a separably closed residue field can also be presented.

Kay

11. Proof of Theorem 2, II (15.01.15).

Aim of the talk: See above.

Kay

12. The structure of the group of rational points of abelian varieties over a p-adic field (22.01.15).

Aim of the talk: State and sketch a proof of Mattuck's theorem [Mat55, Thm 7] saying the following:

Let K be a finite extension of $\mathbb{Q}_p$ and A an abelian variety of dimension d over K. Then $A(K)$ has a subgroup of finite index which is isomorphic to $\mathcal{O}_K^{\oplus d}$.

This talk is independent from the previous talks. It will be used in the talk **14** for the proof of Theorem 1.

Sina

13. Another moving lemma (29.01.15).

Prove the generalization of [SS10, 7.1] given in [GLL13, 6.10 (1)]. To do this, give a detail account of the generic moving lemma [GLL13, Th 6.4].

This talk is independent from the previous talks. It will be used in the last talk for the proof of [SS10, Cor 9.2, Cor 9.5].

Inder

14. Proof of Theorem 1 (05.02.15).

Aim of the talk: State and prove Theorem 1 the main result of this seminar.

Details: Explain the proofs of [SS10, Cor 9.6, Thm 9.7] and [SS10, Cor 9.2, Cor 9.5] in full details.

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