Forschungsseminar
Higher Dimensional Class Field Theory and Ramification

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Introduction

Let K be a 1-dimensional global field, i.e. a number field or the function field of a smooth projective curve over $\mathbb{F}_p$. Then classical global Class Field Theory deals with the question of understanding the *abelian* extensions of K, i.e. those Galois extensions L/K with abelian Galois group, or - what is the same - understanding $\mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{ab}}$, just by looking at K itself. More precisely the main results can be summarized as follows: For a place v of K (finite or infinite) denote by K_v the completion of K at v and by $\mathcal{O}_{K_v}$ the local ring of K_v. (Recall that in the number field case the infinite places are just the embeddings $v : K \hookrightarrow \mathbb{C}$ and then $K_v =: \mathcal{O}_{K_v}$ equals $\mathbb{R}$ or $\mathbb{C}$ depending on whether v is a real or complex embedding and that there are no infinite places in the function field case). Then we set

$$\mathbb{A}_K^\times := \{(a_v) \in \prod_v K_v^\times \,|\, \text{all but finitely many } a_v \in \mathcal{O}_{K_v}^\times\}.$$

Notice that there is a canonical diagonal map $K^\times \to \mathbb{A}_K^\times$ and the idele class group of K is by definition

$$C_K := \mathrm{Coker}(K^\times \to \mathbb{A}_K^\times).$$

It is a topological group. Finally denote by K^{ab} the extension of K which corresponds to the closure of the commutator subgroup in the absolute Galois group, i.e. K^{ab} is the maximal abelian extension of K. The main result of global CFT is now: There is a continuous map, called the Artin- or reciprocity homomorphism,

$$\rho_K : C_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K),$$

such that:

(Isomorphism Theorem) For any finite abelian extension L/K the map ρ_K induces an isomorphism

$$\rho_{L/K} : C_K/\mathrm{norm}_{L/K}(C_L) \to \mathrm{Gal}(L/K).$$

(Existence Theorem) We have a bijection

$$\{\text{open subgroups of } \mathrm{Gal}(K^{\mathrm{ab}}/K)\} \to \{\text{open subgroups of finite index of } C_K\},$$

$$U \mapsto \rho_K^{-1}(U).$$

Notice that this in particular means that the finite abelian extensions correspond exactly to open subgroups of finite index in C_K. Moreover:

(i) If K is a *number field*, we denote by C_K^0 the connected component of zero in C_K and we have an exact sequence of topological groups

$$0 \to C_K^0 \to C_K \xrightarrow{\rho_K} \mathrm{Gal}(K^{\mathrm{ab}}/K) \to 0.$$

(ii) If K is the function field of a smooth curve which is geometrically connected over $\mathbb{F}_q$ we set

$$\mathrm{Gal}(K^{\mathrm{ab}}/K)^0 := \mathrm{Ker}(\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q))$$

(notice that this group classifies exactly the interesting abelian extensions of K, namely those which do not come from extensions of the field of constants) and

$$C_K^0 := \mathrm{Ker}\bigg(C_K \xrightarrow{\deg} \mathbb{Z},\, (a_v) \mapsto \sum_v \mathrm{ord}_v(a_v)[\mathbb{F}_q(v) : \mathbb{F}_q]\bigg).$$

Then ρ_K induces an isomorphism of topological groups

$$\rho_K^0 : C_K^0 \xrightarrow{\simeq} \mathrm{Gal}(K^{\mathrm{ab}}/K)^0.$$

Now any finite (abelian) extension of K ramifies only at finitely many places. Thus one can try to refine the above results by trying to classify those abelian extensions L/K which are unramified outside a given finite set of places. Let us recall how this works in the function field case (the number field case is similar but a little bit more technical due to the existence of infinite places). So from now on we assume that K is the function field of a smooth projective curve X, which is geometrically connected over $\mathbb{F}_q$ and denote by $U \subset X$ a dense open subset. Then the finite abelian extensions of K, which are unramified over U correspond to open subgroups of the abelian fundamental group $\pi_1(U)^{\mathrm{ab}}$. We denote by $|U|$ the set of closed points in U and set

$$C(U) := \mathrm{Coker}\bigg(\prod_{x \in U} \mathcal{O}_{K_x}^\times \times \prod_{x \in |X \setminus U|} \{1\} \to C_K\bigg)$$

$$\cong \mathrm{Coker}\bigg(K^\times \to \bigoplus_{x \in |U|} \mathbb{Z} \oplus \bigoplus_{x \in |X \setminus U|} K_x^\times\bigg).$$

Notice that this group surjects to $CH_0(U)$, the Chow group of zero-cycles on U. Then ρ_K^0 induces an isomorphism (notation similar as above)

$$(1) \quad \rho_U : C(U)^0 \xrightarrow{\simeq} \pi_1(U)^{\mathrm{ab},0},$$

which when we take the inductive limit over all open $U \subset X$ gives back the isomorphism from (ii) above.

But one can be even more precise. In fact a fixed finite abelian extension L/K will not only ramify at only finitely many places but at the places where it ramifies, the ramification will be bounded. Recall if $D = \sum_{x \in |X|} n_x[x]$ is an effective divisor on X, then we say that the ramification of L/K is bounded by D if the following holds: Denote by X_L the normalization of X in L, for a point $y \in |X_L|$ denote by G_y the decomposition group of $\mathrm{Gal}(L/K)$ at y and by G_y^n the n-th ramification subgroup of G_y in the upper numbering (see [Serre, IV, §3]). Then we say that the ramification of L/K is bounded by D if for all $y \in |X_L|$ with image $x \in X$ we have $G_y^{n_x} = 0$. (Notice that G_y^0 is the inertia group, so that L is in particular unramified outside the support of D.) The finite abelian extensions of K whose ramification is bounded by D correspond to open subgroups of $\pi_1(X,D)^{\mathrm{ab}}$ (by definition). Define the Chow group of zero cycles on X with modulus D to be

$$C(X,D) := \mathrm{Coker}\Big(K^\times \to \bigoplus_{x\in|U|}\mathbb{Z}\oplus\bigoplus_{x\in|D|}K_x^\times/(1+\mathfrak{m}_x^{n_x}\mathcal{O}_{K_x})\Big)$$

$$\cong\Big(\bigoplus_{x\in|X\setminus D|}\mathbb{Z}\Big)/<\mathrm{div}(f)\,|\,f\in K^\times \text{ with } f\equiv 1 \bmod D>.$$

Then it follows from [Serre, XV, §2, Thm 2 and the remark thereafter] that the isomorphism (1) above induces an isomorphism

$$(2)\qquad \rho_{X,D}: C(X,D)^0 \xrightarrow{\simeq} \pi_1(X,D)^{\mathrm{ab},0}.$$

This gives a full description of the finite abelian extensions of K together with their ramification behavior. Also taking the inverse limit over all effective divisors D with support equal to $|X\setminus U|$ of the isomorphism (2) we get back isomorphism (1).

This is all classical and the result of the work of many people. The subject of this Forschungsseminar is to understand the generalizations of these classical results to the case of higher dimensions, i.e. where K is either the function field of a regular scheme which is separated, flat and of finite type over $\mathbb{Z}$ or the function field of a generically smooth projective variety over a finite field (in which case we will only see how to describe the extensions, which are unramified over the smooth locus).

Here is a short plan of the talks: The first talk recalls the classical results as above. In talk 2 and 3 we prove a classical result of Katz-Lang. In the talks 4-6 we study Wiesend's approach to higher dimensional CFT, following the presentation in [K] and [KS]. In the talks 7-14 we discuss the description of finite abelian extensions with bounded ramification of function fields over a finite field mainly following [KSa]. The main result will be that we get the isomorphism (2) also for an effective Cartier divisor D on a normal proper variety over a finite field of characteristic unequal 2 such that $U = X\setminus D$ is smooth.

The Talks

Classical Results

1. 1-dimensional Class Field Theory (24.10.13).

Aim of the talk: Explain (without proofs) the statements of 1-dimensional Class Field Theory.
Details: Cover the material from [P, Sec. 0-3]. At the end you should state them in the form as presented in [K, Sec. 1.3].

N.N.

2. First properties of the abelian fundamental group (31.10.13).

Aim of the talk: Introduce the abelian fundamental group, explain its connection to G-torsors and for a morphism of schemes $f: X\to S$ give some first structure results of $\mathrm{Ker}(\pi_1(X)^{\mathrm{ab}}\to\pi_1(S)^{\mathrm{ab}})$
Details: Go through [KL, Sec. 0, I]. In particular explain (1.1) and the Lemmas 1- 3 and their proofs.

N.N.

3. The theorem of Katz and Lang (07.11.13).

Aim of the talk: Prove the Katz-Lang theorem.
Details: Explain the proof of Theorem 1 (bis, ter) of [KL, Sec. II].

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Higher dimensional Class Field Theory for regular arithmetic schemes (after Wiesend)

4. The reciprocity map (14.11.13).

Aim of the talk: For a regular arithmetic scheme X define the class group $C(X)$ and the reciprocity homomorphism

$$\rho : C(X) \to \pi^{\mathrm{ab}}(X).$$

Details: Go through section 4 of [K]. In particular carefully explain the definition of the class group $C(X)$ with its topology and the reciprocity map (see [K, Prop. 4.8] and [KS, Prop. 7.5]). Also explain the definition of the pushforward in [K, Cor. 4.9] as well as Proposition 4.10.

N.N.

5. The Isomorphism - and the Weak Existence Theorem (21.11.13).

Aim of the talk: See title.
Details: Explain [K, Thm 6.1] in detail. Then discuss the Propositions 7.1 - 7.3 and give some ideas of their proof.

N.N.

6. Fundamental Theorems (28.11.13).

Aim of the talk: Explain the proof of the finiteness Theorem ("Every open and normal subgroup of $C(X)$ has finite index") and of the exact sequence

$$0 \to C(X)^0 \to C(X) \xrightarrow{\rho} \pi_1^{\mathrm{ab}}(X) \to 0.$$

Details: Explain the proof of [K, Thm 8.1] and give Corollary 8.2. Then [K, Thm 9.1].

N.N.

Higher dimensional Class Field Theory for smooth varieties over a finite field

7. The Tame case (05.12.13).

Aim of the talk: Explain the tame version of the higher dimensional class field theory for smooth schemes over a finite field.
Details: Introduce the notion of tame covering and define the tame fundamental group (see [KS, Sec. 2]). Give [KS, Thm 2.8]. Define the class group and the tame class group for a smooth scheme over a finite field (see [KS, Sec. 7], in particular the part between Lemma 7.6 and Section 8 for $Z = \operatorname{Spec} \mathbb{F}_p$). Then state [KS, Thm 8.3] (only the part over $\mathbb{F}_p$). You don't need to prove this, but it should be similar to what we did already, see [K, Rmk 8.2].

N.N.

8. Ramification Theory (12.12.13).

Aim of the talk: Introduce the notion of ramification of a character $\chi : \pi_1(U)^{\mathrm{ab}} \to \mathbb{Q}/\mathbb{Z}$, where U is either the spectrum of henselian discrete valuation ring (with not necessary perfect residue field) or a smooth scheme over a perfect field of positive characteristic. Show some properties and define the fundamental group $\pi_1(X, D)^{\mathrm{ab}}$ which classifies finite étale abelian covers of $X \setminus |D|$ whose ramification is bounded by the effective divisor D.
Details: Go through Section 2.1 up to Theorem 2.3 and through Section 2.2 of [KSa] and also through [KSa2, Sec. 2] (there are some intersections.) Explain why we have

$$\pi_1(X, C)^{\mathrm{ab}} \cong \pi_1^{\mathrm{t,ab}}(X \setminus C),$$

where $X/\mathbb{F}_p$ is a normal proper variety and C is a *reduced* effective Cartier divisor on X and the group on the right is the tame fundamental group introduced in the previous talk.

N.N.

9. Lefschetz Theorem for abelian Fundamental group with modulus (09.01.14).

Aim of the talk: Explain the proof of the theorem in the title of this talk.
Details: Go through [KSa2, Sec. 3].

N.N.

10. (16.01.14).

No talk!

11. The Existence Theorem with modulus (23.01.14).

Aim of the talk: Introduce the Chow group of zero cycles with modulus $C(X, D)$ and its inverse limit $C(U)$. Then define the reciprocity homomorphisms $\rho_{X,D} : C(X, D) \to \pi_1(X, D)^{\mathrm{ab}}$ and $\rho_U : C(U) \to \pi_1(U)^{\mathrm{ab}}$ and state the Existence Theorem (= the main theorem). Finally explain how to reduce the proof of this theorem to the following statement:

(*) For all smooth varieties $U/\mathbb{F}_p$ *of dimension 2* the map ρ_U induces a surjection

$$\mathrm{Hom}_{\mathrm{cont}}(\pi_1(U)^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathrm{cont}}(C(U), \mathbb{Q}/\mathbb{Z}).$$

Details: Introduce the group $C(X, D)$ as in [KSa, Def. 1.6] (try to be economical with the notations you use). Then go through [KSa, Sec. 3]. In particular clearly state Theorem 3.3 and Corollary 3.4 and try to explain Proposition 3.2 and Lemma 3.7 carefully.

12. (30.01.14).

No talk!

13. The cycle Conductor (06.02.14).

Aim of the talk: Define the dual μ_x of the local cycle conductor and state the main properties of the cycle conductor.
Details: Go through [KSa, Sec. 4] (and recall the necessary notations form [KSa, Sec. 1]). Clearly state Theorem 4.5 and 4.7 (without proof).

14. Proof of the Existence Theorem with modulus (13.02.14).

Aim of the talk: Finish the proof of the existence theorem by proving (*) above (and assuming the properties of the cycle conductor).

Details: Recall [KSa, Thm 3.3 and Cor 3.4] and that we reduced its proof to show (*). Then go through [KSa, Sec. 5] and explain the proof of (*). It should be made clear how the cycle conductor cc and the refined Artin conductor rar are used in the proof (see last diagram in section 5).

References

[KL] N.M. Katz, S. Lang: *Finiteness theorems in geometric classfield theory*, Enseign. Math. (2), Vol. 27 (1981), p. 285-319.

[K] M. Kerz: *Higher Class Field Theory and the Connected Component*, Manuscripta Math. 135 (2011), p. 63-89.

[KS] M. Kerz, A. Schmidt: *Covering data and higher dimensional global class field theory*, Journal of Number Theory 129 (2009), p. 2569-2599.

[KSa] M. Kerz, S. Saito: *Chow group of 0-cycles with modulus and higher dimensional class field theory*, preprint (2013), http://www.mathematik.uni-regensburg.de/kerz/

[KSa2] M. Kerz, S. Saito: *Lefschetz theorem for abelian fundamental group with modulus*, preprint (2013), http://www.mathematik.uni-regensburg.de/kerz/

[P] B. Poonen: A brief summary of the statements of Class Field Theory, http://www-math.mit.edu/~poonen/papers/cft.pdf.

[Serre] J.-P. Serre: *Local Fields.* Springer.