Research Seminar Talk: 02.02.17

p-adic integration and Igusa Zeta function:

Setting / Notation:

K p-adic field, i.e., complete field $+ |\cdot|_K$ non-archimedean absolute value

and $\mathcal{O}_K / \pi \mathcal{O}_K = \mathbb{F}_q$

where $\mathcal{O}_K = \{a \in K \mid |a|_K \leq 1\}$

Assume image of $K^\times$ under $|\,|_K$ is discrete in $\mathbb{R}^\times$.

- K-analytic manifolds can be defined in the same way as in the case where $K = \mathbb{R}$ (for K complete field)

Recall: Def^n:

(1) $x = (x_1, \dots, x_n) \in K^n$, $U \underset{\text{open}}{\subseteq} K^n$, then $f: U \to K$ is called K-analytic function on U if

$\forall\, a = (a_1, \dots, a_n) \in U$

$\exists\, f_a(x) \in K\langle\langle x - a \rangle\rangle = K\langle\langle x_1 - a_1, \dots, x_n - a_n \rangle\rangle$

(ring of convergent power seri

such that $f(x) = f_a(x)$ for any point x in an open nbhd of a.

(2) $f: U \to K^m$ is called K-analytic map if every f$_i$ in $f = (f_1, \dots, f_m)$ is a K-analytic function on U.

(3) n-dim K-analytic manifold is the following data:

- X Hausdorff topological space
- atlas $\{(U_i, \phi_{U_i})\}_{i \in I}$ s.t. $U_i \underset{\text{open}}{\subseteq} X$, $\bigcup_{i \in I} U_i = X$,

$\phi_{U_i} : U_i \longrightarrow \phi_{U_i}(U_i) \subseteq K^n$ bicontinuous

if $U_i \cap U_j \neq \emptyset$, the map

$\phi_{U'} \circ \phi_U^{-1} : \phi_U(U \cap U') \longrightarrow \phi_{U'}(U' \cap U)$ is K-analytic

(4) $(X, \{(U, \phi_U)\})$, $(Y, \{(V, \phi_V)\})$ K-analytic manifolds
$f: X \to Y$ is K-analytic map if for every U, V such that $U \cap f^{-1}(V) \neq \emptyset$.

$$\psi_V \circ f \circ \phi_U^{-1} : \phi_U(U \cap f^{-1}(V)) \to K^{\dim(Y)}$$

is K-analytic.

(5) $(X, \{(U, \phi_U)\})$ K-analytic manifold, $a \in X$ point

$\mathcal{O}_{X,a} = \mathcal{O}_a :=$ {local ring of germs of K-analytic functions in a nbhd of a}

m_a (maximal ideal) $:= \{f \in \mathcal{O}_a \mid f(a) = 0\}$

if (U, ϕ_U) is a chart s.t $a \in U$, & $\phi_U(x) = (x_1, \ldots, x_n)$

then m_a has $(x_1 - x_1(a), \ldots, x_n - x_n(a))$ as its ideal basis

& $\mathcal{O}_{X,a} \simeq K\langle\langle x - x(a)\rangle\rangle$.

(6) $T_a(X) :=$ {V.S over K of K-linear maps $\partial: \mathcal{O}_a \to K$ s.t $\partial(fg) = (\partial f)g(a) + f(a)\partial g \ \forall f, g \in \mathcal{O}_a$}

$\Omega_a(X) := (T_a(X))^\vee$
$\quad \wr\wr$
m_a/m_a^2

for any $f \in \mathcal{O}_a$, $(df)_a := f - f(a) \bmod m_a^2$

$(dx_1)_a, \ldots, (dx_n)_a$ form a K-basis for m_a/m_a^2

$$\Omega_a^p(X) := \Lambda^p(\Omega_a(X)) := \sum_{i_1 < \ldots < i_p} K(dx_{i_1})_a \wedge \ldots \wedge (dx_{i_p})_a$$

We say α is a K-analytic differential form of degree p on X if for every $a \in X$, $\alpha(a) \in \Omega_a^p(X)$ and has the local form

$$\alpha(x) = \sum_{i_1 < \ldots < i_p} f_{U, i_1, \ldots, i_p}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

where $f_{U, i_1, \ldots i_p}$ are all K-analytic functions on U $\forall$ U

- If f is any K-analytic f^n on X, then df is a K-analytic differential form of degree 1 on X. $(df(a) := (df)_a)$

(7) $f: X \longrightarrow Y$ K-analytic map of K-analytic manifolds

β a K-analytic differential form of degree p on Y

$\{(U, \phi_U)\}$, $\{(V, \psi_V)\}$ charts of X, Y respectively.

for $a \in X$, $f(a) = b$, $a \in U$, $b \in V$

$\phi_U(x) = (x_1, \ldots, x_n)$, $\psi_V(y) = (y_1, \ldots y_m)$

$U' = U \cap f^{-1}(V) \neq 0$ s.t

$$\beta(y) = \sum_{j_1 < \ldots < j_p} g_{V, j_1 \ldots j_p}(y)\, dy_{j_1} \wedge \ldots \wedge dy_{j_p}$$

$g_{V, j_1 \ldots j_p} \longrightarrow$ K-analytic f^n on V.

then

$$(\partial f^*)(\beta)(x) = \sum_{j_1 < \ldots < j_p} (g_{V, j_1 \ldots j_p} \circ f)(x)\, d(y_{j_1} \circ f) \wedge \ldots \wedge d(y_{j_p} \circ f)$$

on U'.

If $\dim X = \dim Y = p = n$. we put $g_V = g_{V, 1, \ldots n}$

then $$(\partial f)^*(\beta)(x) = g_V(f(x)) \frac{\partial(y_1 \ldots y_n)}{\partial(x_1 \ldots, x_n)} \cdot dx_1 \wedge \ldots \wedge dx_n.$$

Remarks: (1) $\mathcal{O}_K$ is a compact ring

(2) $\pi^i \mathcal{O}_K$ are compact open subgroups of the additive group K $\forall i \in \mathbb{Z}$

and $\bigcup_i \pi^i \mathcal{O}_K = K$

$\Rightarrow$ K is locally compact totally disconnected group countable at ∞

(i.e. covered by countably many compact subspaces)

$\Rightarrow$ So is K^n and any K-analytic manifold is locally compact totally disconnected.

$\Rightarrow$ We can put a Haar measure on K^n

<u>More notation</u>: X – K-analytic manifold

$$\mathcal{T}(X) := \left\{ \begin{array}{l} \text{all compact open subset} \\ \text{of } X + \Phi \text{ (empty set)} \end{array} \right\} \qquad A \longmapsto \chi_A \qquad \operatorname{Supp}\varphi \longleftarrow \varphi$$

$$\mathcal{D}(X) = \left\{ \begin{array}{l} \text{all locally constant } \mathbb{C}\text{-valued} \\ \text{functions on } X \text{ with compact support} \end{array} \right\}$$

vector space$/\mathbb{C}$

$\Rightarrow$ $\mathbb{C}$-span of $\mathcal{T}(X) \simeq \mathcal{D}(X)$

$\mathcal{D}(X)' :=$ dual of $\mathcal{D}(X)$

$\quad ||$

simply additive $\mathbb{C}$-valued functions on $\mathcal{T}(X)$.

<u>Defn</u>: A measure μ on X is an element of $\mathcal{D}(X)'$ satisfying $\mu(A) = \mu(\chi_A) \in \mathbb{R}_{\geq 0}$ for every A in $\mathcal{T}(X)$.

for $\varphi \in \mathcal{D}(X)$, $\mu(\varphi) = \int_X \varphi(x)\mu(x) = \int_X \varphi(x)\,dx$

By defn we have

$$\left| \int_X \varphi(x)\mu(x) \right| \leq \int_X |\varphi(x)|\mu(x) \leq \mu(\operatorname{Supp}(\varphi))\, \|\varphi\|_\infty \quad \forall \varphi \in \mathcal{D}(X).$$

Now specialize $X = K^n$. K^n acts on itself by left ~~multiplication~~ addition

<u>Defn</u>: A Haar measure μ_n or μ_{K^n} or μ on K^n is a K^n-invariant measure different from 0.

<u>Thm</u>: Haar measure exists on K^n and is unique upto a factor in $\mathbb{R}_{>0}$.

Prop^n (Change of variable) : μ_n haar measure on K^n & $\mu_n(\mathcal{O}_K^n)=1$

Let $f(x) = (f_1(x), \ldots, f_n(x)) : K^n \to K^n$ s.t

every $f_i(x)$ is K- analytic around $a \in K^n$ and

$$Jac(f) = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}(a) \neq 0$$

so that $\exists\, U \ni a$ and $V \ni f(a) = b$ and

$f : U \to V$ bianalytic

$x \longmapsto y = f(x)$

then $dy = \left| \frac{\partial(f_1 \ldots, f_n)}{\partial(x_1 \ldots, x_n)} \right|_K dx$, where $dx = \mu_n|_U$, $dy = \mu_n|_V$

In other words, $A \in \Upsilon(U)$

$$\int_{f(A)} g(y)\, dy = \int_A g(f(x)) \left| \frac{\partial(f_1 \ldots, f_n)}{\partial(x_1 \ldots x_n)}(x) \right| dx$$

$\square$

– Define: $\Omega(K^\times) := Hom(K^\times, \mathbb{C}^\times)$ continuous homomorphi

& $\Omega(K) := Hom(K, \mathbb{C}^\times)$

Fact – $\varphi \in \Omega(K)$ or $\Omega(K^\times)$ is locally constant.

$\Rightarrow$ In particular, φ is integrable

Explicitly, we can describe $\Omega(K^\times)$ as follows

$a \in K^\times$, then $a = \pi^e u$, $e \in \mathbb{Z}$, $u \in \mathcal{O}_K^\times$
(unique upto choice of π)

$e :=$ order of $a =: ord(a)$

$u =$ angular component of $a =: ac(a)$

$\Rightarrow K^\times \simeq \mathbb{Z} \times \mathcal{O}_K^\times$ (bicontinuous)

$a \longmapsto (ord(a), ac(a))$

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$$\Rightarrow \Omega(K^\times) \xrightarrow{\sim} \mathbb{C}^\times \times (\mathcal{O}_K^\times)^*$$
$$\omega \longmapsto (\omega(\pi), \omega|_{\mathcal{O}_K^\times})$$

$$\Rightarrow \Omega(K^\times) = \coprod_{(\mathcal{O}_K^\times)^*} \mathbb{C}^\times$$

Note that $(\mathcal{O}_K^\times)^* = \bigcup_{e \in \mathbb{N}} (\mathcal{O}_K^\times / (1+\pi^e \mathcal{O}_K))^*$
finite groups

Define $\omega_s \in \Omega(K^\times)$, $s \in \mathbb{C}$
$$\omega_s(a) := |a|_K^s = q^{-\mathrm{ord}(a)s}.$$

Let $\omega \in \Omega(K^\times)$, then we can choose $s \in \mathbb{C}$ such that
$$\omega(\pi) = q^{-s}$$
$$\Rightarrow \omega(a) = \omega_s(a)\chi(ac(a))$$
where $\chi = \omega|_{\mathcal{O}_K^\times}$

⚠ s is not unique!!
In fact, we have a bicontinuous isomorphism
$$\mathbb{C}/(2\pi i/\log q)\mathbb{Z} \longrightarrow \mathbb{C}^\times$$
$$s \longmapsto t = q^{-s}$$

Observation: • A $\mathbb{C}$-valued function of t is holomorphic on $\mathbb{C}^\times$ iff it is a holomorphic f^n on the s-plane $\mathbb{C}$.

• Note that $Re(s)$ depends only on ω, even though s is not uniquely determined by ω.
We denote $\sigma(\omega) := Re(s)$
$$\Rightarrow |\omega(a)| = \omega_{\sigma(\omega)}(a).$$

Define: an open subset $\Omega_\sigma(K^\times) \subset \Omega(K^\times)$ by $\sigma(\omega) > \sigma$ for any $\sigma \in \mathbb{R}$

Lemma: $a \in K$, $e \in \mathbb{Z}$, $\omega \in \Omega_0(K^\times)$, $N, n \in \mathbb{N}$, $t = \omega(\pi)$, $\chi = \omega|_{\mathcal{O}_K^\times}$. Then

$$\int_{a+\pi^e \mathcal{O}_K} \omega(x)^N |x|_K^{n-1}\, dx = \begin{cases} (1-q^{-1})(q^{-n} t^N)^e / (1-q^{-n}t^N) & \text{if } a \in \pi^e \mathcal{O}_K,\ \chi^N = 1 \\ q^{-e}\omega(a)^N |a|_K^{n-1} & \text{if } a \notin \pi^e \mathcal{O}_K,\ \chi^N|_{U'} = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $U' = 1 + \pi^e a^{-1} \mathcal{O}_K$.

$f \in K[x_1, \dots x_n] \setminus K$, $\mathrm{Char}(K) = 0$

$\rightsquigarrow f : \mathbb{A}^{n}_K \longrightarrow \mathbb{A}^1_K$

choose an embedded resolution of singularities for morphisms

i.e : $h : Y \longrightarrow \mathbb{A}^n_K$ projective birational morphism of K varieties w/ Y smooth/K

$$h : Y \setminus \{h^{-1}(f^{-1}(0))\} \xrightarrow[\text{iso}]{\sim} \mathbb{A}^d_K \setminus \{f^{-1}(0)\}$$

and the divisor $E = (f \circ h)$ is sncd

such an embedded resolution always exist.

Jac_h : Jacobian ideal sheaf of h on Y

note that the divisor (Jac_h) is supported on E.

let $\{E_i\}_{i \in I}$ irreducible components of E,

N_i multiplicity of $f \circ h$ along E_i

$\nu_i - 1$ multiplicity of the Jac_h along E_i.

(N_i, ν_i) - numerical data of resolution of h.

Main Thm: (Igusa) $\omega \in \Omega_0(K^\times)$, $\Phi \in \mathcal{D}(X)$, $X = K^n$

$$Z_\Phi(\omega) := \omega(f)(\Phi) = \int_{X \setminus f^{-1}(0)} \omega(f(x))\, \Phi(x)\, dx$$

$Z(\Phi, X, s, f)$

$$= \int_{X \setminus f^{-1}(0)} |f(x)|_K^s\, \chi(ac(f(x)))\, \Phi(x)\, dx$$

defines a holomorphic f^n

$$Z_\Phi(f) : \Omega_0(K^\times) \longrightarrow S(X)' \longrightarrow \mathbb{C}$$

$$\omega \longmapsto \omega(f) \longrightarrow \omega(f)(\Phi)$$

and has a meromorphic continuation

$$Z_{\Phi}(f) : \Omega(K^\times) \longrightarrow S(X)'$$
$$\omega \longmapsto \omega(f)$$

such that $\omega(f)(\Phi)$ is a rational function of $t=\omega(\pi)$ for each $\chi = \omega|_{\mathcal{O}_K^\times}$ and Φ.

Furthermore: $h: Y \to X$ is a resolution as above.

then
$$\prod_{i\in I}(1-q^{-\nu_i}t^{N_i})\,\omega(f)$$
becomes holomorphic on the punctured t-plane $\mathbb{C}^\times$ for all χ in $(\mathcal{O}_K^\times)^*$.

Proof: Fix χ and Φ. Consider $Z(\Phi,\chi,s,f)$ as a function of s, $Z(s)$.
Now choose any compact subset $C \subseteq \Omega_0(K^\times)$

Put $\sigma_0 = \max_{\omega\in C}(\sigma(\omega))$, $A = \operatorname{supp}(\Phi)$

$$M = \max\left(1, \sup_{x\in A}|f(x)|_K\right)$$

define $\phi_C := \|\Phi\|_\infty M^{\sigma_0}\chi_A$

where $\chi_A \to$ characteristic function of A

Then
- $|\omega(f(x))\Phi(x)| \le \phi_C(x) \quad \forall x\in X\setminus f^{-1}(0)$, $\omega\in C$
- $\displaystyle\int_{X\setminus f^{-1}(0)} \phi_C(x)\,dx < \infty$

Then using Lebesgue theorem, we get $Z(s)$ is continuous.

In fact, if $s_0 \in U$, where U openset of $\mathbb{C}^\times$ with $Re(z)>0$

$$\lim_{s \to s_0} Z(s) = \lim_{s \to s_0} \int_X |f(x)|_K^s \, \chi(ac(f(x))\, \Phi(x)\, dx$$

$$= \int_X \lim_{s \to s_0} |f(x)|_K^s \, \chi(ac(f(x))\, \Phi(x)\, dx$$

$$= \int_X |f(x)|_K^{s_0} \, \chi(ac(f(x))\, \Phi(x)\, dx$$

$$= Z(s_0)$$

Moreover,

$\omega \longmapsto \omega(f(x))$ for every x in $X \setminus f^{-1}(0)$ is a holomorphic function on $\Omega(K^\times)$ hence on $\Omega_0(K^\times)$ hence on U.

so if C is any closed curve of finite length in U, then

$$\int_C Z(s)\, ds = \int_C \left(\int_X |f(x)|_K^{s_0} \chi(ac(f(x))\, \Phi(x)\, dx \right) ds$$

$$\overset{\text{Fubini}}{=} \int_X \left(\int_C |f(x)|_K^{s_0} \chi(ac(f(x))\, \Phi(x)\, ds \right) dx$$

$$= 0 \quad \text{(Cauchy)}$$

$\Rightarrow$ F is holomorphic on U by Morera's theorem.

This gives the first part of the theorem

As for the main part, at every point b of Y we can choose a chart (U, ϕ_U) s.t $b \in U$, $\phi_U(y) = (y_1, \dots y_n)$ &

$$f \circ h = \varepsilon \cdot \prod_{j \in J} y_j^{N_j} \quad , \quad h^*\left(\bigwedge_{1 \le k \le n} dx_k \right) = \eta \cdot \prod_{j \in J} y_j^{\nu_j - 1} \bigwedge_{1 \le k \le n} dy_k$$

$J \subseteq I$ such that all E_j, $j \in I$ contain b.

& $\varepsilon, \eta \in \mathcal{O}_b^\times \cong K\langle\langle y_1, \dots y_n \rangle\rangle^\times$.

with $\varepsilon(y)$, $\eta(y)$ having expansions:

$$\varepsilon(y) = \varepsilon(0)\left(1 + \sum_{|i|>0} c_i y^i\right)$$

w/ $\varepsilon(0) \in K^\times$, $c_i \in K$ $\forall\, i \neq 0 \in \mathbb{N}^n$ s.t $\sum c_i y^i$ is convergent on $\pi^k \mathcal{O}_K^n$ for some $k \in \mathbb{N}$.

By making k larger if necessary, we may assume

$$\varepsilon(0)^{-1}\varepsilon(y) - 1 \in \pi^{e(\chi)}\mathcal{O}_K \quad \forall\, y \in \pi^k \mathcal{O}_K^n$$

where $\chi = \omega|_{\mathcal{O}_K^\times}$ & $e(\chi)$ smallest e such that

$$\chi|_{(1+\pi^e \mathcal{O}_K)} = 1$$

Then we have

$$|u(y)|_K = |u(0)|_K, \quad \omega(u(y)) = \omega(u(0))$$

$$\forall\, y \in \pi^k \mathcal{O}_K^n.$$

Since h is proper, $A = \mathrm{supp}(\Phi)$ compact open

$\Rightarrow B = h^{-1}(A) \in \mathcal{T}(Y)$

$\Rightarrow B = \bigsqcup_{\alpha=1}^{\ell} B_\alpha$, $\quad B_\alpha \in \mathcal{T}(Y)$ s.t

$B_\alpha \subseteq U$ for some U.

Since Φ is locally constant, after subdividing B_α we may assume

$$(\Phi \circ h)|_{B_\alpha} = \Phi(h(b))$$

Further $\quad \phi_U(B_\alpha) = \beta + \pi^e \mathcal{O}_K^n \quad$, $\beta = (\beta_1, \dots, \beta_n) \in K^n$, $e \in \mathbb{N}$

Since $h : Y \setminus (f \circ h)^{-1}(0) \longrightarrow X \setminus f^{-1}(0)$ is bianalytic

we have using the change of variable formula

$$Z(s) = \omega(f)(\Phi) = \sum_{\alpha} \Phi(h(b))\, \omega(\varepsilon(b))\, |\eta(b)|_K \cdot \prod_{1\le i\le n} \int_{\beta_i + \pi^{e}\mathcal{O}_K} \omega(y_i)^{N_i} |y_i|_K^{\nu_i - 1}\, dy_i$$

with the understanding that $N_i = 0$, $\nu_i = 1$ $\forall i \notin J$.

Then the previous lemma shows that RHS is a rational function of $t = \omega(\pi)$.

Furthermore, the denominator of each term is the product of $1 - q^{-\nu_j} t^{N_j}$ $\forall j \in J$ multiplied possibly by a power of t, which is holomorphic on $\mathbb{C}^\times$.

$\square$

Poincaré series $P(t)$ of a polynomial $f(x) \in \mathcal{O}_K[x_1, \dots, x_n] \setminus \mathcal{O}_K$

Observe: $\xi, \xi' \in \mathcal{O}_K^n$, $i \in \mathbb{N}$, $\xi \equiv \xi' \bmod \pi^i$

& $f(\xi) \equiv 0 \bmod \pi^i$

Then, $f(\xi') \equiv 0 \bmod \pi^i$

Therefore, $c_i = \#\{\xi \bmod \pi^i \mid f(\xi) \equiv 0 \bmod \pi^i\}$ is well defined.

Consider, the following power series

$$P(t) = \sum_{i \geq 0} c_i (q^{-n} t)^i$$

in a complex variable t.

- $P(t)$ is dominated by series $\sum t^i$

$\Rightarrow$ $P(t)$ is convergent for $|t| < 1$.

- We can express $P(t)$ by $Z(s)$ for $f(x)$ as above as follows:

$$f : \mathcal{O}_K^n \longrightarrow \mathcal{O}_K$$

then $f^{-1}(\pi^i \mathcal{O}_K) = \bigcup \{\xi + \pi^i \mathcal{O}_K^n \mid \xi \in \mathcal{O}_K^n + f(\xi) \equiv 0 \bmod \pi^i\}$

$$\Rightarrow \mu_n(f^{-1}(\pi^i \mathcal{O}_K)) = c_i \cdot \mu_n(\pi^i \mathcal{O}_K^n) = c_i \cdot q^{-ni},$$

$$\Rightarrow \mu_n(f^{-1}(\pi^i \mathcal{O}_K^\times)) = \mu_n(f^{-1}(\pi^i \mathcal{O}_K \setminus \pi^{i+1} \mathcal{O}_K))$$
$$= c_i q^{-ni} - c_{i+1} \cdot q^{-n(i+1)} \quad \forall i \in \mathbb{N}$$

Therefore, if $t = \omega_s(\pi) = q^{-s}$ for $\mathrm{Re}(s) > 0$, i.e., $|t| < 1$.

$$\Rightarrow Z(s) = \int_{\mathcal{O}_K^n \setminus f^{-1}(0)} |f(x)|_K^s \, dx = \sum_{i \geq 0} \int_{f^{-1}(\pi^i \mathcal{O}_K^\times)} |f(x)|_K^s \, dx$$

$$= \sum_{i\geq 0} (c_i q^{-ni} - c_{i+1} q^{-n(i+1)}) t^i$$

$$= P(t) - t^{-1}(P(t) - 1).$$

<u>Thm</u> : $P(t)$ is a rational function of t. $\square$