Retraction onto the skeleton

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1 Notations and setting

We fix a complete DVR R, with maximal ideal $\mathfrak{m}$, fraction field K and residue field k. We fix a uniformizer π of R. The discrete valuation on $K^\times$ will be denoted by v_K, and the associated norm $|-|_K := \exp(-v_K(-))$. We also fix a smooth connected separated K-variety X.

Example 1.1. *Let $N_1, N_2 \geq 0$. Assume for simplicity N_1, N_2 relatively prime and fix $a_1, a_2 \in \mathbb{Z}$ with $a_1N_1 + a_2N_2 = 1$. The running example for this talk will be*

$$Y = \mathrm{Spec}(K[T_1, T_2]/(T_1^{N_1}T_2^{N_2} - \pi).$$

Note that, as a K variety, it is isomorphic to $\mathbb{G}_{m,K} = \mathrm{Spec}(K[W, W^{-1}])$ via the maps

$$T_1 \mapsto \pi^{a_1}W^{N_2}, T_2 \mapsto \pi^{a_2}W^{-N_1}; W \mapsto T_1^{a_2}T_2^{-a_1}$$

but we are using this presentation because it suggests a specific formal model. Note that, if N_1, N_2 are not relatively prime, the variety Y is not geometrically connected.

1.1 Generic fiber and Berkovich analytification

We can associate to X its Berkovich analytification; this is a Berkovich space X^{an} together with a map $\iota : X^{\mathrm{an}} \to X$ of locally ringed spaces, which is universal for this property. As as set, we have

$$X^{\mathrm{an}} = \{(x, |-|_x) |\, x \in |X|,\ |-|_x : \kappa(x) \to \mathbb{R}_{\geq 0} \text{ multiplicative semi-norm extending } |-|_K\}$$

and the topology is the weakest topology which is finer than the Zariski topology (via ι) and such that, for all $U \subset X$ open and $f \in \mathcal{O}_X(U)$, the function

$$X^{\mathrm{an}} \to \mathbb{R}_{\geq 0}, (x, |-|_x) \mapsto |f(x)|_x$$

is continuous. The space X^{an} is path-connected and locally compact; it is compact iff X is proper.

Let $(x, |-|_x) \in X^{\mathrm{an}}$. We define $\mathcal{H}(x)$ to be the completion of the residue field $\kappa(x)$ with respect to the semi-norm $|-|_x$; we extend the semi-norm to the completion, and we define $\mathcal{H}(x)^\circ$ as the value ring of $|-|_x$, i.e. $\mathcal{X}(x)^\circ = \{y \in \mathcal{H}(x) | |y|_x \leq 1\}$. The subset X^{bir} of birational points of X^{an} is by definition $\iota^{-1}(\theta)$ with θ the generic point of X.

Example 1.2. *The Berkovich space Y^{an} is isomorphic to $\mathbb{G}_m^{\mathrm{an}}$, hence can be described as the set of multiplicative seminorms on $K[W, W^{-1}]$ which extend $|-|_K$. We have a closed cover by affinoid annuli*

$$\mathbb{G}_m^{\mathrm{an}} = \cup_{r>0} \mathcal{M}(K\{r^{-1}T, rT^{-1}\}).$$

1.2 Models, rigid generic fiber, skeleton

Let $\mathcal{X}$ be an sncd model of X, that is, a flat separated finite type regular R-scheme together with a fixed isomorphism $\mathcal{X}_K \simeq X$ and such that $(\mathcal{X}_k)_{\mathrm{red}}$ is a strict normal crossings divisor. We write $\{E_i\}_{i\in I}$ for the regular irreducible components of $(\mathcal{X}_k)_{\mathrm{red}}$, and write N_i for the multiplicity of E_i in $\mathcal{X}_k$. The combinatorics of the reduced special fiber are encoded in the dual complex $\Delta(\mathcal{X}_k)$, a Δ-complex (small generalization of a simplicial complex, in which a set of vertices can bound several distinct faces) whose vertices are in bijection with I and the higher dimensional faces correspond to connected components of the E_J for $J \subset I$.

Remark 1.3. *If R is of equal characteristic 0, such a model always exists by resolution of singularities in its strong form. In general, we assume the existence of such a model.*

The formal completion of $\mathcal{X}$ along $\mathfrak{m}$ is a formal scheme $\widehat{\mathcal{X}}$ over $\mathrm{Spf}(R)$, which is flat, separated and topologically of finite type. To such a formal scheme, we can associate its rigid analytic generic fiber $\widehat{\mathcal{X}}_\eta$, which is also a Berkovich space. Unlike X^{an}, this Berkovich space is always compact. Because the formal scheme came from the model $\mathcal{X}$, there is an injective continuous quasi-compact map of Berkovich spaces

$$\widehat{\mathcal{X}}_\eta \to X^{\mathrm{an}}$$

(actually an immersion in the Berkovich sense; we will not use this), which is an isomorphism if $\mathcal{X}$ is proper over R. We will always identify $\widehat{\mathcal{X}}_\eta$ with its image in X^{an}, which as a set is simply

$$\widehat{\mathcal{X}}_\eta = \{(x, |-|_x) |\ \mathrm{Spec}(\mathcal{H}(x)) \to X \text{ extends to } \mathrm{Spec}(\mathcal{H}(x)^\circ) \to \mathcal{X}\}.$$

Remark 1.4. *For a formal scheme $\mathcal{Z}$ of the type above, not necessarily coming from a R-model, the construction of the rigid generic fiber goes as follows. If $\mathcal{Z} \simeq \mathrm{Spf}(A)$ with A topological R-algebra, with A flat topologically of finite type, then $A \otimes_R K$ has a natural structure of K-affinoid Banach algebra, and we have $\mathcal{Z}_\eta \simeq \mathcal{M}(A \otimes_R K)$. In general, $\mathcal{Z}_\eta$ is defined by a gluing procedure.*

Because of the definition of $\widehat{\mathcal{X}}_\eta \subset X^{\mathrm{an}}$, there is a specialisation map

$$\mathrm{sp}_{\mathcal{X}} : \widehat{\mathcal{X}}_\eta \to |\mathcal{X}_k|$$

which is anti-continuous.

Example 1.5. *The model*

$$\mathcal{Y} = \mathrm{Spec}(R[T_1, T_2]/(T_1^{N_1} T_2^{N_2} - \pi)).$$

is an sncd model if N_1 and N_2 are coprime (otherwise, several components of the generic fiber come together in the special fiber). Its special fiber has two irreducible components $E_i = V(T_i)$ with multiplicities N_i, and intersection point O. The dual complex is just an interval. The rigid generic fiber is then the affinoid annulus

$$\widehat{\mathcal{Y}}_\eta = \{(x, |-|_x) | x \in \mathrm{Spec}(K[T_1, T_2]/(T_1^{N_1} T_2^{N_2} - \pi)), |T_1(x)|_x \leq 1, |T_2(x)|_x \leq 1\}.$$

Note that the equation implies $\exp(-1/N_i) \leq |T_i|_x \leq 1$*. The specialisation map sends points with $|T_i|_x = 1$ to points in $E_i \setminus 0$, while it sends all points with $|T_1|_x < 1, |T_2|_x < 1$ to O.*

1.3 Monomial points and skeleton

Using sncd models, we can define two subsets $X^{\mathrm{div}} \subset X^{\mathrm{mon}}$ of X^{bir}: the subsets of divisorial and monomial points. A divisorial point is the valuation on $\kappa(X)$ attached to an irreducible component E of an sncd model $\mathcal{X}$. Monomial points are more complicated to define. Such a point is attached to the datum $(\mathcal{X}, J \subset I, (\alpha_j)_{j\in J}, C)$ with $\mathcal{X}$ sncd model, J a subset of the irreductible components of the special fiber, α_j a family with $\sum_{j\in J} \alpha_j N_j = 1$ and C a connected component of the intersection E_J.

The set of all monomial points associated to a given model $\mathcal{X}$ is called the skeleton of $\mathcal{X}$ and denoted by $\mathrm{Sk}(\mathcal{X}) \subset X^{\mathrm{mon}} \cap \widehat{\mathcal{X}}_\eta$. We write $i_{\mathcal{X}} : \mathrm{Sk}(\mathcal{X}) \to \widehat{\mathcal{X}}_\eta$. We equip it with the induced topology from X^{an}. By construction of monomial points, there is a map

$$\Phi : |\Delta(\mathcal{X}_k)| \to \mathrm{Sk}(\mathcal{X})$$

which was shown, in the previous talk, to be an homeomorphism. For a point $z \in \mathrm{Sk}(\mathcal{X})$ attached to the datum $(\mathcal{X}, J \subset I, (\alpha_j)_{j\in J}, C)$, the point $\mathrm{sp}_{\mathcal{X}}(z)$ is the generic point of C.

Example 1.6. *For the sncd model $\mathcal{Y}$, the homeomorphism*

$$\Phi : \{(\lambda, 1-\lambda) \in \mathbb{R}^2_{\geq 0}\} \to \mathrm{Sk}(\mathcal{Y})$$

sends $(0, 1)$ to the divisorial point associated to $(\mathcal{Y}, E_1)$, $(1, 0)$ to the divisorial point associated to $(\mathcal{Y}, E_2)$, and $(\lambda, 1-\lambda)$ to the monomial point associated with $(\mathcal{Y}, (E_1, E_2), (\frac{\lambda}{N_1}, \frac{1-\lambda}{N_2}), O)$.

2 Retraction

We can now state the main theorem of this talk.

Theorem 2.1. *[NX16] There exists a continuous retraction*

$$\rho_{\mathcal{X}} : \widehat{\mathcal{X}}_\eta \to \mathrm{Sk}(\mathcal{X})$$

which makes $\mathrm{Sk}(\mathcal{X})$ *into a strong deformation retract of* $\widehat{\mathcal{X}}_\eta$*.*

In particular, if $\mathcal{X}$ *is a proper* R*-scheme, then* $\mathrm{Sk}(\mathcal{X})$ *is homotopy equivalent to* X^{an}*, and in particular the homotopy type of* $|\Delta(\mathcal{X}_k)|$ *is independent of the proper sncd model.*

Let us mention an important related result.

Theorem 2.2. *[Ber99] The Berkovich space* X^{an} *is locally contractible (i.e., every point has a system of contractible neighbourhoods). More generally, any Berkovich space which is locally embeddable in a smooth Berkovich space is locally contractible.*

Remark 2.3. *The proof of this theorem of Berkovich is closely related to the one of the main theorem here, and in fact inspired it. The proof of 2.2 also involves retractions onto skeleta of models, contructed as in the proof of 2.1 below via actions of analytic tori.*

If X *is a curve, then 2.2 follows essentially from 2.1, since the latter implies that* X^{an} *is uniquely path-connected.*

In higher dimensions, the main difference is that for 2.2 one uses the theory of "polystable reduction" by alterations of de Jong (hence the result does not depend on resolution of singularities), and that local contractibility requires more careful geometric arguments since having a strong deformation retract to a locally contractible space like a Δ*-complex does not guarantee local contractibility, only the existence of "big" contractible neighbourhoods.*

In this section, we discuss the construction and basic properties of ρ. Let $z \in \widehat{\mathcal{X}}_\eta$. By definition, $\bar{z} := sp_{\mathcal{X}}(z) \in \mathcal{X}_k$ is well-defined. We put $J := \{i \in I | \bar{z} \in E_i\}$, and we define ξ as the generic point of the connected component of E_J containing $\bar{z}$. Choose local equations T_i of E_i at ξ. The data $(\mathcal{X}, J, (v_x(T_i))_{i\in J}, \xi)$ defines a monomial point in $\mathrm{Sk}(\mathcal{X})$, and we define $\rho_{\mathcal{X}}(z)$ to be this point.

Alternatively, $\rho_{\mathcal{X}}(z)$ is the unique monomial point in $\mathrm{Sk}(\mathcal{X})$ such that the same irreducible components of the special fiber pass through $\mathrm{sp}_{\mathcal{X}}(z))$ and $\mathrm{sp}_{\mathcal{X}}(\rho_{\mathcal{X}})$, we have $\mathrm{sp}_{\mathcal{X}}(z)) \in \overline{\mathrm{sp}_{\mathcal{X}}(\rho_{\mathcal{X}})}$ and with the same valuations of the local equations.

Lemma 2.4. *The map* $\rho_{\mathcal{X}}$ *is a continuous retraction of* $i_{\mathcal{X}}$*.*

Example 2.5. *We prove the lemma in our running example.*

- *Let* $]a, b[$ *be an open interval strictly contained in* $[0, 1]$*. Then we have*

  $$\rho_{\mathcal{Y}}^{-1}(]a,b[) = \{(x, |-|_x) | x \in \mathrm{Spec}(K[T_1,T_2]/(T_1^{N_1}T_2^{N_2} - \pi)), \exp(-b/N_1) < |T_1(x)|_x < \exp(-a/N_1), |T_2$$

  which is an open annulus, open in the closed annulus $\widehat{\mathcal{Y}}_\eta$*.*

- *Let $[0,c[$ be an open interval in $[0,1]$ containing 0, with $c<1$. We have*

  $\rho_{\mathcal{Y}}^{-1}([0,c[) = \{(x,|-|_x)|x \in \mathrm{Spec}(K[T_1,T_2]/(T_1^{N_1}T_2^{N_2}-\pi)), \exp(-c/N_1) < |T_1(x)|_x \leq 1, |T_2(x)|_x \leq 1\}$

  which is open in the closed annulus $\widehat{\mathcal{Y}}_\eta$.

We look at the dependency of the skeleton and the retraction in the model.

Proposition 2.6. *Let $h : \mathcal{X}' \to \mathcal{X}$ be a morphism of sncd models. Then*

1. *We have $\widehat{\mathcal{X}}'_\eta \subset \widehat{\mathcal{X}}_\eta$ inside X^{an}. Moreover, $\rho_{\mathcal{X}} \circ \rho_{\mathcal{X}'} = \rho_{\mathcal{X}}$ on $\widehat{\mathcal{X}}'_\eta$.*
2. *If h is proper, then $\widehat{\mathcal{X}}'_\eta = \widehat{\mathcal{X}}_\eta$ and $\mathrm{Sk}(\mathcal{X}) \subset \mathrm{Sk}(\mathcal{X}')$.*

Proof. We clearly have $\widehat{\mathcal{X}}'_\eta \subset \widehat{\mathcal{X}}_\eta$, and if h is proper the equality follows from the valuative criterion of properness. Let $z \in \widehat{\mathcal{X}}'_\eta$, put $y = \rho_{\mathcal{X}'}(z)$. We want to show that $\rho_{\mathcal{X}}(z) = \rho_{\mathcal{X}}(y)$. By construction, we have $\mathrm{sp}_{\mathcal{X}'}(z) \in \overline{\{\mathrm{sp}_{\mathcal{X}'}(y)\}}$ in $\mathcal{X}'_k$. We have a commutative diagram

$$\begin{tikzcd} \widehat{\mathcal{X}}'_\eta \arrow[r, "\mathrm{sp}_{\mathcal{X}'}"'] \arrow[d] & {|\mathcal{X}'_k|} \arrow[d] \\ \widehat{\mathcal{X}}'_\eta \arrow[r, "\mathrm{sp}_{\mathcal{X}}"'] & {|\mathcal{X}'_k|} \end{tikzcd}$$

where the vertical maps are continuous, which shows that $\mathrm{sp}_{\mathcal{X}}(z) \in \overline{\{\mathrm{sp}_{\mathcal{X}}(y)\}}$. By a similar argument, the irreducible components of $\mathcal{X}_k$ passing through $\mathrm{sp}_{\mathcal{X}}(z)$ and $\mathrm{sp}_{\mathcal{X}}(y)$ are the same.

It remains to show that for every such component E with local equation T, we have $v_z(T) = v_y(T)$. Write $h^*E = \sum_{i=1}^r a_i E'_i$. If T'_i are local equations of the E'_i, then we have $T = u \prod_i^r T_i^{a_i}$ in $\kappa(X)$ with u unit, so that

$$v_z(T) = \sum_{i=1}^r a_i v_z(T'_i) = \sum_{i=1}^r a_i v_y(T'_i) = v_y(T).$$

This concludes the proof of 1).

We now assume h proper. Let $z \in \mathrm{Sk}(\mathcal{X})$. Put $\xi = \mathrm{sp}_{\mathcal{X}}(z) \in \mathcal{X}_k$, $z' = \rho_{\mathcal{X}'}(z)$ and $\xi' = \mathrm{sp}_{\mathcal{X}'}(z') \in \mathcal{X}'_k$. We need to show that $z = z'$. It is enough to show that, for all $f \in \mathcal{O}_{\mathcal{X},h(\xi')}$, we have $v_z(f) = v_{z'}(f)$. Write an admissible expansion of f with respect to the model $\mathcal{X}$.

$$f = \sum_\beta c_\beta T^\beta$$

We can choose local parameters $T'_1, \dots T'_s$ in $\mathcal{O}_{\mathcal{X}',\xi'}$ with $u\prod_{j=1}^s (T'_j)^{N'_j}$ uniformizer in R. We have $T_i \in \mathcal{O}_{\mathcal{X}',\xi'}$, hence we can write

$$T_i = v_i \prod_{j=1}^s (T'_j)^{\gamma_{i,j}}$$

with v_i unit and $\gamma_{i,j}$ positive integer. We write

$$f = \sum_\beta c_\beta v^\beta (T')^{\sum_{i=1}^r \beta_i \gamma_i}$$

We want to show that this is an admissible expansion of f for $\mathcal{X}'$ at ξ'.

In fact, it turns out that the multi-exponents $\sum_{i=1}^r \beta_i\gamma_i$ are all distinct, so that the property that all coefficients are units or zero is preserved (note that units in $\mathcal{O}_{\mathcal{X},h(\xi')}$ are mapped to units in $\mathcal{O}_{\mathcal{X}',\xi'}$).. Assume $\sum_{i=1}^r \beta_i\gamma_i = \sum_{i=1}^r \beta_i'\gamma_i$. This exactly says that the monomials T^β and $T^{\beta'}$ have the same multiplicity along each components of $\mathcal{X}'_k$. Using properness of h and normality of the models, one can compute the Stein factorisation of h locally around ξ and show

$$\mathcal{O}_{\mathcal{X},\xi} \simeq \mathcal{O}(\mathcal{X}' \times_{\mathcal{X}} \mathrm{Spec}(\mathcal{O}_{\mathcal{X},\xi})).$$

Computing in the right hand side and using the equality of multiplicities, we see that $z^{\beta-\beta'}$ has to be a unit in $\mathcal{O}_{\mathcal{X},\xi}$, which implies $\beta = \beta'$. This proves that the expansion of f is admissible, and from this it is clear that $v_z(f) = v_{z'}(f)$. $\square$

Example 2.7. *If we blow up $\mathcal{Y}$ at a closed point $y \in \mathcal{Y}_k$, we obtain a proper morphism of sncd models $h : \mathcal{Y}' \to \mathcal{Y}$ of Y with an extra component in the special fiber. There are two possibilities:*

- *The point y is O. Then $\mathrm{Sk}(\mathcal{Y}) = \mathrm{Sk}(\mathcal{Y}')$ (as dual complexes, this is a barycentric subdivision of an interval).*
- *The point y is not O. Then $\mathrm{Sk}(\mathcal{Y}) \subsetneq \mathrm{Sk}(\mathcal{Y}')$, obtained by extending the interval on one side.*

We suggest, as an exercise, to go through the proof of the previous proposition for the blow up at O, paying particular attention to the middle point of the interval.

3 Strong deformation retract

The proof in [NX16] of the main theorem on the strong deformation property of ρ requires a stronger hypothesis.

Hypothesis 3.1. *There exists a smooth algebraic curve C over k and a point $O \in C$ such that $R \simeq \widehat{\mathcal{O}}_{C,O}$ (so that, in particular, we are in the equal characteristic case), and a model $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ over C.*

Theorem 2.1 should hold in the stated generality, but the proof, using logarithmic geometry, is unpublished.

The strategy of the proof goes as follows.

- One first constructs the required homotopy H in the case where the pair $(\widetilde{\mathcal{X}}, \mathcal{X}_k)$ is of toric type, i.e. $\widetilde{\mathcal{X}}$ is a toric variety over k, and the $\mathcal{X}_k$ is the toric boundary (=complement of the dense open torus). The construction in this case is quite explicit and uses the torus action.

- One then proves that the pair $(\widetilde{\mathcal{X}}, \mathcal{X}_k)$ is Zariski locally of toric type. This is a standard property of scnd divisors, and we sketch the argument.

- One finally shows that the homotopies constructed on the so-called relative étale charts, can be glued down to an homotopy on $\widetilde{\mathcal{X}}$.

3.1 The toric case

We look at the toric construction in our running example. Since everything is explicit, no knowledge of toric geometry is required. We consider the following variety

$$\widetilde{\mathcal{Y}} = \mathrm{Spec}(k[\pi, T_1, T_2]/(T_1^{N_1} T_2^{N_2} - \pi)) \simeq \mathbb{A}_k^2.$$

which is an sncd model of its generic fiber Y. We denote by E_i the component given by $T_i = 0$. The skeleton $\mathrm{Sk}(\mathcal{Y})$ is simply a 1-simplex. Assume N_1, N_2 coprime for simplicity, and choose a_1, a_2 integers with $a_1 N_1 + a_2 N_2 = 1$.

Enters toric geometry. The variety $\widetilde{\mathcal{Y}}$ is toric, with dense torus

$$\widetilde{\mathcal{T}}' = \mathrm{Spec}(k[\pi, \pi^{-1}, T_1, T_1^{-1}, T_2, T_2^{-1}]/(T_1^{N_1} T_2^{N_2} - \pi))$$

Let us write an explicit isomorphism of $\widetilde{\mathcal{T}}'$ with $\mathbb{G}_m^2$:

$$k[\pi, \pi^{-1}, V, V^{-1}] \simeq k[\pi, \pi^{-1}, T_1, T_1^{-1}, T_2, T_2^{-1}]/(T_1^{N_1} T_2^{N_2} - \pi), W \mapsto T_1^{a_2} T_2^{-a_1}$$

This toric variety comes with a natural toric morphism given by π:

$$\phi_\pi : \widetilde{\mathcal{Y}} \to \mathbb{A}_k^1.$$

such that the special fiber $\mathcal{Y}_k$ is $\phi_\pi^{-1}(0)$ There is a natural subtorus

$$\widetilde{\mathcal{T}}_1 = V(\pi - 1) = \mathrm{Spec}(k[T_1, T_1^{-1}, T_2, T_2^{-1}]/(T_1^{N_1} T_2^{N_2} - 1) \simeq \mathrm{Spec}(k[V, V^{-1}])$$

which acts on the fibers of ϕ_π; we can promote this action to an action of the constant group scheme over $\mathbb{A}_k^1$ on $\widetilde{\mathcal{Y}} \to \mathbb{A}_k^1$ given

$$\widetilde{\mathcal{T}} = \mathrm{Spec}(k[\pi, T_1, T_1^{-1}, T_2, T_2^{-1}]/(T_1^{N_1} T_2^{N_2} - 1)$$

We have the notion of fiber product over K of Berkovich spaces and of fiber product over R of formal schemes, with

$$Z^{\mathrm{an}} \times_K (Z')^{\mathrm{an}} \simeq (Z \times_K Z')^{\mathrm{an}}$$

and

$$\widehat{\mathcal{Z}}_\eta \times_K \widehat{\mathcal{Z}}'_\eta \hookrightarrow (\widehat{\mathcal{Z}} \hat{\times}_R \widehat{\mathcal{Z}}')_\eta$$

We deduce an action of the Berkovich group variety over K

$$\mathbb{G}_K := \widehat{\mathcal{T}}_\eta = \{t \in T^{\mathrm{an}} |\ |T_1(t)| = |T_2(t)| = 1\}$$

on

$$\widehat{\mathcal{Y}}_\eta = \{z \in Y^{\mathrm{an}} |\ |T_1(z)| \leq 1, |T_2(z)| \leq 1\}.$$

We have to be careful about what such an action means, because as with schemes, the underlying topological space of the fiber product is not the product of the topological spaces. We only look at the action at the level of the rational points of $\widehat{\mathfrak{Y}}_\eta$, i.e., the points with $\mathcal{H}(x) = K$. We write $\widehat{\mathfrak{Y}}_\eta(K)$ for that set. Since we want to act on all points, we need to extend scalars. Let L be a complete valued field extension of K. We thus have an action map

$$\mathbb{G}_L \times \widehat{\mathfrak{Y}}_\eta(L) \to \widehat{\mathfrak{Y}}_\eta$$

such that, whenever I fix an L-rational point y, the induced map $\mathbb{G}_L \to \widehat{\mathfrak{Y}}_\eta$ is continuous.

For $t \in [0,1]$, we define a point $\gamma_L(t) \in \mathbb{G}_L$ which is the sup-norm on the closed disk of radius t around the identity. More concretely, for all $f \in L[V]$, written as

$$f(V) = \sum_{i \geq 0} c_i (V-1)^i$$

we have

$$|f(\gamma_L(t))| = \max_{i \geq 0} |c_i|_L t^i.$$

We do have $|T_1(\gamma_L(t))| = |V^{N_2}(\gamma_L(t))| = 1$ hence $\gamma_L(t)$ lies in $\mathbb{G}_L$. The induced map

$$[0,1] \to \mathbb{G}_L, t \mapsto \gamma_L(t)$$

is clearly continuous since the formula for the norm is continuous in t. We have $\gamma_L(0) = 1$, whereas $\gamma_L(1)$ is the Gauss point of the affine line (since the closed disk on radius 1 around 1 is the closed disk of radius 1 around 0!).

Let $x \in \widehat{\mathfrak{Y}}_\eta$. The point x induces a unique rational point in $\widehat{\mathfrak{Y}}_\eta \times_K \mathcal{H}(x)$, which is in the x-fiber of the continuous projection map

$$\pi_{\mathcal{H}(x)} \widehat{\mathfrak{Y}}_\eta \times_K \mathcal{H}(x) \to \widehat{\mathfrak{Y}}_\eta.$$

We put

$$H(x,t) = \pi_{\mathcal{H}(x)}(\gamma_{\mathcal{H}(x)} \cdot x)$$

This defines a map

$$H : \widehat{\mathfrak{Y}}_\eta \times [0,1] \to \widehat{\mathfrak{Y}}_\eta$$

which can be shown to be continuous (it is, at least, clearly continuous separately in both variables).

More explicitely, for $f \in K[T_1, T_2]$, we write

$$f(|T_1(x)|V^{N_2}, |T_2(x)|V^{-N_1}) = \frac{1}{V^j} \sum_{i \geq 0} c_i V^i$$

and we have

$$|f(H(x,t))| = \max_i |c_i| t^i$$

so that in particular

$$|f(H(x,t))| \geq c_0 = |f(x)|.$$

Proposition 3.2. *The map H is an homotopy which makes ρ a strong deformation retract of $\widehat{\mathcal{Y}}_\eta$ onto $\mathrm{Sk}(\mathcal{Y})$.*

Proof. We have to establish that, for $y \in \widehat{\mathcal{Y}}_\eta$ and $t \in [0,1]$,

- $H(y,0) = y$,
- $H(y,1) = \rho_{\mathcal{Y}}(y)$ and
- $H(y,t) = y$ if $y \in \mathrm{Sk}(\mathcal{Y})$.

The first point is obvious. For the second, we use the fact that the closed disk centered at 1 of radius 1 is the closed disk centered at 0 of radius 1, hence for every $g \in \mathcal{H}(x)[V]$, we have

$$|g(\gamma_{\mathcal{H}(x)})| = \max_i |c_i|_x$$

hence for $f = \sum_{i,j} d_{i,j} T_1^i T_2^j$ in $K[T_1, T_2]$, we have

$$|f(H(x,1))| = \max_{i,j} |d_{i,j}|_x |T_1(x)|_x^i |T_2(x)|_x^j$$

which caracterizes the monomial point on Y^{an} associated to $(\mathcal{Y}, (E_1, E_2), (v_x(T_1), v_x(T_2), O)$, which is precisely $\rho_{\mathcal{Y}}(y)$.

We omit the proof of the last property, which follows from the easy computation

$$|T_i(H(x,t))| = |T_i(x)|,$$

the inequality

$$|f(H(x,t))| \geq c_0 = |f(x)|$$

and a minimality property of valuation attached to monomial points which we have not discussed. □

One can show from the formula for H that, for a point in $y \in \widehat{\mathcal{Y}}_\eta \cap Y^{\mathrm{bir}}$ (which implies , there exists a $t_0 > 0$ such that for all $t \leq t_0$, we have $H(y,t) = y$.

3.2 Sncd models and relative toroidal embeddings

Definition 3.3. *Let k be a field, and X be a normal variety together with a function $\phi : X \to C$ to a smooth curve C. Put $Z = f^{-1}(0)$ and $X_0 = X \setminus Z$. The pair (X, ϕ) is a* (simple) relative toroidal embedding *if for all $x \in X$, there exists compatible open neighbourhood V of x and W of $\phi(x)$ and a commutative diagram*

$$\begin{tikzcd} V \arrow[r, "\gamma"'] \arrow[d, "\phi"'] & Y \arrow[d, "\psi"'] \\ W \arrow[r, "\gamma'"'] & \mathbb{A}^1 \end{tikzcd}$$

with γ, γ' étale morphisms, Y affine toric variety (with Y_0 dense open torus), ψ toric morphism such that $\psi^{-1}(0)$ is the toric boundary $Y \setminus Y_0$ of Y, and $\gamma^{-1}(Y_0) = V \cap X_0$. Such a diagram is called a relative étale chart.

Proposition 3.4. *Let $\phi : \tilde{\mathcal{X}} \to C$ be an sncd model of its generic fiber X. Assume that k is perfect. Then $(\tilde{\mathcal{X}}, \phi)$ is a relative toroidal embedding.*

Proof. We only do check the condition for a closed point $x \in \tilde{\mathcal{X}}$; the general case follows by a specialisation argument. Let $T_1, \ldots, T_{d-1}$ be a regular system of parameters in $\mathcal{O}_{\tilde{\mathcal{X}},x}$ such that $(\mathcal{X}_k)_x$ is defined by the equation $T_1^{N_1} \ldots T_d^{N_d}$. There exists an open neighbourhood V of x such that $T_i \in \mathcal{O}_{\tilde{\mathcal{X}}}(V)$. Write $J = \{i | N_i > 0\}$. We define $\gamma : V \to \mathbb{A}_k^d$ (with coordinates U_i) by

$$\gamma^*(U_i) = \left\{ \begin{array}{c} T_i, \ i \in I \\ T_i + 1, \ i \notin I \end{array} \right\}$$

Because the field extension $k(x)/k$ is separable by perfection of k and by the Jacobian criterion, the morphism γ is étale at x. We restrict V such that γ is everywhere étale and the sections $T_i + 1$ for $i \notin I$ are all invertible on V. We define $\psi : \mathbb{A}_k^n \to \mathbb{A}_k^1$ (with coordinate U) by

$$\psi^*(U) = U_1^{T_1} \ldots U_d^{T_d}$$

We also define $W = \phi(V)$ (open since an sncd model is flat). It is easy to see that, up to restricting V, there is an étale map γ' completing the commutative diagram, and that $\gamma^{-1}(\mathbb{G}_m^d) = V \cap X_0$. □

In fact, we see that the result is more precise and that we get the smooth toric variety $\mathbb{A}_k^d$ together with the function $T_1^{N_1} \ldots T_d^{N_d}$ as the étale local model. This is, up to some extra notational complexity, exactly the example we have treated above.

3.3 Gluing homotopies

By the previous step, one can find a Zariski open cover $\widetilde{\mathcal{U}_i}$ of $\widetilde{\mathcal{X}}$ together with relative étale charts

$$\begin{array}{ccc} \widetilde{\mathcal{U}_i} & \xrightarrow[\gamma]{} & Y_i \\ \phi \downarrow & & \downarrow \psi \\ W_i & \xrightarrow[\gamma']{} & \mathbb{A}^1 \end{array}$$

We have seen in Pedro's talk that the open covering $\{\mathcal{U}_i\}$ of the model $\mathcal{X}$ induces a closed covering $\{\mathrm{Sk}(\mathcal{U}_i)\}$ of the skeleton $\mathrm{Sk}(\mathcal{X})$.

By the first step of the proof, we have homotopies H_i, defined via the actions of rigid generic fibers of formal tori acting on $\widehat{Y}_i$, which make $\rho_{\mathcal{U}_i}$ into a strong deformation retract of $\widehat{Y}_{i,\eta}$ onto $\mathrm{Sk}((Y_i)_R)$. We have to show that the homotopies can be lifted along the étale charts, and that the retractions and the lifted homotopies are compatible on the closed covering $\{\mathrm{Sk}(\mathcal{U}_i)\}$ and finish the proof.

We will not say anything about this aspect of the proof, except that, in [Thu07], descending the formal torus action along an étale chart is handled by using the following kind of observation.

Proposition 3.5. *Let $\mathcal{G}$ an R-group scheme acting on an R-scheme $\mathcal{Z}$. For $\gamma : \mathcal{U} \to \mathcal{Z}$ an étale morphism, there exists an action of the formal group scheme $\widehat{\mathcal{G}}$ on $\widehat{\mathcal{U}}$, compatible with the induced action on $\widehat{\mathcal{Z}}$.*

4 Affine structure

The skeleton $\mathrm{Sk}(\mathcal{X})$ is not just a topological space, but inherits from its embedding in X^{an} a canonical *integral affine structure*, which also encodes the multiplicities of the components of the special fiber. This structure can be described in two different ways: one intrinsic in terms of rational functions on X, and one using the Δ-complex structure on $\mathrm{Sk}(\mathcal{X})$ coming from $\Delta(\mathcal{X}_k)$ through the homeomorphism Φ. We use the second perspective as our definition.

Definition 4.1. *Let $U \subset \mathrm{Sk}(\mathcal{X})$ be any subset, and $f : U \to \mathbb{R}$ be a continuous function. The function f is*

- (integral) affine *if for every closed face σ of $\mathrm{Sk}(\mathcal{X})$, we can cover $\sigma \cap U$ by open sets V such that f_V is an affine function with coefficients in $\mathbb{Z}$ in the variables w_i/N_i with w_i the barycentric coordinates of σ. Note that, in many cases (say U open and $\sigma \cap U$ connected), there is no need to pass to a cover and the whole of $f_{\sigma\cap U}$ has the required shape.*
- (integral) piecewise affine *if we can cover each face of $\mathrm{Sk}(\mathcal{X})$ by finitely many polytopes P such that the vertices of P have rational barycentric coordinates and f_P is affine.*

We now come to the second description.

Proposition 4.2. *Let h be a non-zero rational function on X. Then the function*

$$f_h : \mathrm{Sk}(\mathcal{X}) \to \mathbb{R}, z \mapsto v_z(h)$$

is piecewise affine. Conversely, a function $f : U \to \mathbb{R}$ is piecewise affine if it is possible to find an open cover of U by $V \subset U$ such that the f_V can be written as $(f_h)_V$ for some rational function h.

Proof. Let us prove that f_h is piecewise affine. Let σ be a face of $\mathcal{X}_k$ corresponding to a generic point ξ. We can reduce easily to the case where h lies in $\mathcal{O}_{\mathcal{X},\xi}$. Choose an admissible expansion of h with respect to ξ:

$$h = \sum_\beta c_\beta T^\beta$$

Then for every $z \in \sigma$, we can compute

$$v_z(h) = \min\{\sum_i v_z(T_i)\beta_i | c_\beta \neq 0\}.$$

This is a minimum of integral affine functions, hence a concave integral piecewise affine function. ☐

The proof actually establishes some more precise properties of f_h.

Corollary 4.3. *Let h be a non-zero rational function on X, and σ a face of $\mathcal{X}_k$ corresponding to a generic point ξ of an intersection of irreducible components of $\mathcal{X}_k$. The function $(f_h)_\sigma$ is*

- *concave if ξ is not contained in the closure of the locus of poles of h on X.*
- *convex if ξ is not contained in the closure of the locus of zeros of h on X.*
- *affine if ξ is not contained in the closure of the locus of zeros and poles of h on X.*

Another corollary is

Corollary 4.4. *Let $\mathcal{X}$ and $\mathcal{X}'$ be two sncd models of X. Then the map*

$$\rho_{\mathcal{X}} : \widehat{\mathcal{X}}_\eta \cap \mathrm{Sk}(\mathcal{X}') \to \mathrm{Sk}(\mathcal{X})$$

is compatible with the piecewise affine structures. As a corollary, the notion of piecewise affine function is independent of the model, in a suitable sense.

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