SEMINAR WS 16/17
BERKOVICH SPACES, BIRATIONAL GEOMETRY AND MOTIVIC ZETA FUNCTIONS
Talk II on Birational geometry and MMP

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Introduction

The goal of this talk is to present the notions that have to do with the Minimal model Program and will be used in the course of the seminar, as well as the main theorems of this area. For this we rely entirely on the references listed in the end.

Given a projective variety X defined over an algebraically closed field k, one could try to find another projective variety birationally isomorphic to X which is the simplest possible. We define on the set C_X of all (isomorphism classes of) projective varieties birationally isomorphic to X a relation as follows: if X_0, X_1 are in C_X , we write $X_0 \succcurlyeq X_1$, whenever there is a birational morphism $X_0 \to X_1$. This defines an ordering on C_X and we look for minimal elements in C_X or for the smallest element of it.

For example, each irreducible curve is birational to a unique smooth projective curve, thus the investigation of smooth projective curves is equivalent to the study of all curves up to birational equivalence.

When X is a smooth surface, it has a smooth minimal model obtained by contracting all exceptional curves on it. If X is not uniruled, this minimal model has nef canonical divisor and is the smallest element in c_X. When X is uniruled, this minimal model is not unique.

Smooth projective varieties with nef canonical bundles are minimal in the above sense:

Proposition. *Let X, Y be smooth proj. varieties and $\pi : X \to Y$ a birational morphism which is not an isomorphism. Then there exists a rational curve C on X contracted by π such that $(K_X.C) < 0$. In particular, if K_X is nef, X is a minimal element in C_X.*

Starting from X, Mori's idea is to try to simplify X by contracting K_X -negative extremal rays, hoping to end up with a variety X_0 which either has a fiber contraction (in which case X_0 and hence X is covered by rational curves) or has nef canonical divisor (hence no K_{X_0}- negative extremal rays). However, three problems can arise:

- The end-product of a contraction is usually singular. This means that to continue Mori's program, we must allow singularities.
- One has to determine the kind of singularities allowed. Whichever choices we make, the singularities of the target of a small contraction are too severe and one needs to perform a flip. So Problem: Existence of flips!
- One needs to know that this process terminates.

The first two problems have been overcome, but the second is still open in full generality.

Notation-Terminology

1. A **$\mathbb{Q}$-divisor** on a normal scheme X is a formal linear combination $D = \sum d_i D_i$ of irreducible, codimension one subschemes, with $d_i \in \mathbb{Q}$. D is **$\mathbb{Q}$ - Cartier** if some integral multiple of it is Cartier (locally defined by a regular function). X is called **$\mathbb{Q}$-factorial** if every $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier.
2. For a birational morphism $f : X \dashrightarrow Y$, the **exceptional set** $Ex(f) \subset X$ is the set of points $\{x \in X\}$ where f is not biregular, that is f^{-1} is not a morphism at $f(x)$. We usually view $Ex(f)$ as a subscheme with the induced reduced structure.

   In other words, if $\Sigma \subset X$ is the smallest closed subset of X outside of which $f : (Y \setminus f^{-1}(\Sigma)) \to (X \setminus \Sigma)$ is an isomorphism, the exceptional set is defined to be $E = f^{-1}(\Sigma)$.
3. A **resolution** of a scheme X is a proper birational morphism $g : Y \to X$ with Y smooth.
4. Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on X. A **log resolution** of (X, D) is a proper birational morphism $g : Y \to X$ such that Y is smooth, $Ex(g)$ is a divisor and $Ex(g) \cup g^{-1}(SuppD)$ is a snc divisor. Log resolutions exist for varieties over a field of characteristic zero.

Singularities

In this part we review some of the main notions concerning singularities that come up during the process of finding minimal models.

Definition. ***Discrepancy*** *is a way to measure how singular a variety is:*

Let (X, Δ) be a pair where X is a normal variety and $\Delta = \sum a_i D_i$ is a sum of distinct prime divisors. (We allow the a_i to be arbitrary rational numbers.) Assume that $m(Kx + \Delta)$ is Cartier for some $m > 0$. Suppose $f : Y \dashrightarrow X$ is a birational morphism from a normal variety Y. Let $E \subset Y$ denote the exceptional locus of f and $E_i \subset E$ the irreducible exceptional divisors. We can write using numerical equivalence

$$K_Y \equiv f^*(K_X + \Delta) + \sum a(E_i, X, \Delta) E_i$$

The numbers $a(E, X, \Delta)$ are called the discrepancy of the divisor wrt (X, Δ).

The discrepancy of (X, Δ) is defined as the $inf\{a(E, X, \Delta) : E$ is an exceptional divisor over $X\}$. (where $E \subset Y$ runs through all the irreducible exceptional divisors for all birational morphisms to X and through all the irreducible divisor on X.)

Definition. *Let (X, Δ) as before with $m(Kx + \Delta)$ is Cartier for some $m > 0$.*

(X, Δ) is:

- ***terminal*** *if $discrep(X, \Delta) > 0$.*

  For $\Delta = 0$ this is the smallest class that is necessary to run the minimal model program for smooth varieties.

- ***canonical*** *if $discrep(X, \Delta) \geq 0$*

  Assuming $\Delta = 0$, these are precisely the singularities that appear on the canonical models of varieties of general type.

- ***kawamata log terminal*** *(klt) if $discrep(X, \Delta) > -1$ and $\lfloor \Delta \rfloor := \sum \lfloor d_i \rfloor D_i \leq 0$.*

  *If $\Delta = 0$, then it is called **log terminal** (lt).*

- ***log canonical*** *if $discrep(X, \Delta) \geq -1$*

- ***divisorial log terminal*** *(dlt) if there exists a log resolution $f : X' \to X$ such that $a(E_i, X, \Delta) > -1$ for every exceptional divisor $E_i \subset X'$.*

Definition. *Let $E \subset Y$ be a prime divisor over X.*

*The closure of the image of E on X is called the **center** of E on X.*

*If (X, Δ) is a pair as before, there exists the largest Zariski open set U of X such that the pair is lc on U. We set $Nlc(X, \Delta) = X \setminus U$. If $a(E, X, \Delta) = -1$ and the center of E is not contained in Nlc, then the center of E is called an **lc center** of (X, Δ).*

Definition. *Let (X, Δ) be an lc pair, $Z \subset X$ a closed subscheme and D an effective $\mathbb{Q}$-Cartier divisor on X. The **log canonical threshold** (or lc-threshold) of D along Z with respect to (X, Δ) is the number*

$$c_{Z(X,\Delta,D)} := sup\{c | (X, \Delta + cD) \text{ is lc in an open neighborhood of } Z\}.$$

The following proposition ([5]) motivates an alternative definition of the log canonical threshold, which will come up and be used in a later talk.

Proposition. *Let (X, Δ) be an lc pair, $Z \subset X$ a closed subscheme and D an effective $\mathbb{Q}$-Cartier divisor on X. Let $h : Y \to X$ be a proper birational morphism. Recall that we can write*

$$K_Y \equiv h^*(K_X + \Delta) + \sum a_i E_i, \text{ and } h^*D = \sum b_i E_i.$$

Then:

$$c_Z(X, \Delta, D) \leq \min_{i: h(E_i) \cap Z \neq \emptyset} \left\{ \frac{a_i + 1}{b_i} \right\}$$

One can indeed define the notion of threshold as in (NX15):

Let k be a field of char. 0 and X a smooth k -variety, Δ an effective $\mathbb{Q}$ -divisor on X, v a divisorial valuation with center contained in Δ, i.e. it is a real valuation on the function fiels $k(X)$ and there exists a birational morphism of k- varieties $h : Y \to X$, with Y normal and a prime compontent E of $h^*\Delta$ s.t. v is a real multiple of the valuation ord_E associated to E.

Then one can make the equivalent definition for $x \in X$ as in ([8])

$$lct_x(X, \Delta) := \inf_v \left\{ \frac{\text{multiplicity of} K_{Y/X} \text{ along } E + 1}{\text{multiplicity of } h^*\Delta \text{ along } E} \right\}$$

Minimal Model Program

Cone of curves

Let X be a variety. It will be convenient to define a curve on X as a morphism $\rho : C \to X$, where C is a smooth (projective) curve. Given any, possibly singular, curve in X, one may consider its normalization as a "curve on X." For any Cartier divisor D on X, we set $(D.C) := deg(\rho^* D)$, where for a divisor $E = \sum n_p p$ on a curve C we define the degree of E in C as $\sum n_p dim_k(\mathcal{O}_{C,p}/m_{C,p})$

This definition extends to $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors, but this intersection number is then only a rational number in general.

Definition. *We say that two divisors D, D' on a proper scheme are **numerically equivalent** if $(D.C) = (D'.C)$ for every curve C inside X.*

The quotient of the group of Cartier divisors by this equivalence class is denoted by $N^1(X)_{\mathbb{Z}}$. Set $N^1(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Z}} \otimes \mathbb{Q}$ and $N^1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$. These spaces are finite dimensional vector spaces and their dimension is called the Picard number of X.

Definition. *Analogously, for two 1-cycles C, C' on X we define the equivalence relation $C \sim C'$ iff they have the same intersection number with every divisor* *on X. We denote the quotient group by $N_1(X)_{\mathbb{Z}}$ and the respective $\mathbb{Q}, \mathbb{R}$ vector spaces by $N_1(X)_{\mathbb{Q}}$ and $N_1(X)_{\mathbb{R}}$. We have a non degenerate intersection pairing $N^1(X)_{\mathbb{Z}} \times N_1(X)_{\mathbb{Z}} \to \mathbb{Z}$.*

*An element $[C] \in \mathbb{R} \otimes Z_1(X)$ can be represented by $\sum_i a_i C_i$, with $a_i \in \mathbb{R}, C_i$ irreducible curves. It is called effective if all $a_i \geq 0$. A class $[C]$ in $N_1(X)_{\mathbb{R}}$ is called effective if its numerical equivalence class contains one effective 1-cycle. The **cone of curves** NE(X) is the subset of effective 1-cycles in $N_1(X)_{\mathbb{R}}$.*

Definition. *If $\pi : X \to Y$ morphism of projective varieties, we define the relative cone of curves as the convex subcone $NE(\pi) \subset NE(X)$ that is generated by the classes of curves that are contracted by π.*

Now let X be a normal projective irreducible variety.

Definition. *A line bundle on X is called **very ample** if $L = \phi^* \mathcal{O}_{\mathbb{P}^n}(1)$ for some embedding $\phi : X \hookrightarrow \mathbb{P}^n$.*

*It is called **ample** if a positive multiple of L is very ample.*

*A divisor on X is **ample** if the line bundle associated to a sufficiently divisible, positive multiple of it is ample. This is also a numerical property.*

*The tensor product of two ample line bundles is ample. Moreover, the tensor product of any line bundle with a sufficiently high multiple of an ample line bundle is ample. Hence, the classes of ample divisors form an open, convex cone, called the **ample cone** in $N^1(X)$.*

*A divisor is called **NEF** if $(D.C) \geq 0$ for every irreducible curve C in X. The set of NEF divisors also forms a closed convex cone in $N^1(X)$ called the NEF cone. It contains the ample cone.*

For L a line bundle on X, the semigroup $N(X, L)$ is defined as:

$$N(X, L) := \{m \geq 0 : h^0(X, L^{\otimes m}) > 0\}$$

*To $m \in N(X, L)$ we associate ϕ_m the map associated to $L^{\otimes m}$. The Itaka dimension is the maximum dimension of the image of ϕ_m for $m \in N(X, L)$. If $N(X, L) = 0$ then we set it as $-\infty$. L is called **big** if its Itaka dimension is equal to dim(X). Note that in general, $0 <$ Itaka dim $\geq dim(X)$. Such line bundles form the big cone in N^1.*

*The closure of the big cone consists of all divisor classes that are limits off effective divisor classes. This closed, convex cone is the **pseudoeffective cone**.*

To sum up we get this picture concerning the cones:

$$Amp(X) \subset NEF(X) \subset Psef(X) \subset N^1(X)_{\mathbb{R}}$$

$$Amp(X) \subset Big(X) \subset Eff(X) \subset Psef(X)$$

Denote the closure of the convex cone $NE(X)$ of classes of effective 1-cycles by $\overline{NE(X)}$.

Lemma (Negativity Lemma)**.** *Let $h : Y \to X$ a proper birational morphism between normal varieties. Let $-B$ be an h-nef $\mathbb{Q}$-Cartier divisor on Y. (Being h-nef means B has non-positive intersection with every curve contracted by h.)*

Then B is effective iff h_*B is and if B is effective, then $\forall x \in X, h^{-1}(x) \subset Supp(B)$ or $h^{-1}(x) \cap Supp(B) = \emptyset$.

Theorem (Base point free theorem)**.** *Let (X, Δ) be a klt pair and let L be a nef Cartier divisor on X. Assume that there exists $p > 0$ such that $pL - (K_X + \Delta)$ is nef and big. Then the linear system $|nL|$ is basepoint free for all $n \geq 0$.*

Definition. *For any $\mathbb{Q}$ or $\mathbb{R}$ - vector space V, and $N \subset V$ a cone (has 0 and is closed under multiplication by positive scalars) we have:*

*A subcone $M \subset N$ is called extremal if $u, v \in N, u + v \in M \Rightarrow u, v \in M$. If M has dimension one, it is called an **extremal ray**. An extremal ray M (or face if it's higher dimensional) is called $(K_X + B)$-negative if $M \cap \overline{NE(X)}_{K_X+B\geq 0} = \{0\}$*

Mori's cone theorem shows that $\overline{NE(X)}$ is generated by countably many extremal rays and that each one of these is generated by classes of rational curves. They only accumulate on the plane $K_X = 0$. Formulated for dlt pairs:

Theorem (Cone theorem [2])**.** *Let (X, Δ) be a dlt pair over Z. Then there are $(K_X + \Delta)$- negative rational curves $C_i \subset X$ such that*

$$\overline{NE(X/Z)} = \overline{NE(X/Z)}_{(K_X+\Delta)\geq 0} + \sum_{finite} R_i$$

where the $(K_X + \Delta)$- negative extremal rays R_i are spanned by the classes of C_i and locally discrete in $\overline{NE(Z)}_{(K_X+\Delta)<0}$.

It is central to the realization of mori's program that rays can be contracted.

Theorem (Contraction theorem)**.** *Let (X, Δ) be a dlt pair over Z and F a $(K_X + \Delta)$- negative extremal ray.*

Then there is a morphism $contF : X \to Y$ to a projective variety such that $(contF)_*\mathcal{O}_X = \mathcal{O}_Y$ and an irreducible curve $C \subset X$ is mapped to a point by contF iff $[C] \in F$.

*Then contF is called the **contaction of** F.*

Sketch of proof. [3]. The proof is based on the Basepoint free theorem. One can prove that the subcone

$$V = \overline{NE(X)}_{K_X+\Delta\geq 0} + \sum_{F' \in \mathcal{F}\setminus\{F\}} F'$$

for $\mathcal{F}$ the set of all $(K_X + \Delta)$-negative extremal rays, is closed, hence there exists a nef $\mathbb{R}$-divisor M that is positive on $V \setminus \{0\}$ but vanishes at some nonzero point of $\overline{NE(X)}$, which must be in F. One can assume M has integral coefficients and show that $mM - (K_X + \Delta)$ is ample for all m sufficiently large. By the Basepoint-free theorem, any sufficiently large multiple of M defines a morphism, whose Stein factorization is the contraction of F. $\square$

There are three types of contractions that appear:

Definition. *Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair, and let $\phi : X \to Y$ be the contraction of a $(K_X + \Delta)$-negative extremal ray F.*
*1. If $dimY < dimX$, then ϕ is a **Mori fibration**;*
*2. If $dimY = dimX$ and $Exc(\phi)$ is an irreducible divisor, then ϕ is a **divisorial contraction**;*
*3. If $dimY = dimX$ and $codim_X\ Exc(\phi) \geq 2$, i.e. ϕ is **small**, then ϕ is a **flipping contraction**.*

If $\phi : X \to Y$ is a small extremal contraction, Y is not $\mathbb{Q}$-factorial, in fact may be disconnected. In higher dimensions, a surgery operation in codimension 2 is necessary to proceed with the MMP. This operation is the flip of ϕ.

Definition. *[2] Let $\phi : X \to Y$ be a flipping contraction. A **flip** of ϕ is a commutative diagram*

$$\begin{array}{ccc} (X, \Delta_X) & \longrightarrow & (X^+, \Delta^+) \\ & \searrow^{\phi} \quad \swarrow_{\phi^+} & \\ & Z & \end{array}$$

with ϕ^+ is small and projective and $K_{X^+} + \Delta_{X^+}$ is ϕ^+-ample, with $\Delta_{X^+} = ((\phi^+)^{-1} \circ \phi)_* \Delta_X$

It is conjectured that there does not exist an infinite sequence of flips, although we have for example a bound to the number to divisorial contractions.

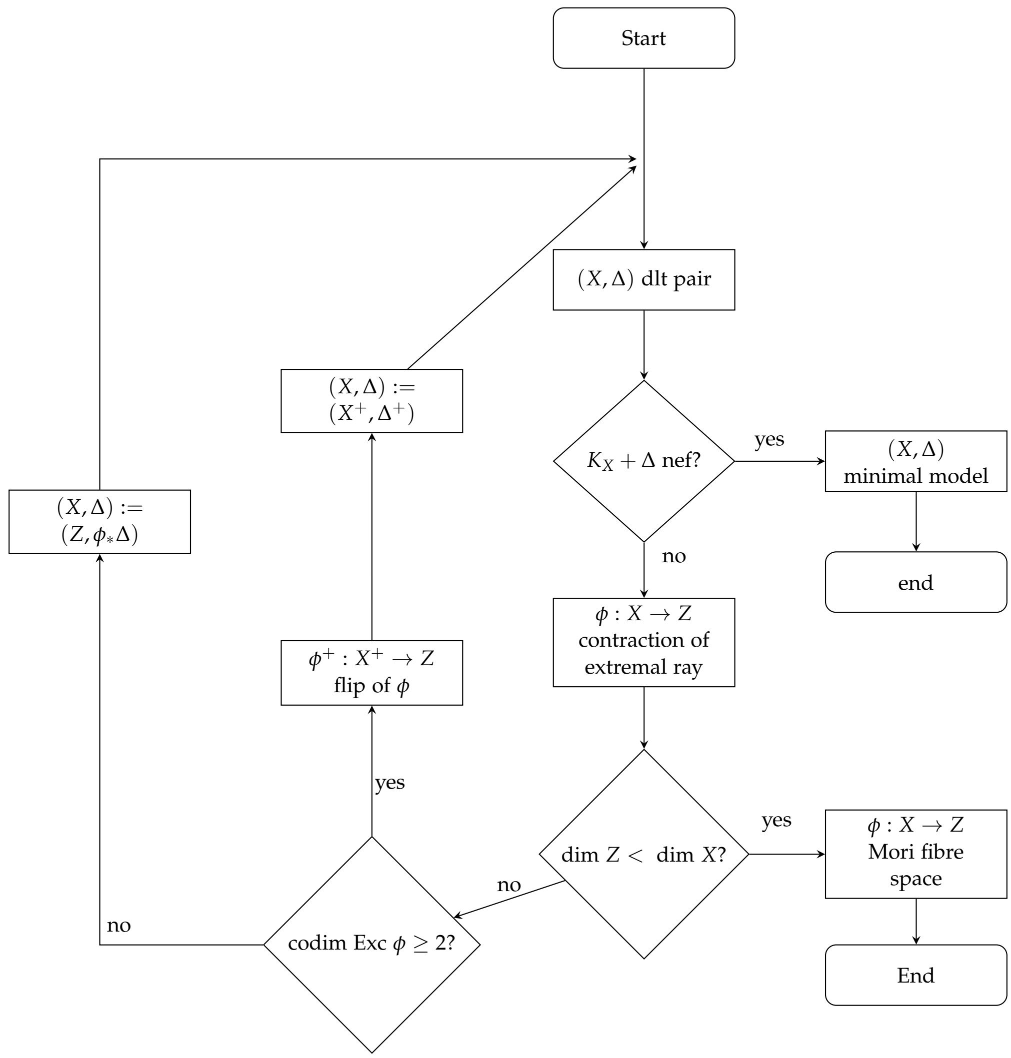
Relative MMP

We present the Minimal Model Program (MMP) in a relative setting, that is, given a birational projective morphism $f : X \to Z$ we consider the MMP over Z. Let (X, Δ) be a dlt pair over Z. If existence and termination of flips hold, there is a sequence of birational maps

$$X = X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_n = X'$$

over Z, where (X', Δ) is either a minimal model (i.e. $K_X + \Delta$ is nef) or a Mori fibre space, i.e. $K_X + \Delta$ is not nef and there is a Mori fibration $(X, \Delta) \to Y$.

The sequence is obtained as presented in the following flowchart(as found in [2]):



Definition. *Let $(X, \Delta + A)$ be a $\mathbb{Q}$- factorial dlt pair over Z such that $K_X + \Delta + A$ is nef. A $(K_X + \Delta)$-**MMP with scaling of** A is a sequence*

$$(X_q, l_1) \dashrightarrow^{\phi_1} \dots (X_i, l_i) \dashrightarrow^{\phi_i}$$

where for each i, A_i, Δ_i are the strict transforms of A and Δ on X_i and $l_i := min\{t \in \mathbb{R} : K_{X_i} + \Delta_i + tA_i \text{ is nef}\}$.

Each ϕ_i is a divisorial contraction or a flip associated to a $(K_{X_i} + \Delta_i)$-negative extremal ray $R_i \subset \overline{NE(X_i/Z)}$ which is $K_{X_i} + \Delta_i + l_i A_i$-trivial.

Theorem. *[1] Let $(X, \Delta + A)$ be a klt pair projective over Z, where Δ is big over Z. If there is an effective divisor D such that $K_X + \Delta \sim_{\mathbb{R},Z} D$, then (X, Δ) has a log* *terminal model over Z.*

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