ON A VANISHING THEOREM OF S. SAITO AND K. SATO

by

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Introduction

This text is an expanded version of a talk given for the Winter research seminar *Chow groups of zero cycles over p-adic fields* that I co-organized with Kay Rülling. The main reference for this seminar was [**SS10**]. We give here a detailed account of Saito-Sato's proof of the vanishing theorem [**SS10**, 3.2].

0.1. Let B be an henselian discrete valuation ring. Let $s = \operatorname{Spec} k$ be the closed point of B, and $i_s : s \longrightarrow B$ the canonical inclusion(1). We suppose k to be separably closed or finite. For a prime number ℓ invertible on B, we denote $\Lambda := \mathbb{Z}/\ell^n\mathbb{Z}$ and $\Lambda_\infty = \varinjlim \mathbb{Z}/l^k\mathbb{Z}$.

0.2. Let (X, Y) be a $\mathcal{QS}$-pair(2), f (resp. g) the structural morphism of X (resp. Y) over B, and $\dim X = d+1$. If one defines $Z = X_{s,\mathrm{red}}$ and $W = Y_{s,\mathrm{red}}$, we have a cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{\kappa'} & Z \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow[\kappa]{} & X \end{array}$$

with W an effective Cartier divisor of Z. Let $j : U = X \setminus Y \longrightarrow X$ be the canonical open inclusion and let $h : U \longrightarrow B$ be the structural morphism of U.

(1)Note that in [**SS10**], the residue field of s is denoted by F and the function field of B is denoted by k.

(2)See [**SS10**, 1.15] and [**SS10**, 3.1] for the terminology.

In parts 1.1 and 1.2, we motivate Saito-Sato's vanishing theorem 1.2.5 (1), (2) and explain why it does not follow from known results on the étale cohomology of schemes. Let us consider the following cartesian diagram

$$\begin{array}{ccc} U_s & \longrightarrow & U \\ {\scriptstyle h_s}\downarrow & & \downarrow{\scriptstyle h} \\ s & \underset{i_s}{\longrightarrow} & B \end{array}$$

We explain in part 1.3 why Saito-Sato's vanishing theorem is equivalent to the fact that the (non proper and non smooth !) base change morphism

$$i_s^* R^q h_* \Lambda_U(a) \longrightarrow R^q h_s \Lambda_{U_s}(a)$$

is an isomorphism. Using further dévissages relying on the proper base change theorem, we reduce in part 2.1 the proof of this base change statement to the proof of the fact that the Gysin morphism

$$\mathrm{Gys}_{\kappa'} : \Lambda_W(-1)[-2] \longrightarrow R\kappa'^! \Lambda_Z$$

is an isomorphism. Since Z and W are not regular in general, Gabber's purity theorem does not apply to this situation, so some extra work carried out in 2.2 is needed. The key device for that is a resolution (2.2.3) of Λ_Z in terms of push-forwards of sheaves living on the iterated intersections of the irreducible components of Z, which are regular by assumption on Z. Thus, any cohomological statement on Λ_Z and its Tate twists can be reduced via the hypercohomology spectral sequence to an analogous statement on a regular scheme.

In part 3, we give a detailed account of Saito-Sato's refinement 1.2.5 (3) of 1.2.5 (2) in the case where the base dvr has finite residue field. The key to this refinement is a beautiful weights argument relying on [**Del80**, Th I].

In the proof of [**SS10**, 3.2], Saito and Sato use several times the compatibility with pull-back of the fundamental class to deduce the commutativity of squares involving Gysin morphisms. In (the purely technical) part 4, we prove such a commutativity along these lines. We thank Kay Rülling for helping us to demystify this commutativity check.

1. Motivation

1.1. Saito-Sato prove in [**SS10**] the following

Theorem 1.1.1. — *If $X \in \mathcal{QSP}$ with $\dim X \geq 2$ and if k is separably closed or finite, the cycle class map*

$$\rho_X : \mathrm{CH}_1(X) \otimes \Lambda \longrightarrow H_2(X, \Lambda)$$

is an isomorphism. The same statement holds if Λ is replaced by Λ_∞.

The proof of the surjectivity uses a recursion on $\dim X$, the case where $\dim X \geq 2$ being treated with specific arguments. The two ingredients of the recursion are

(1) A Bertini theorem according to which there exists a closed subscheme Y in X such that (X,Y) is an ample $\mathcal{QSP}$-pair.

(2) The vanishing of $H_2(U,\Lambda_U)$ for $\dim X = d+1 > 2$.

In what follows, we explain why (2) doesn't follow immediately from known results on the cohomology of schemes.

1.2. By purity [**ILO14**, XVI 3.1.1], we know from [**SS10**, 1.8] that there is a canonical identification

$$H_2(U,\Lambda_U) \xrightarrow{\sim} H^{2(d+1)-2}(U,\Lambda_U(d)) \tag{1.2.1}$$

So we will work with cohomology from now on.

The Leray spectral sequence for the structural morphism $h : U \longrightarrow B$ reads

$$E_2^{pq} = H^p(B, R^q h_* \Lambda_U(d)) \Longrightarrow H^{p+q}(U,\Lambda_U(d)) \tag{1.2.2}$$

Since B is local noetherien henselian excellent, Gabber's version of Lefschetz affine theorem [**ILO14**, XVI 1.1.2] says that $R^q h_*\Lambda(d) \simeq 0$ for $q > d+1$. Since B is an henselian trait with perfect residue field, we know from [**Maz74**] that B and k have the same cohomological dimension. If k is separably closed, (1.2.2) degenerates in page 2 and we have canonical isomorphisms

$$H^q(U,\Lambda_U(d)) \xrightarrow{\sim} H^0(B, R^q h_*\Lambda_U(d)) \tag{1.2.3}$$

for every q. In particular, the right hand side of (1.2.3) is zero for $q = 2(d+1)-2$ if $2(d+1)-2 > d+1$, that is $d+1 > 2$, so condition (2) is always satisfied. If k is finite, k has cohomological dimension 1, so for $q = 0, \ldots, d+1$ there is an exacte sequence (1.2.4)

$$0 \longrightarrow H^1(B, R^{q-1}h_*\Lambda_U(d)) \longrightarrow H^q(U,\Lambda_U(d)) \longrightarrow H^0(B, R^q h_*\Lambda_U(d)) \longrightarrow 0$$

and an identification

$$H^{d+2}(U,\Lambda_U(d)) \simeq H^1(B, R^{d+1}h_*\Lambda_U(d))$$

and the vanishing

$$H^q(U,\Lambda_U(d)) \simeq 0$$

for $q > d+2$. So $H^q(U,\Lambda_U(d)) \simeq 0$ for $q = 2(d+1)-2$ if $2(d+1)-2 > d+2$, that is $d+1 > 3$. Thus there is some extra work to be done in the case where k is finite and $\dim X = 3$.

Saito-Sato prove more generally the following

Theorem 1.2.5. — *Let (X,Y) be an ample $\mathcal{QSP}$-pair with $\dim X = d+1 \geq 2$.*

(1) *If k is separably closed, $H^q(U,\Lambda_U(d)) \simeq 0$ for $q \geq d+1$*(3)*.*

(2) *If k is finite, $H^q(U,\Lambda_U(d)) \simeq 0$ for $q \geq d+2$*(4)*.*

(3) *If k is finite, and if $d+1 \geq 3$, then $H^{d+1}(U,\Lambda_\infty(d)) \simeq 0$.*

(3) That is $H_q(U,\Lambda_U) \simeq 0$ for $q \leq d+1$.

(4) That is $H_q(U,\Lambda_U) \simeq 0$ for $q \leq d$.

1.3. Reformulation of 1.2.5 (1) and (2) in terms of base change. — Since B is an henselian trait with perfect residue field, we know from [**Maz74**] that for every abelian sheaf $\mathcal{F}$ on $B_{\text{ét}}$, the canonical restriction morphism

$$H^q(B, \mathcal{F}) \longrightarrow H^q(s, i_s^* \mathcal{F})$$

is an isomorphism. The spectral sequence (1.2.2) can thus be rewritten as

$$E_2^{pq} = H^p(s, i_s^* R^q h_* \Lambda_U(d)) \Longrightarrow H^{p+q}(U, \Lambda_U(d)) \tag{1.3.1}$$

If one considers the cartesian diagram

$$\begin{array}{ccc} U_s & \longrightarrow & U \\ {\scriptstyle h_s}\downarrow & & \downarrow{\scriptstyle h} \\ s & \xrightarrow[i_s]{} & B \end{array}$$

we have the following base change morphism [**AGV73**, XII 4.2]

$$i_s^* R^q h_* \Lambda_U(a) \longrightarrow R^q h_s \Lambda_{U_s}(a) \tag{1.3.2}$$

for every $a, q \in \mathbb{Z}$. In the conditions of 1.2.5, if (1.3.2) is an isomorphism, one can apply the affine Lefschetz theorem [**AGV73**, XIV 3.1] to U_s/k, where $\dim U_s = d$, and 1.2.5 (1), (2) follow immediately via (1.3.1).

2. The base change map (1.3.2) is an isomorphism

In what follows, we suppose that (X, Y) is a $\mathcal{QSP}$-pair without making any assumption on U, and we show that (1.3.2) is an isomorphism.

2.1. A dévissage. — We start with the following local cohomology triangle on X

$$\kappa_* R\kappa^! \Lambda_X(a) \longrightarrow \Lambda_X(a) \longrightarrow Rj_* \Lambda_U(a) \tag{2.1.1}$$

By applying $i_s^* Rf_*$, we obtain the following triangle on $s_{\text{ét}}$

$$i_s^* Rg_* R\kappa^! \Lambda_X(a) \longrightarrow i_s^* Rf_* \Lambda_X(a) \longrightarrow i_s^* Rh_* \Lambda_U(a)$$

On the other hand, we have the following local cohomology triangle on Z

$$\kappa'_* R\kappa'^! \Lambda_Z(a) \longrightarrow \Lambda_Z(a) \longrightarrow Rj_{s*} \Lambda_{U_s}(a) \tag{2.1.2}$$

There is a morphism of triangles $i_s^* Rf_*(2.1.1) \longrightarrow Rf_{s*}(2.1.2)$
(2.1.3)

$$\begin{array}{ccccc} & i_s^* Rg_* R\kappa^! \Lambda_X(a) & \longrightarrow & i_s^* Rf_* \Lambda_X(a) & \longrightarrow & i_s^* Rh_* \Lambda_U(a) \\ \swarrow^{(a)} & & & \downarrow (b) & & \downarrow \\ Rg_{s*} i'^* R\kappa^! \Lambda_X(a) & & & & & \\ \searrow & & & & & \\ & Rg_{s*} R\kappa'^! \Lambda_Z(a) & \longrightarrow & Rf_{s*} \Lambda_Z(a) & \longrightarrow & Rh_{s*} \Lambda_{U_s}(a) \end{array}$$

so that (a) is obtained by applying Rg_{s*} to the base change morphism [**AGV73**, XVIII 3.1.14.2]

$$i^* R\kappa^! \Lambda_X(a) \longrightarrow R\kappa'^! \Lambda_Z(a) \tag{2.1.4}$$

By proper base change theorem [**AGV73**, XII 5.1], the morphisms (a) and (b) are isomorphisms[(5)]. Thus, to show that (1.3.2) is an isomorphism, it is enough to show that (2.1.4) is an isomorphism.

We have the following commutative square

$$\begin{array}{ccc} i'^* \Lambda_Y(a-1)[-2] & \xrightarrow{i'^* \operatorname{Gys}_\kappa} & i'^* R\kappa^! \Lambda_X(a) \\ \downarrow \wr & & \downarrow \\ \Lambda_W(a-1)[-2] & \xrightarrow[\operatorname{Gys}_{\kappa'}]{} & R\kappa'^! \Lambda_Z(a) \end{array}$$

The schemes X and Y are regular by hypothesis. By purety [**ILO14**, XVI 3.1.1], we deduce that $\operatorname{Gys}_\kappa$ is an isomorphism. Thus to show that (2.1.4) is an isomorphism, it is enough to show the following

Proposition 2.1.5. — *The Gysin morphism*

$$\operatorname{Gys}_{\kappa'} : \Lambda_W(-1)[-2] \longrightarrow R\kappa'^! \Lambda_Z$$

is an isomorphism.

The proposition 2.1.5 does not follow immediately from Gabber's purity theorem since W and Z are not regular in general. However, the irreducible components of W and Z are regular.

2.2. Proof of 2.1.5. — In this paragraph, we suppose that (X, Y) is a $\mathcal{QS}$-pair, without any projectivity hypothesis.

Let $Z_1, \ldots, Z_m$ be the irreducible components of Z. For $I \subset [\![1, m]\!]$, we denote

(5) Note that this is will be the only use of the projectivity assumption on X.

$Z_I := \cap_{i\in I} Z_i$. By assumption on Z, the scheme Z_I is empty or regular of dimension $\dim X - |I|$. If $q \in [\![1,m]\!]$, we define

$$Z^{(q)} = \bigsqcup_{\substack{I\subset [\![1,m]\!] \\ |I|=q}} Z_I$$

Let $u_q : Z^{(q)} \longrightarrow Z$ be the canonical morphism. It is finite. We do the same with W and obtain a cartesian diagram

$$\begin{array}{ccc} W^{(q)} & \xrightarrow{\kappa'^{(q)}} & Z^{(q)} \\ {\scriptstyle v_q}\downarrow & & \downarrow{\scriptstyle u_q} \\ W & \xrightarrow[\kappa']{} & Z \end{array} \tag{2.2.1}$$

For every (I,J) with $I \subset J \subset [\![1,m]\!]$, the canonical inclusion $u_{I,J} : Z_J \longrightarrow Z_I$ induces an adjunction morphism

$$\Lambda_{Z_I} \longrightarrow u_{I,J*}\Lambda_{Z_J} \tag{2.2.2}$$

If we define $p = \sharp I$ and $q = \sharp J$ and if we note $\iota_I : Z_I \longrightarrow Z^{(p)}$ the canonical inclusion, we have the following commutative diagram

$$\begin{array}{ccc} Z_J & \xrightarrow{u_{I,J}} & Z_I \\ {\scriptstyle \iota_J}\downarrow & & \downarrow{\scriptstyle \iota_I} \\ Z^{(q)} & & Z^{(p)} \\ & {\scriptstyle u_q}\searrow \quad \swarrow{\scriptstyle u_p} & \\ & Z & \end{array}$$

Applying $u_{p*}\iota_{I*}$ to (2.2.2), we obtain a morphism

$$d_{I,J} : u_{p*}\iota_{I*}\Lambda_{Z_I} \longrightarrow u_{q*}\iota_{J*}\Lambda_{Z_J}$$

Since

$$\Lambda_{Z^{(p)}} = \bigoplus_{\substack{I\subset [\![1,m]\!] \\ |I|=p}} \iota_{I*}\Lambda_{Z_I}$$

we define

$$\begin{array}{rcl} d_p : u_{p*}\Lambda_{Z^{(p)}} & \longrightarrow & u_{p+1*}\Lambda_{Z^{(p+1)}} \\ s_I & \longrightarrow & \displaystyle\sum_{i\in [\![1,m]\!]\setminus I} (-1)^{\sharp(I\cap [\![1,i-1]\!])} d_{I,I\cup\{i\}}(s_I) \end{array}$$

So we have a complex

(2.2.3)

$$0 \longrightarrow \Lambda_Z \longrightarrow u_{1*}\Lambda_{Z^{(1)}} \longrightarrow u_{2*}\Lambda_{Z^{(2)}} \longrightarrow \cdots \longrightarrow u_{d+1*}\Lambda_{Z^{(d+1)}} \longrightarrow 0$$

We observe that $d_0 = \mathrm{adj}_{u_1}$. We have the following

Lemma 2.2.4. — *The complex* (2.2.3) *is exact.*

Proof. — Since the étale topos has enough points [**AGV72**, VIII 3.5], it is enough to prove 2.2.4 at the level of germs. Let $x \in Z$ and let $\overline{x} \longrightarrow Z$ be a geometric point of Z over x. Let $i_1 < \cdots < i_k$ be the indices for which $x \in Z_{i_j}$. For $I \subset \{i_1, \ldots, i_k\}, |I| = p$, let us denote by $\Lambda(I)$ a copy of Λ and by e_I the canonical generator of $\Lambda(I)$. Then, we have

$$(u_{p*}\Lambda_{Z^{(p)}})_{\overline{x}} \simeq \bigoplus_{\substack{I\subset\{i_1,\ldots,i_k\}\\ |I|=p}} \Lambda(I)$$

Through this identification, the morphism d_p reads

$$e_I \longrightarrow \sum_{i\in\{i_1,\ldots,i_k\}\setminus I} (-1)^{\sharp(I\cap[\![1,i-1]\!])} e_{I\cup\{i\}}$$

So the germ of (2.2.3) at $\overline{x}$ is the Koszul complex $K((1,\ldots,1),\Lambda^k)$ as defined in [**Eis95**, 17.2]. The exactness follows then immediately from [**Eis95**, 17.11] and the exactness of $K(1,\Lambda)$. □

As a consequence of 2.2.4, the sheaf Λ_Z is quasi-isomorphic to

$$0 \longrightarrow u_{1*}\Lambda_{Z^{(1)}} \longrightarrow u_{2*}\Lambda_{Z^{(2)}} \longrightarrow \cdots \longrightarrow u_{d+1*}\Lambda_{Z^{(d+1)}} \longrightarrow 0 \tag{2.2.5}$$

so the hypercohomology spectral sequence for $R\kappa'^!$(2.2.5) reads

$$E_1^{ab} = R^b\kappa'^! u_{a+1*}\Lambda_{Z^{(a+1)}} \Longrightarrow R^{a+b}\kappa'^!\Lambda_Z \tag{2.2.6}$$

Since (2.2.1) is cartesian, we have

$$R^b\kappa'^! u_{a*}\Lambda_{Z^{(a)}} \simeq \mathcal{H}^b R\kappa'^! Ru_{a*}\Lambda_{Z^{(a)}} \simeq_{bc^{-1}} \mathcal{H}^b Rv_{a*}R\kappa'^{(a)!}\Lambda_{Z^{(a)}} \simeq v_{a*}R^b\kappa'^{(a)!}\Lambda_{Z^{(a)}}$$

where the first and last identifications come from the fact that finite direct images are exact, and the second identification comes from the face that the base change morphism [**AGV73**, XVIII 3.1.12.3]

$$\mathrm{bc} : Rv_{a*}R\kappa'^{(a)!}\Lambda_{Z^{(a)}} \longrightarrow R\kappa'^! Ru_{a*}\Lambda_{Z^{(a)}}$$

is an isomorphism. Since $W^{(a)}$ et $Z^{(a)}$ are regular, purity [**ILO14**, XVI 3.1.1] implies

$$\begin{aligned} E_1^{ab} &\simeq v_{a+1*}\Lambda_{W^{(a+1)}}(-1) \quad \text{if } b = 2 \text{ and } a \geq 0 \\ &\simeq 0 \quad \text{otherwise} \end{aligned}$$

So (2.2.6) degenerates at page 2, so we deduce that for all a, $R^{a+2}\kappa'^!\Lambda_Z$ is the a-th cohomology sheaf of

(2.2.7)

$$R^2\kappa'^! u_{1*}\Lambda_{Z^{(1)}} \xrightarrow{R^2\kappa'^! d_1} R^2\kappa'^! u_{2*}\Lambda_{Z^{(2)}} \xrightarrow{R^2\kappa'^! d_2} \cdots \longrightarrow R^2\kappa'^! u_{d+1*}\Lambda_{Z^{(d+1)}}$$

where $R^2\kappa'^! u_{1*}\Lambda_{Z^{(1)}}$ sits in degree 0. In particular, $R^k\kappa'^!\Lambda_Z \simeq 0$ for $k < 2$. Moreover, the isomorphisms

$$R^2\kappa'^! u_{a*}\Lambda_{Z^{(a)}} \xrightarrow{\mathcal{H}^2 v_{a*}\operatorname{Gys}^{-1}_{\kappa'(a)} \circ \operatorname{bc}^{-1}} v_{a*}\Lambda_{W^{(a)}}(-1)$$

give rise to an isomorphism of complexes[6] between (2.2.7) and

$$0 \longrightarrow v_{1*}\Lambda_{W^{(1)}}(-1) \xrightarrow{d_1} v_{2*}\Lambda_{W^{(2)}}(-1) \xrightarrow{d_2} \cdots \xrightarrow{d_{d-1}} v_{d*}\Lambda_{W^{(d)}}(-1) \longrightarrow 0$$

Thus, $R^k\kappa'^!\Lambda_Z \simeq 0$ for $k > 2$. Finally, the square

$$\begin{array}{ccccccc}
0 & \longrightarrow & R^2\kappa'^!\Lambda_Z & \xrightarrow{R^2\kappa'^!\operatorname{adj}_{u_1}} & R^2\kappa'^! u_{1*}\Lambda_{Z^{(1)}} & \xrightarrow{R^2\kappa'^! d_1} & \\
 & & \uparrow{\scriptstyle \mathcal{H}^2\operatorname{Gys}_{\kappa'}} & & \wr\uparrow{\scriptstyle \mathcal{H}^2 bc\circ v_{1*}\operatorname{Gys}_{\kappa'(1)}} & & \\
0 & \longrightarrow & \Lambda_W(-1) & \xrightarrow[\operatorname{adj}_{v_1}]{} & v_{1*}\Lambda_{W^{(1)}}(-1) & \xrightarrow[d_1]{} &
\end{array}$$

commutes according to 4.3.2. We deduce that $\mathcal{H}^2\operatorname{Gys}_{\kappa'}$ is an isomorphism, so $\mathcal{H}^k\operatorname{Gys}_{\kappa'}$ is an isomorphism for every k, and the proposition 2.1.5 is proved.

3. Proof of 1.2.5 (3)

3.1. We suppose that k is finite and $d+1 \geq 3$. Let $\overline{k}$ be an algebraic closure for k. Let us note $G_k := \operatorname{Gal}(\overline{k}/k)$ et $\varphi : \overline{k} \longrightarrow \overline{k}$ the Frobenius of $\overline{k}$. We note $\overline{Z}$, $\overline{W}$... the base changes to $\overline{k}$ of Z, W....

3.2. We have an exact sequence of sheaves on U_{proet}

$$0 \longrightarrow \mathbb{Z}_\ell(d) \longrightarrow \mathbb{Q}_\ell(d) \longrightarrow \Lambda_\infty(d) \longrightarrow 0 \tag{3.2.1}$$

From (1.2.5) (2), we have

$$H^{d+2}(U, \mathbb{Z}_\ell(d)) \simeq 0$$

so the long exact sequence associated to $R\Gamma(U, (3.2.1))$ gives a surjection

$$H^{d+1}(U, \mathbb{Q}_\ell(d)) \longrightarrow H^{d+1}(U, \Lambda_\infty(d)) \longrightarrow 0$$

Thus, it is enough to prove that $H^{d+1}(U, \mathbb{Q}_\ell(d)) \simeq 0$. From **[Maz74]**, the exact sequence (1.2.4) for $q = d+1$ reads

$$0 \longrightarrow H^1(s, i_s^* R^d h_*\mathbb{Q}_\ell(d)) \longrightarrow H^{d+1}(U, \mathbb{Q}_\ell(d)) \longrightarrow H^0(s, i_s^* R^{d+1} h_*\mathbb{Q}_\ell(d)) \longrightarrow 0$$

(6) given some delicate commutativity checks that we have not done...

Since the base change morphism (1.3.2) is an isomorphism, and since $V := U_s$ has dimension d, we deduce from the affine Lefschetz

$$\begin{aligned} H^{d+1}(U, \mathbb{Q}_\ell(d)) &\simeq H^1(s, R^d h_{s*}\mathbb{Q}_\ell(d)) \\ &\simeq H^1(G_k, H^d(\overline{V}, \mathbb{Q}_\ell(d))) \\ &\simeq \operatorname{Coker}\left(H^d(\overline{V}, \mathbb{Q}_\ell(d)) \xrightarrow{1-\varphi} H^d(\overline{V}, \mathbb{Q}_\ell(d))\right) \end{aligned}$$

Thus, it is enough to show that the action of φ on $H^d(\overline{V}, \mathbb{Q}_\ell(d))$ has no fixed points. By applying $R\Gamma(Z, \quad)$ to the following local cohomology triangle on $\overline{Z}$

$$\kappa'_* R\kappa'^! \mathbb{Q}_\ell(d) \longrightarrow \mathbb{Q}_\ell(d) \longrightarrow Rj_{s*}\mathbb{Q}_\ell(d)$$

we obtain a long exact sequence

$$\cdots \longrightarrow H^d(\overline{Z}, \mathbb{Q}_\ell(d)) \xrightarrow{d_1} H^d(\overline{V}, \mathbb{Q}_\ell(d)) \xrightarrow{d_2} H^{d+1}_{\overline{W}}(\overline{Z}, \mathbb{Q}_\ell(d)) \longrightarrow \cdots$$

So we have an exact sequence

$$0 \longrightarrow H^d(\overline{Z}, \mathbb{Q}_\ell(d))/\operatorname{Ker} d_1 \longrightarrow H^d(\overline{V}, \mathbb{Q}_\ell(d)) \longrightarrow \operatorname{Im} d_2 \longrightarrow 0$$

Thus, it is enough to prove the

Lemma 3.2.2. — *The action of φ on*

$$H^d(\overline{Z}, \mathbb{Q}_\ell(d)) \quad \text{and} \quad H^{d+1}_{\overline{W}}(\overline{Z}, \mathbb{Q}_\ell(d))$$

has no fixed points.

Proof. — After base change to $\overline{k}$ and twist of (2.2.3), we obtain the following exact sequence on $\overline{Z}$

$$0 \longrightarrow \mathbb{Q}_\ell(d) \longrightarrow u_{1*}\mathbb{Q}_\ell(d) \longrightarrow u_{2*}\mathbb{Q}_\ell(d) \longrightarrow \cdots \longrightarrow u_{d+1*}\mathbb{Q}_\ell(d) \longrightarrow 0$$

from which we deduce the following hypercohomology spectral sequence

$$E_1^{ab} = H^b(\overline{Z^{(a+1)}}, \mathbb{Q}_\ell(d)) \Longrightarrow H^{a+b}(\overline{Z}, \mathbb{Q}_\ell(d)) \tag{3.2.3}$$

Since $Z^{(a+1)}$ is smooth, Deligne's theorem on weights [**Del80**, Th I] says that $E_1^{a,b}$ is pure of weight $b - 2d$. Hence, the spectral sequence (3.2.3) degenerates at page 2. Thus, we have an identification

$$H^d(\overline{Z}, \mathbb{Q}_\ell(d)) \simeq \bigoplus_{a=0}^{d} \operatorname{Ker}(d_1^{a,d-a})/\operatorname{Im} d_1^{a-1,d-a}$$

where $\operatorname{Ker}(d_1^{a,d-a})$ is pure of weights $-d-a \neq 0$ for $a = 0, \ldots, d$.

To treat the case of $H^{d+1}_{\overline{W}}(\overline{Z}, \mathbb{Q}_\ell(d))$, we use 2.1.5 to get an isomorphism of G_k-modules

$$H^{d+1}_{\overline{W}}(\overline{Z}, \mathbb{Q}_\ell(d)) \xrightarrow{\sim} H^{d-1}(\overline{W}, \mathbb{Q}_\ell(d-1))$$

In the following hypercohomology spectral sequence

$$E_1^{ab} = H^b(\overline{W^{(a+1)}}, \mathbb{Q}_\ell(d-1)) \Longrightarrow H^{a+b}(\overline{W}, \mathbb{Q}_\ell(d-1)) \tag{3.2.4}$$

E_1^{ab} has weight $b-2d+2$. Hence, (3.2.4) degenerates at page 2. Thus, we have an identification

$$H^{d-1}(\overline{W},\mathbb{Q}_\ell(d-1)) \simeq \bigoplus_{a=0}^{d-1} \mathrm{Ker}(d_1^{a,d-1-a})/\operatorname{Im} d_1^{a-1,d-1-a}$$

where $\mathrm{Ker}(d_1^{a,d-1-a})$ is pure of weights $-d-a+1 \leq -d+1 < 0$ since d is supposed to be > 1. □

4. A technical commutativity check

4.1. Let $n \geq 1$ be an integer. We define $\Lambda := \mathbb{Z}/n\mathbb{Z}$. For a morphism of complexes of Λ-sheaves $f : K_1 \longrightarrow K_2$, we will denote by $[f]$ the morphisms of $D(X_{\text{ét}},\Lambda)$ and $\mathcal{K}(X_{\text{ét}},\Lambda)$ induced by f. If $s : K_2 \longrightarrow K_1$ is a quasi-isomorphism, and if $f : K_2 \longrightarrow K_3$ is a morphism of $\mathcal{K}(X_{\text{ét}},\Lambda)$, we note $[(s,f)]$ the induced morphism of $D(X_{\text{ét}},\Lambda)$.

4.2. Cohomology and Hom **in the derived category.** — Let X be a scheme, and let $K \in D(X_{\text{ét}},\Lambda)$ and $\mathrm{qis} : K \longrightarrow I$ a quasi-isomorphism with I a complex of injectives sheaves. We have a square

$$\begin{array}{ccc} H^k(X,K) & \xrightarrow{\sim} & H^k(X,I) \\ \downarrow & & \downarrow \wr \\ \mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(\Lambda,K[k]) & \xrightarrow[\mathrm{qis}\,\circ]{\sim} & \mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(\Lambda,I[k]) \end{array}$$

whose commutativity defines the vertical left arrow, and where the vertical right arrow is

$$[x] \in H^k\Gamma(X,I) \longrightarrow [\mathrm{id}_\Lambda,[1_X \longrightarrow x]] \tag{4.2.1}$$

It is an isomorphism. We will prove the

Lemma 4.2.2. — *Let $f : K_1 \longrightarrow K_2 \in \mathrm{Hom}_{D(X_{\acute{e}t},\Lambda)}(K_1,K_2)$. Then the following diagram*

$$\begin{array}{ccc} H^k(X,K_1) & \xrightarrow{H^k(X,f)} & H^k(X,K_2) \\ \wr\downarrow & & \downarrow\wr \\ \mathrm{Hom}_{D(X_{\acute{e}t},\Lambda)}(\Lambda,K_1[k]) & \xrightarrow[f[k]\circ]{} & \mathrm{Hom}_{D(X_{\acute{e}t},\Lambda)}(\Lambda,K_2[k]) \end{array} \tag{4.2.3}$$

commutes.

Proof. — Let $\mathrm{qis}_1 : K_1 \longrightarrow I_1$ (resp. $\mathrm{qis}_1 : K_1 \longrightarrow I_1$) a quasi-isomorphism of complexes with I_1 (resp. I_2) a complex of injective sheaves. Let $F : I_1 \longrightarrow I_2$ a morphism of complexes such that $[F] \in \mathrm{Hom}_{\mathcal{K}(X_{\text{ét}},\Lambda)}(I_1,I_2)$ corresponds to $[\mathrm{qis}_2]f[\mathrm{qis}_1]^{-1}$ via the canonical identification

$$\mathrm{Hom}_{\mathcal{K}(X_{\text{ét}},\Lambda)}(I_1,I_2) \xrightarrow{\sim} \mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(I_1,I_2)$$

This means we have the following equality

$$[\mathrm{qis}_2] f [\mathrm{qis}_1]^{-1} = [F] \tag{4.2.4}$$

in $\mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(I_1, I_2)$. In the cube

$$\begin{array}{ccccccc}
H^k(X,K_1) & & \xrightarrow{H^k(X,f)} & & H^k(X,K_2) & & \\
& \searrow^{\sim} & & & & \searrow^{\sim} & \\
\downarrow \wr & & H^k(X,I_1) & \xrightarrow{H^k\Gamma(X,F)} & \downarrow \wr & & H^k(X,I_2) \\
\mathrm{Hom}(\Lambda, K_1[k]) & & \downarrow \wr & \xrightarrow{?} & \mathrm{Hom}(\Lambda, K_2[k]) & & \downarrow \wr \\
& \searrow^{\sim} & & & & \searrow^{\sim} & \\
& & \mathrm{Hom}(\Lambda, I_1[k]) & & \xrightarrow[{[F][k]\circ}]{} & & \mathrm{Hom}(\Lambda, I_2[k])
\end{array}$$

the left and right faces are commutatives by definition. Applying $H^k R\Gamma(X, \quad)$ to (4.2.4), we obtain that the top face of the cube commutes. By definition (4.2.1) the front face is commutative. We deduce that ? is $f[k]\circ$. ☐

4.3. A compatibility for Gysin morphisms. — Consider a cartesian diagram

$$\begin{array}{ccc}
Y' & \xrightarrow{i'} & X' \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\
Y & \xrightarrow[i]{} & X
\end{array}$$

of $\mathbb{Z}[\frac{1}{n}]$ noetherian schemes with i, i' closed regular immersions with same codimension c. According to [**ILO14**, XVI], we have a Gysin morphism

$$\mathrm{Gys}_i : \Lambda_Y \longrightarrow Ri^!\Lambda_X(c)[2c] \quad \text{et} \quad \mathrm{Gys}_{i'} : \Lambda_{Y'} \longrightarrow Ri'^!\Lambda_{X'}(c)[2c]$$

By [**AGV73**, XVIII 3.1.12.3], the base change morphism

$$\mathrm{bc} : Rv_* Ri'^! K \longrightarrow Ri^! Ru_* K$$

is an isomorphism for every $K \in D^+(X', \Lambda)$. So we have a square

$$\begin{array}{ccccc}
\Lambda_Y & & \xrightarrow{\mathrm{adj}_v} & & Rv_*\Lambda_{Y'} \\
{\scriptstyle \mathrm{Gys}_i}\downarrow & & & & \downarrow{\scriptstyle Rv_* \mathrm{Gys}_{i'}} \\
Ri^!\Lambda_X(c)[2c] & \xrightarrow[Ri^! \mathrm{adj}_u]{} & Ri^! Ru_* \Lambda_{X'}(c)[2c] & \xrightarrow[\mathrm{bc}^{-1}]{} & Rv_* Ri'^! \Lambda_{X'}(c)[2c]
\end{array} \tag{4.3.1}$$

Proposition 4.3.2. — *The square* (4.3.1) *commutes.*

Proof. —

(4.3.3)
$$
\begin{array}{ccc}
H^{2c}(Y, Ri^!\Lambda_X(c)) & \xrightarrow{\ \sim\ } & \mathrm{Hom}(\Lambda_Y, Ri^!\Lambda_X(c)[2c]) \\
{\scriptstyle H^{2c}(Y,Ri^!\,\mathrm{adj}_u)}\downarrow & & \downarrow{\scriptstyle Ri^!\,\mathrm{adj}_u\,\circ} \\
H^{2c}(Y, Ri^!Ru_*\Lambda_{X'}(c)) & \xrightarrow{\ \sim\ } & \mathrm{Hom}(\Lambda_Y, Ri^!Ru_*\Lambda_{X'}(c)[2c]) \\
{\scriptstyle H^{2c}(Y,\mathrm{bc}^{-1})}\downarrow & & \downarrow{\scriptstyle \mathrm{bc}^{-1}\,\circ} \\
H^{2c}(Y, Rv_*Ri'^!\Lambda_{X'}(c)) & \xrightarrow{\ \sim\ } & \mathrm{Hom}(\Lambda_Y, Rv_*Ri'^!\Lambda_{X'}(c)[2c]) \\
 & & \uparrow{\scriptstyle \circ\,\mathrm{adj}_v} \\
\wr\downarrow & & \mathrm{Hom}(Rv_*\Lambda_{Y'}, Rv_*Ri'^!\Lambda_{X'}(c)[2c]) \\
 & & \uparrow{\scriptstyle Rv_*} \\
H^{2c}(Y', Ri'^!\Lambda_{X'}(c)) & \xrightarrow{\ \sim\ } & \mathrm{Hom}(\Lambda_{Y'}, Ri'^!\Lambda_{X'}(c)[2c])
\end{array}
$$

The commutativity of the squares in (4.3.3) comes from 4.2 and some general property of adjoint functors. The composite of the vertical left morphisms is the pull-back morphism

(4.3.4)
$$H^{2c}_Y(X, \Lambda_X(c)) \longrightarrow H^{2c}_{Y'}(X', \Lambda_{X'}(c))$$

The proposition 4.3.2 is thus deduced from the fact [**ILO14**, XVI 2.3.2] that (4.3.4) sends Cl_i on $\mathrm{Cl}_{i'}$. □

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