A MOVING LEMMA, AFTER GABBER, LIU, LORENZINI

by

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Introduction

This text is a written version of a talk given for the Winter research seminar *Chow groups of zero cycles over p-adic fields* that I co-organized with Kay Rülling. The main reference for this seminar was [**SS10**]. We give here an account of Gabber-Liu-Lorenzini proof [(1)] of [**GLL13**, 2.3].

Let X be a scheme, and let $\sim_{\text{rat}}$ be the rational equivalence relation on the group of cycles $Z(X)$ of X.

Theorem 0.0.1. — *Let S be an excellent trait, let $s \in S$ be the closed point of S and let $f : X \longrightarrow S$ be a separated morphism of finite type with X regular and FA* [(2)]. *Let C be a 1-cycle of X whose support is finite over S. Let F be a closed subset of X such that the irreducible components of $F \cap X_s$ meeting C are not irreducible components of X_s.*

Then, there is a 1-cycle C' on X such that $C \sim_{\text{rat}} C'$ and the support of C' is finite over S and disjoint from F.

1. The proof

1.1. Reduction. — One can always suppose that C is irreducible and meets F. Since X is FA and C is semi-local, C is a closed subset of an affine open subset V of X. So C is affine. Let Γ be an irreducible component of $(F \cap V)_s$ such that $\Gamma \cap C \neq \emptyset$. Then, $\overline{\Gamma}$ is an irreducible component of F_s such that $\overline{\Gamma} \cap C \neq \emptyset$. Hence, $\overline{\Gamma}$ is a strict closed subset of an irreducible component of X_s, so Γ is a strict closed subset of an

1. We stick here to the excellent base case, although [**GLL13**, 2.3] is slightly more general.
2. That is, every finite subset of X is contained in an affine open subset of X. Ex: X quasi-projective over S.

irreducible component of V_s. Thus, we can suppose that X is affine.

Let $D \longrightarrow C$ be the normalization of C. Since we are in an excellent situation, $D \longrightarrow C$ is a finite morphism. Thus, one can find an integer $N > 0$ and a factorization

$$\begin{array}{ccccc} D & \longrightarrow & C \times_S \mathbb{A}^N_S & \longrightarrow & X \times_S \mathbb{A}^N_S \\ & \searrow & \downarrow & & \downarrow \\ & & C & \longrightarrow & X \end{array}$$

where the horizontal arrows are closed immersions. Define $U := X \times_S \mathbb{A}^N_S$ with $\mathbf{F} = F \times_S \mathbb{A}^N_S$. The morphism $U \longrightarrow X$ is smooth since it is a base change of a smooth morphism. By permanence property [**Gro71**, 3.1], we deduce that U is regular.

On the other hand, an irreducible component $\mathbf{\Gamma}$ of $\mathbf{F}_s = F_s \times_s \mathbb{A}^N_s$ is of the form $\Gamma \times_s \mathbb{A}^N_s$ where Γ is an irreducible component of F_s. If $\mathbf{\Gamma}$ meets D, then Γ meets C. Hence, Γ is a strict closed subset of an irreducible component of X_s. So $\mathbf{\Gamma}$ is a strict closed subset of an irreducible component of U_s. Thus, the triple $(X, \mathbf{F}, D)$ satisfies the hypothesis of 0.0.1. Let us suppose that 0.0.1 is true for $(X, \mathbf{F}, D)$. Let D' be the 1-cycle given by 0.0.1. We have a commutative diagram

$$\begin{array}{ccc} D, D' & \longrightarrow & X \times_S \mathbb{P}^N_S \\ & \searrow & \downarrow p \\ & & X \end{array}$$

where the upper arrow is a closed immersion. So $D \sim_{\text{rat}} D'$ in $X \times_S \mathbb{P}^N$. Since S is universally catenary and equidimensional at every point, [**Thr90**] ensures that

$$C = p_* D \sim_{\text{rat}} p_* D'$$

The 1-cycle $C' := p_* D'$ is the one we are looking for. Hence, we can suppose that X is affine, that C is irreducible and regular and that $C \cap F_s \neq \emptyset$.

Since one can replace X by its connected components, one can suppose that X is connected. Thus, one can suppose that X is irreducible. If we had $C = X$, every point in $C \cap F_s$ would be an irreducible component of X_s, which contradict the hypothesis made on F. Thus, C is a strict closed subset of X, so $d := \operatorname{codim}_X C > 0$.

Let $x \in C \cap F_s$. We know from [**GD65**, 5.1.5] that catenarity for a scheme can be tested at the level of local rings. From [**Mat86**, 17.8] and [**Mat86**, 17.9], we deduce that X is catenary. Hence,

$$\operatorname{codim}_X x = \operatorname{codim}_C x + \operatorname{codim}_X C = 1 + d$$

Let Γ be an irreducible component of F_s containing x. Γ is a strict closed subset of an irreducible component Γ' of X_s. Moreover, Γ' is a strict closed subset of X, otherwise C would be finite over the point s, so C would have dimension 0, which is not the case. Thus, we have a chain of closed irreducible subset

$$\{x\} \subset \Gamma \subsetneq \Gamma' \subsetneq X$$

so

$$\operatorname{codim}_X \Gamma \geq 2$$

We know from [**GD65**, 5.2.1] that the closed points of Γ all have codimension $\dim \Gamma$. So the formula

$$\operatorname{codim}_\Gamma x + \operatorname{codim}_X \Gamma = \operatorname{codim}_X x = 1 + d$$

gives $\dim \Gamma \leq d - 1$. We are thus left to prove the

Theorem 1.1.1. — *Let S be a local noetherian excellent scheme, and let $s \in S$ be the closed point of S. Let $U \longrightarrow S$ be a morphism of finite type with U affine. Let C be an integral closed subscheme of U with codimension $d \geq 1$. Let F be a closed subset of U such that the irreducible components of F_s meeting C have dimension at most $d-1$. Then, there is a 1-cycle C' of U, rationally equivalent to C and such that*

(1) $\operatorname{Supp} C'$ *is disjoint from C.*

(2) $\operatorname{Supp} C'$ *is disjoint from F.*

(3) $\operatorname{Supp} C'$ *does not contain any irreducible component of U_s.*

(4) $\operatorname{Supp} C'$ *is finite over S.*

The fundamental case is the case where $d = 1$. This is the one on which we will focus. We write $U = \operatorname{Spec} A$.

1.2. The case $d = 1$ of 1.1.1. — We have to find $g \in \mathcal{K}(U)$ such that the support of

$$C' := C + \operatorname{div} g$$

satisfies the conditions $(1) - (4)$. Note that (2) is satisfied if C' does not meet F_s. We have the following dichotomy: for an irreducible component Γ of F_s, either $\Gamma \subset C$ or Γ is a point not contained in C. Thus, to avoid F_s and to avoid C are now disjoint problems.

Let us choose a closed point in each irreducible component of U_s not contained in C. Let us denote by Z the union of this set of points with the set of points of F_s not in C. The condition

(1) is equivalent to $\operatorname{ord}_C g = -1$ and $\operatorname{ord}_{Y_1} g = 0$ for every integral divisor Y_1 of U different from C and meeting C.

(2) and (3) are fulfilled if $\operatorname{ord}_{Y_2} g = 0$ for every integral divisor Y_2 of U containing a point of Z.

If we look for g of the form $1 + h$ with $h \in \Gamma(U, \mathcal{I}_Z) \subset A$, the conditions (2) et (3) are satisfied. Indeed, we have $\operatorname{ord}_{Y_2}(1+h) \geq 0$ and if we had $\operatorname{ord}_{Y_2}(1+h) > 0$, this would mean that $1 + h$ would vanish generically on Y_2, thus by irreducibility would vanish on Y_2, which is not possible since the value of $1 + h$ at a point of $Y_2 \cap Z \neq \emptyset$ is 1. Hence, the $1 + h$ with $h \in \Gamma(U, \mathcal{I}_Z)$ are well-adapted for our problem. However, one cannot say anything on $\operatorname{ord}_C(1+h)$ and $\operatorname{ord}_{Y_1}(1+h)$ a priori.

To obtain $\operatorname{ord}_C g = -1$, it is natural to multiply $(1 + h)$ by a function defining C in U. Such a function does not exist globally in general but using 1.3.1, we can find a global equation φ for C in an affine neighbourhood V of C. In particular, the support of $\operatorname{div}\varphi - C$ is disjoint from C. The problem arising with a random choice of φ is the lack of control of φ away from V. Let us summarize the situation.

	ord_C	ord_{Y_1}	ord_{Y_2}
$1+h$	?	?	0
φ	1	0	?
$\varphi^{-1}(1+h)$	?	?	?

A way to get rid of the first indetermination for $1+h$ is to rather consider $\varphi+h$ with h invertible in a neighbourhood of C. Indeed, $\operatorname{ord}_C h = 0$ and $\operatorname{ord}_C \varphi > 0$ would give $\operatorname{ord}_C(\varphi+h) = 0$. So $\varphi^{-1}(\varphi+h) = 1+\varphi^{-1}h$ with $h \in \Gamma(U, \mathcal{I}_Z)$ and h invertible on C is a better candidate. Still, we don't know anything on $\operatorname{ord}_{Y_2} 1+\varphi^{-1}h$ since we don't have any control on the poles of φ away from C.

	ord_C	ord_{Y_1}	ord_{Y_2}
$\varphi+h$	0	?	?
$1+\varphi^{-1}h$	-1	?	?

To constrain the pole locus of $\varphi^{-1}h$ and keep the vanishing along Z, we will look for a global section of $\mathcal{I}_Z(C) := \mathcal{I}_Z \otimes_{\mathcal{O}_U} \mathcal{I}_C^{-1}$. Since $\mathcal{I}_C$ is locally free, we have an exact sequence

$$0 \longrightarrow \mathcal{I}_Z(C) \longrightarrow \mathcal{I}_C^{-1}$$

Given the construction of φ in 1.3.1, the sought function g is of the type 1+a global section of $\mathcal{I}_Z(C)$ sent to $\overline{\varphi}^{-1}$ by

$$\mathcal{I}_Z(C) \longrightarrow i_* i^* \mathcal{I}_Z(C) \longrightarrow 0 \tag{1.2.1}$$

where $i : C \longrightarrow U$ is the canonical inclusion. Indeed, such a g will be of the form "$1+\varphi^{-1}$" in a neighbourhood of C, and of the form "$1+h$" in a neighbourhood of a point of Z. Such a g always exists since U is affine.

With a random lift of $\overline{\varphi}^{-1}$, we can unfortunately end up with a divisor with vertical components over S, which is forbidden by (4). Since U is affine, a way to produce a closed subset of U which is finite over S is to ask for properness over S. Thus, one is naturally led to invoke Nagata **[Nag62]**,**[Nag63]** to work with a compactification

$$\begin{array}{ccc} U & \hookrightarrow & X \\ & \searrow & \downarrow{\scriptstyle \text{propre}} \\ & & S \end{array}$$

and look for $g \in \mathcal{K}(X) = \mathcal{K}(U)$. Since we don't want the locus at infinity to interfere with our problem, we look for a g satisfying the extra condition

$$\operatorname{Supp} g \cap (X \setminus U) = \emptyset \tag{1.2.2}$$

This condition is simply obtained by adding $X \setminus U$ in the definition of Z.

Let us prove that C is closed in X. The morphism $i : C \longrightarrow X$ is a monomorphism since it is a composition of the closed immersion $C \longrightarrow U$ with the open immersion $U \longrightarrow X$. Moreover, with have a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{i} & X \\ & \underset{\text{finite}}{\searrow} & \downarrow \text{proper} \\ & & S \end{array}$$

So we deduce from [**GD61a**, 5.4.3] that i is proper. Thus, we have by [**GD67**, 18.12.6] that i is a closed immersion.

The problem arising with the compactified situation is that the surjectivity of sheaves (1.2.1) does not automatically induce a surjective morphism at the level of global sections. We have a short exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{I}_Z(C) \longrightarrow i_*i^*\mathcal{I}_Z(C) \longrightarrow 0 \tag{1.2.3}$$

Since C is affine, we have

$$H^1(X, i_*i^*\mathcal{I}_Z(C)) \simeq H^1(C, i^*\mathcal{I}_Z(C)) \simeq 0$$

so the long exact sequence associated to (1.2.3) reads

(1.2.4)

$$H^0(X, \mathcal{I}_Z(C)) \longrightarrow H^0(X, i_*i^*\mathcal{I}_Z(C)) \longrightarrow H^1(X, \mathcal{I}_Z) \xrightarrow{a_1} H^1(X, \mathcal{I}_Z(C)) \longrightarrow 0$$

And the point is that a_1 may not be injective. The observation of Gabber-Liu-Lorenzini is that by twisting (1.2.3) with a high enough power of $\mathcal{I}_C^{-1}$, the canonical surjection

$$a_n : H^1(X, \mathcal{I}_Z((n-1)C)) \longrightarrow H^1(X, \mathcal{I}_Z(nC)) \longrightarrow 0$$

induced by the short exact sequence

$$0 \longrightarrow \mathcal{I}_Z(n-1) \longrightarrow \mathcal{I}_Z(nC) \longrightarrow i_*i^*\mathcal{I}_Z(nC) \longrightarrow 0$$

is an isomorphism.

To see it, note that by the fundamental theorem for proper morphisms [**GD61b**, 3.2.3], the space $H^1(X, \mathcal{I}_Z(C))$ is a $\mathcal{O}_S(S)$-module of finite type. The arrows a_n induce by composition a surjection

$$A_n : H^1(X, \mathcal{I}_Z(C)) \longrightarrow H^1(X, \mathcal{I}_Z(nC)) \longrightarrow 0$$

for every n. Since $\mathcal{O}_S(S)$ is noetherian, the increasing sequence $\operatorname{Ker} A_n$ is stationnary for $n \geq N$. For $n > N$, pick $x \in \operatorname{Ker} a_n$. By surjectivity, we have $x = A_{n-1}(y)$. Applying a_n, we deduce $y \in \operatorname{Ker} A_n$. So $y \in \operatorname{Ker} A_{n-1}$, so $x = 0$.

For a given $n > N$, let us choose a lift f_n of $\overline{\varphi}^{-n}$ by

$$H^0(X, \mathcal{I}_Z(nC)) \longrightarrow H^0(C, i^*\mathcal{I}_Z(nC)) \longrightarrow 0$$

and a lift f_{n+1} of $\overline{\varphi}^{-(n+1)}$ by

$$H^0(X, \mathcal{I}_Z((n+1)C)) \longrightarrow H^0(C, i^*\mathcal{I}_Z((n+1)C)) \longrightarrow 0$$

We conclude with

	ord_C	ord_{Y_1}	ord_{Y_2}
$1+f_n$	$-n$	0	0
$1+f_{n+1}$	$-(n+1)$	0	0
$(1+f_{n+1})/(1+f_n)$	-1	0	0

1.3. Local global equation for a semi-local divisor. —

Proposition 1.3.1. — *Let S be a local noetherian scheme and let $U = \operatorname{Spec} A \longrightarrow S$ be a morphism of finite type. Let $i : C \longrightarrow U$ be a regular closed immersion of codimension 1 with C finite over S. Denote by $\mathcal{I}$ the ideal of definition of C. Then, there is an affine open V in U such that $\mathcal{I}_{|V}$ is trivial.*

Proof. — Since C is finite over S local, C is semi-local. Let $c \in C$ be a closed point in C. The hypothesis says one can find an open neighbourhood V of c in U such that $\mathcal{I}_{|V}$ is free of rank 1. We deduce that $i^*\mathcal{I}_{|V} \simeq (\mathcal{I}/\mathcal{I}^2)_{|V\cap C}$ is free of rank 1 over $V \cap C$.

But we know that a projective module over a semi-local ring whose localizations at maximal ideals have the same rank is free, so $\mathcal{I}/\mathcal{I}^2$ is free of rank 1 over C. Since U is affine, a global trivialisation of $\mathcal{I}/\mathcal{I}^2$ lifts to a global section f of $\mathcal{I}$ over U. By Nakayama lemma, $f_c \in \mathcal{I}_c$ generates $\mathcal{I}_c$. Since $\mathcal{I}_c$ is a free $\mathcal{O}_{U,c}$-module of rank 1, we deduce that f_c is a trivialisation of $\mathcal{I}_c$. By noetherianity argument, one can find a neighbourhood V_c of c in U such that $\mathcal{I}_{|V_c}$ is free of rank 1 generated by $f_{|V_c}$.

Since every finite subset of an affine scheme admits a fundamental system of open *affine* neighbourhood, one can choose an affine open $V \subset \cup V_c$ containing the closed points of C, so containing C. The affine open V is the one we were looking for. □

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