CHAPTER 7

The Hales-Jewett theorem

This chapter presents a deep combinatorial theorem by Hales and Jewett, stated in its classical form in 7.9. In this version, it deals with colourings of free semigroups and may seem difficult to understand, at first reading. In these notes, we give a rather straightforward proof of a seemingly more general theorem (the abstract Hales-Jewett theorem 7.2), from which the classical version follows easily. Quite surprisingly, the abstract version can be deduced, vice versa, from the classical one; cf. 7.14. The proof of 7.2 (and also of 7.9, if one wants to prove it directly) heavily depends on the application of minimal idempotents in βS, for a semigroup S, which was established in the preceding chapter.

There are several attractive and more easily understandible consequences of the Hales-Jewett theorem, which we present in the theorems 7.3, 7.5, and 7.6. A rather general one is Gallai's theorem for commutative semigroups, with van der Waerden's theorem for the additive semigroups $\mathbb{N}$ or $\mathbb{N}^k$ as its special cases.
Another special case of 7.3 with a quite intuitive content runs as follows. For a K-vector space V, a *homothetical map* is a function $f : V \to V$ with $f(x) = a + dx$, for some fixed $a \in V$ and some fixed $d \in K$. Gallai's theorem for the semigroup $(V, +)$ then says that for every colouring of V with finitely many colours and every finite subset E of X, there is a monochromatic homothetical copy of E.

In Section 1, we state the abstract version 7.2 of the Hales-Jewett theorem and derive the consequences mentioned above. The proof of 7.2 is given in Section 2. In the last section, we derive the classical version 7.9 of the Hales-Jewett theorem and explain how it implies the abstract one.

The authors Graham, Rothschild und Spencer, in their book ??? on Ramsey theory, emphasize the importance of the Hales-Jewett theorem as follows.

> The Hales-Jewett theorem strips van der Waerden's theorem of its unessential elements and reveals the hart of Ramsey theory. It provides a focal point from which many results can be derived and acts as a cornerstone for much of the more advanced work.

1. The abstract version of the Hales-Jewett theorem

The formulation of this version of the Hales-Jewett theorem requires a couple of additional notions.

7.1. Definition Let $(V, \cdot)$ be a semigroup and W a subsemigroup of V.
(a) W is called a *nice subsemigroup* of V if the "remainder" $R = V \setminus W$ of V with respect to W is a two sided ideal of V. (I.e. if W is a proper subset of V, and the

product of two elements x, y of V is in W if and only if both x and y are in W.)
(b) A *retraction* from V to W is a semigroup homomorphism $\sigma : V \to W$ such that the restriction of σ to W is the identity map on W (i.e. such that $\sigma(x) = x$ holds for all $x \in W$).

We will see examples of nice semigroups and retractions in the subsequent applications of the Hales-Jewett theorem, but let us first state the reult.

7.2. Theorem *(the Hales-Jewett theorem, abstract version) Assume that W is a nice subsemigroup of V, and Σ is a finite set of retractions from V to W. Moreover assume that $W = B_1 \cup \cdots \cup B_r$ is a colouring of W with finitely many colours. Then there is some $v \in R = V \setminus W$ such that $\{\sigma(v) : \sigma \in \Sigma\}$ is monochromatic, i.e. for some index $j \in \{1, \ldots, r\}$, $\{\sigma(v) : \sigma \in \Sigma\} \subseteq B_j$.*

We proceed to prove the Gallai – van der Waerden type consequences of the theorem. For a commutative semigroup $(S, +)$, an element e of S and $d \in \mathbb{N}$, we denote by de the sum $e + \cdots + e$ in which e is added d times.

7.3. Theorem *(Gallai's theorem for commutative semigroups) Assume that $(S, +)$ is a commutative semigroup, E is a finite subset of S, and assume that $S = A_1 \cup \cdots \cup A_r$ is a colouring of S with finitely many colours. Then there are $a \in S$ and $d \in \mathbb{N}$ such that $\{a + de : e \in E\}$ is monochromatic, i.e. for some index $j \in \{1, \ldots, r\}$, $\{a + de : e \in E\} \subseteq B_j$.*

Proof. We work in the (commutative) semigroup $V = S \times \omega$ with coordinate-wise addition. I.e. ω is the additive semigroup of natural numbers, including 0, and for $a, a' \in S$ and $d, d' \in \omega$, we define

$$(a, d) + (a', d') = (a + a', d + d').$$

To apply 7.2, we note that $W = S \times \{0\}$ is a nice subsemigroup of V. Moreover every $e \in S$ gives rise to a map $\sigma_e : V \to W$ defined by

$$\sigma_e(a, d) = (a + de, 0).$$

Now σ_e is a semigroup homomorphism (here we use commutativity of $(S, +)$), and in fact a retraction from V to W. Finally, the colouring $S = A_1 \cup \cdots \cup A_r$ of S induces a colouring $W = B_1 \cup \cdots \cup B_r$ of W, where

$$(a, 0) \in B_i \text{ iff } a \in A_i.$$

By the abstract Hales-Jewett theorem , pick v in $V \setminus W$ and some j such that $\{\sigma_e : e \in E\} \subseteq B_j$. Say $v = (a, d)$, where $a \in S$ and $d \in \omega$; here $d \neq 0$ because $v \notin W$. Then for every $e \in E$, $\sigma_e(a, d) \in B_j$ means that $a + de \in A_j$. □

We note several easy special cases of 7.3. In a vector space V over a field K, a map f from V into itself is called *homothetic* if there are $a \in V$ and $\lambda \in K$ such that $f(x) = a + \lambda x$ holds for $x \in V$. (So a homothetic map is a very special example of an affine map.) Let us call f non-trivial if $\lambda \neq 0$.

7.4. Corollary *(Gallai's theorem for vector spaces) Assume that K is a field with characteristic 0, V is a vector space over K, $V = A_1 \cup \cdots \cup A_r$ is a colouring of V with finitely many colours, and E is a finite subset of V. Then some A_j contains a non-trivial homothetic image of E.*

In fact, an element $\lambda \in K$ satisfying $f[E] \subseteq A_j$, where $f(x) = a + \lambda x$, can be taken as $\lambda = d \cdot 1_K$, where 1_K is the identity element of K and $d \in \mathbb{N}$. Here $\lambda \neq 0$ since K has characteristic 0.

7.5. Corollary *(van der Waerden's theorem) Assume that $\omega = A_1 \cup \cdots \cup A_r$ is a colouring of ω with finitely many colours and that $n \in \mathbb{N}$. Then there are $a \in \omega, d \in \mathbb{N}$ and some j such that $\{a + di : 0 \leq i \leq n\} = \{a, a+d, a+2d, \ldots, a+dn\} \subseteq A_j$.*

Proof. This is the special case of 7.3 for the additive semigroup ω of natural numbers and its finite subset $E = \{0, 1, \ldots, n\}$. □

A subset of ω of the form $\{a, a+d, a+2d, \ldots, a+dn\}$ is called an *arithmetic progression* (of length n); 7.5 says that at least one of the colours A_j includes an arithmetic progression of length n.

7.6. Corollary *(the k-dimensional van der Waerden theorem) Assume that $k \in \mathbb{N}$, $\omega^k = A_1 \cup \cdots \cup A_r$ is a colouring of ω^k with finitely many colours, and that $n \in \mathbb{N}$. Then there are $a \in \omega^k, d \in \mathbb{N}$ and some j such that for every k-tuple $e = (e_1, \ldots, e_k) \in \omega^k$ where $0 \leq e_i \leq n$, $a + de \in A_j$. I.e. some A_j includes a homothetic copy of the k-dimensional cube $\{0, 1, \ldots, n\}^k$.*

Proof. Apply 7.3 to the semigroup ω^k, with coordinatewise addition, and its finite subset $E = \{0, 1, \ldots, n\}^k$. □

We end this section with a few additional remarks on Gallai's theorem 7.3. So we consider a commutative semigroup $(S, +)$. For $A \subseteq S$ and E a finite subset of S, we say that A is E-good if there are $a \in S$ and $d \in \mathbb{N}$ such that $\{a + de : e \in E\} \subseteq A$.

7.7. Remark (a) If A is E-good then it is F-good for every subset F of E. This explains why we stated the van der Waerden theorems 7.5 and 7.6 only for the seemingly very special cases $E(n) = \{0, \ldots, n\}$ resp. $E(n)^k = \{0, \ldots, n\}^k$: every finite subset of ω resp ω^k is included in $E(n)$ resp. $E(n)^k$, for some $n \in \mathbb{N}$.
(b) A seemingly more general formulation of Theorem 7.3 is that, for a colouring $A_1, \ldots, A_r$ of S with finitely many colours, some A_j is E-good for *all* finite subsets E of S. Because otherwise, we can choose for every $j \in \{1, \ldots, r\}$ a finite subset E_j of S such that A_j is not E_j-good. Then by (a) above, no A_j is E-good for $E = E_1 \cdots \cup E_r$, a contradicton to 7.3. Similarly for every colouring of ω with finitely many colours, one of the colours includes arbitrarily long arithmetic progressions, i.e. progressions of length n for all $n \in \mathbb{N}$.

2. Proof of the abstract Hales-Jewett theorem

We assume in this section that V, $W = B_1 \cup \cdots \cup B_r$ and Σ are given as in 7.2, i.e. W is a nice subsemigroup of V, $B_1, \ldots, B_r$ is a colouring of W with finitely many colours, and Σ is a finite set of retractions from V to W. Moreover we have $R = V \setminus W$, a two-sided ideal of V.

Step 1. We first note how the assumptions of the theorem on V, W, R, and Σ carry over to the Stone-Čech compactifications βV, βW and βR and to the maps

$\tilde{\sigma}$ for $\sigma \in \Sigma$. By 4.8, $\hat{W}$ is a subsemigroup of βV which is canonically isomorphic to βW; we identify βW with the subsemigroup $\hat{W}$ of βV. We will write βW when using the compact right topological semigroup structure on $\hat{W}$ and $\hat{W}$ when thinking about it as a clopen subset of βV and apply the theory of compact right topological semigroups to both βV and βW. Similarly, βR is a two sided ideal of βV, and $\beta V = \hat{W} \cup \hat{R} = \beta W \cup \beta R$, where the unions are disjoint, says that βW is, in fact, a nice subsemigroup of βV. Next, every $\sigma \in \Sigma$ is a homomorphism from V into W, so by 4.4, its Stone-Čech extension $\tilde{\sigma}$ is a homomorphism from βV into βW, and by continuity of $\tilde{\sigma}$, it is even a retraction from βV into βW. Finally, $W = B_1 \cup \cdots \cup B_r$ implies that $\beta W = \hat{B}_1 \cup \cdots \cup \hat{B}_r$.

Step 2. Applying the theory of the preceding chapter to the semigroup βW, we fix a minimal idempotent q of βW. It is also an idempotent of the larger semigroup βV, but we cannot be sure that it is minimal in βV. However by Theorem 6.13(d), we can fix a minimal idempotent p of βV satisfying $p \leq q$. The following properties of the ultrafilters p and q in βV are what makes the proof work.

Step 3. We note that R is an element of p. – For consider the set $I = \beta V \cdot p$, a minimal left ideal of βV by 6.13(c) and hence included in the two sided ideal $\beta R = \hat{R}$ of βV by 6.5. So $p \in I \subseteq \hat{R}$, i.e. $R \in p$.

Step 4. We now claim that each of the maps $\tilde{\sigma}$, where $\sigma \in \Sigma$, maps p to q. For a proof, fix $\sigma \in \Sigma$ and consider $u = \tilde{\sigma}(p)$. In the semigroup βV, p and q are idempotents with $p \leq q$. Applying the retraction $\tilde{\sigma} : \beta V \to \beta W$, we see that in βW, $\tilde{\sigma}(p) = u$ and $\tilde{\sigma}(q) = q$ are idempotents with $u \leq q$. (More precisely, $p^2 = p = pq = qp$ implies that $u^2 = u = uq = qu$.) But q was a minimal idempotent of βW, and thus $q = u = \tilde{\sigma}(p)$ holds.

Step 5. We are ready to pick $v \in R = V \setminus W$ and $j \in \{1, \ldots, r\}$ as desired, i.e. such that $\{\sigma(v) : \sigma \in \Sigma\} \subseteq B_j$.
Since q is an ultrafilter on $W = \mathrm{B}_1 \cup \cdots \cup B_r$, let j be the index satisfying $B_j \in q$. For every $\sigma \in \Sigma$, $B_j \in q = \tilde{\sigma}(p)$, which by Exercise 6 in Chapter 3 means that the preimage $\sigma^{-1}[B_j]$ of B_j under σ is in p. It follows from this and Step 3 that the intersection

$$D = R \cap \bigcap_{\sigma \in \Sigma} \sigma^{-1}[B_j]$$

is an element of p, hence non-empty; pick an arbitrary element v of D. I.e. $v \in R$, and for every $\sigma \in \Sigma$, $v \in \sigma^{-1}[B_j]$ says that $\sigma(v) \in B_j$.

3. The classical Hales-Jewett theorem

The classical form 7.9 of the Hales-Jewett theorem is a combinatorial result on free semigroups which turns out to be a special case of the abstract one. We recall from Chapter 1 the notion of the free semigroup L^* on a set L: L^* is the set of all finite words on the alphabet L, a semigroup under the operation of concatenation of words.

The formulation of 7.9 and its variants reqires an additional definition.

7.8. Definition (a) Assume the alphabet L is split into two disjoint subsets C and X. The elements of C are called *constant letters*, the elements of X are the

variable letters. A word $w \in L^*$ is *constant* if $w \in C^*$, i.e. if w contains only constant letters, and *variable* otherwise.
(b) For $v \in L^*$, $x \in X$ and $a \in C$, $v(x/a)$ denotes the word arising from v by replacing (substituting) the variable letter x by the constant letter a, in all places where x occurs in v.
As an example, suppose $a, b, c \in C$, $x \in X$, and $v = abcxxxcbax$. Then $v(x/a) = abcaaacbaa$, $v(x/b) = abcbbbcbab$, and $v(x/c) = abccccbac$.
(c) A crucial point about substitutions is that, if $L = C \cup \{x\}$ and $a \in C$, the map $s_a : L^* \to C^*$ defined by $s_a(v) = v(x/a)$, is a semigroup homomorphism, the *substitution homomorphism* – in fact a retraction from L^* to C^*.

7.9. Theorem *(the Hales-Jewett theorem, classical version) Assume that $L = C \cup \{x\}$ (i.e. x is the only variable letter) where $C \neq \emptyset$, $C^* = B_1 \cup \cdots \cup B_r$ is a colouring of C^* with finitely many colours, and that F is a finite subset of C. Then there is a variable word $v \in L^*$ such that $\{v(x/a) : a \in F\}$ is monochromatic, i.e. for some index $j \in \{1, \ldots, r\}$, $\{v(x/a) : a \in F\} \subseteq B_j$.*

Proof. Consider $V = L^*$, the free semigroup over L, and $W = C^*$, a nice subsemigroup of V. Every constant letter $a \in F$ gives the retraction $s_a : V \to W$; put $\Sigma = \{s_a : a \in F\}$. Applying 7.2 to this situation yields the result. □

In the much same style, we derive from the abstract Hales-Jewett theorem some generalizations of the classical one – we allow substitution of a constant word u for x , instead of just a constant letter a.

7.10. Definition Assume $L = C \cup X$. Then for $v \in L^*$ and $u \in C^*$, $v(x/u)$ denotes the word arising from v by replacing (substituting) the variable letter x by the constant word u, in all places where x occurs in v.
E.g., for $v = abcxxxcbax$ and $u = aba$, we obtain $v(x/u) = abcabaabaabacbaaba$.

7.11. Theorem *Assume that $L = C \cup \{x\}$, $C^* = B_1 \cup \cdots \cup B_r$ is a colouring of C^* with finitely many colours, and that U is a finite set of constant words. Then there is a variable word $v \in L^*$ such that $\{v(x/u) : u \in U\}$ is monochromatic, i.e. for some index $j \in \{1, \ldots, r\}$, $\{v(x/u) : u \in F\} \subseteq B_j$.*

Proof. As in the proof of 7.9, consider $V = L^*$ and the nice subsemigroup $W = C^*$. For every constant word $u \in U$, define $s_u : V \to W$ by $\sigma_u(v) = v(x/u)$, a retraction from V to W, and apply 7.2. □

We can prove a further generalization of 7.11, substituting every variable letter x occurring in v by a constant word $f(x)$ which may depend on x.

7.12. Definition Assume $L = C \cup X$, and $f : X \to C^*$ is a map assigning to every $x \in X$ a constant word $f(x)$. (We might call f a *substitition assignment.*) Then for $v \in L^*$, $v(X/f)$ denotes the word arising from v by replacing (substituting) every variable letter x by the constant word $f(x)$, in all places where x occurs in v.
E.g., for $v = abcxyxcbayx$, $f(x) = ab$ and $f(y) = ca$, we obtain $v(X/f) = abcabcaabcbacaab$.

7.13. Theorem *Assume that $L = C \cup X$, $C^* = B_1 \cup \cdots \cup B_r$ is a colouring of C^* with finitely many colours, and that F is a finite set of substitution assignments.*

Then there is a variable word $v \in L^*$ *such that* $\{v(X/f) : f \in F\}$ *is monochromatic, i.e. for some index* $j \in \{1, \dots, r\}$, $\{v(X/f) : f \in F\} \subseteq B_j$.

PROOF. Again put $V = L^*$ and $W = C^*$. For every $f \in F$, $s_f : V \to W$ defined by $s_f(v) = v(X/f)$ is a retraction from V to W, so we can apply 7.2. □

The three versions of the classical Hales-Jewett theorem 7.9, and in particular 7.9 itself, look much more special that the abstract version 7.2. But surprisingly, the abstract version can be derived from the classical one.

7.14. Remark Assume V, W, and Σ are given as in 7.2, plus a partition $W = B_1 \cup \cdots \cup B_r$.
We work with the finite alphabet $L = C \cup X$ where C consists of distinct constant letters a_σ, for $\sigma \in \Sigma$, and X of just one variable letter x.
Fix an arbitrary element u of $V \setminus W$. Every $\sigma \in \Sigma$ is a map from V to W; so $\sigma(u)$ is an element of W. Using the freeness of L^* over L, fix

$$f : L^* \to V$$

to be the unique semigroup homomorphism which maps x to u and every constant letter a_σ to $\sigma(u)$. We note that the preimage of W under f is C^*, since $f(x) = u \notin W$ and W is a nice subsemigroup of V.
The partition $W = B_1 \cup \cdots \cup B_r$ of W induces, via f, a partition $C^* = A_1 \cup \cdots \cup A_r$ of C^* where A_i is the preimage of B_i under f. The classical Hales-Jewett theorem now gives a variable word w and some $j \in \{1, \dots, r\}$ such that $w(x/a_\sigma) \in A_j$ holds for every $\sigma \in \Sigma$. And since $w \notin C^*$, $v = f(w)$ is an element of $V \setminus W$; we will see that v works for our theorem, i.e. that $\sigma(v) \in B_j$ holds for every $\sigma \in \Sigma$.
To this end, note that the letter a_σ induces the substitution homomorphism $s_{a_\sigma} : L^* \to C^*$ mapping every word z to $z(x/a_\sigma)$. The homomorphism f commutes with the homomorphisms σ resp. s_{a_σ} in the sense that

$$\sigma \circ f = f \upharpoonright C^* \circ s_{a_\sigma}.$$

(For this commutativity condition holds for all letters of L, i.e. for the free generators of L^*, and hence on the whole of L^*; cf. 1.14.) Applying both sides of this equality to the word w, we obtain on the left hand side $\sigma(f(w)) = \sigma(v)$ and on the right one $f(s_{a_\sigma}(w))$. But $s_{a_\sigma}(w) = w(x/a_\sigma) \in A_j$ which means that $f(s_{a_\sigma}(w)) = \sigma(v) \in B_j$, as desired.