CHAPTER 6

Minimal ideals and minimal idempotents

The abstract version of Hindman's Theorem 5.2 is the paradigmatic example how idempotents of the semigroup βS can be used to prove combinatorial results on a discrete semigroup S. For more refined results, however, just idempotent elements of βS are not enough – we will have to use minimal idempotents. These are intimately connected with minimal left ideals in βS.

We thus turn to a more refined investigation of arbitrary semigroups. Most of the work to be done here is purely algebraic, quite computational and may seem somewhat non-intuitive. To ensure that the objects of our interest, i.e. minimal left ideals and minimal idempotents, do exist, we will make an extra assumption on our semigroup S which we will call abundantness. Fortunately, compact right topological semigroups turn out to be abundant. Hence the theory developed in this chapter applies to our standard example of a compact right topological semigroup, the Stone-Čech compactification βS of an arbitrary discrete semigroup S.

1. Ideals

In this section, we assume that $(S, \cdot)$ is an arbitrary (discrete) semigroup. Up to now, we have dealt with subsemigroups of S; we know consider subsets of S with a stronger closure property.

6.1. Definition Assume that I is a subset of S.
(a) I is a *left ideal* of S if it is non-empty and $SI \subseteq I$, i.e. $i \in I$ and $s \in S$ implies that $si \in I$.
(b) Similary, I is a *right ideal* of S if it is non-empty and $IS \subseteq I$, i.e. $i \in I$ and $s \in S$ implies that $is \in I$.
(c) I is a *two sided ideal* of S if it is both a left ideal and a right ideal of S.

I.e. left ideals are closed under left multiplication with arbitrary elements of S, right ideals under right multiplication, and two sided under both left and right multiplication. Of course, in a commutative semigroup, the notions of left ideal, right ideal, and two sided ideal coincide.

6.2. Example (a) In the commutative semigroup $(\mathbb{N}, +)$, the ideals are the sets $[a, \infty) = \{n \in \mathbb{N} : a \leq n\}$, where a is an arbitrary element of $\mathbb{N}$.
(b) In an arbitrary semigroup $(S, \cdot)$, fix some $a \in S$. Then $Sa = \{sa : s \in S\}$ is a left ideal, $aS = \{as : s \in S\}$ is a right ideal, and $SaS = \{sat : s, t \in S\}$ is a two sided ideal. If a happens to be in Sa, i.e. if $a = sa$ for some $s \in S$ (e.g., if a is idempotent), then Sa is clearly the smallest left ideal of S containing a, the *left ideal generated* by a. In general, i.e. without assuming that $a \in Sa$, we call Sa the *left ideal quasi-generated* by a.

(c) Similar to (b), Ia is a left ideal, for $a \in S$ and I a left ideal, because $S(Ia) = (SI)a) \subseteq Ia$. We leave it to the reader to formulate and prove similar statements for right resp. two sided ideals.
(d) Obviously, the union of any non-empty family of left (right, two sided) ideals is again a left (right, two sided) ideal. Hence, for M a non-empty subset of S, $SM = \{sa : s \in S, a \in M\} = \bigcup_{a \in M} Sa$ is a left ideal, $MS = \{as : s \in S, a \in M\} = \bigcup_{a \in M} aS$ is a right ideal, and $SMS = \{sat : s, t \in S, a \in M\} = \bigcup_{a \in M} SaS$ is a two sided ideal.
(e) The proof of 5.16 shows that in the semigroup $(\beta\mathbb{N}, \cdot)$, the topological closure P of the set $E = E(\beta\mathbb{N}, +) = \{p \in \beta\mathbb{N} : p + p = p\}$ of idempotents is a right ideal.

6.3. Definition A left ideal M of S is *minimal* if every left ideal I of S included in M coincides with M; similarly for right ideals.

Minimal (left) ideals do not necessarily exist – e.g. the semigroup $(\mathbb{N}, +)$ has no minimal ideal, as is shown by Example 6.2(a). However, there are some useful characterizations of minimal left ideals.

6.4. Lemma *Let L be a left ideal of S. The following are equivalent:*
(a) L is a minimal left ideal
(b) $Sx = L$ holds for every $x \in L$
(c) $Lx = L$ holds for every $x \in L$.

Proof. (a) implies (c): Let L be minimal and $x \in L$. Then $Lx \subseteq L$, since L is a left ideal. Moreover, Lx is a left ideal, as noted in 6.2(c); by minimality of L, $Lx = L$.
(c) implies (b): For $x \in L$, we have $Lx \subseteq Sx \subseteq L$, since L is a left ideal. Now $Lx = L$ by (c), and thus $Sx = L$ holds.
(b) implies (a): Assume that I is a left ideal included in L; we have to prove that $L \subseteq I$. Pick an arbitrary element x of I. Then $x \in L$ and hence $L = Sx \subseteq I$ holds by (b), since $x \in I$ and I is a left ideal. □

Minimal left ideals are intimately connected to two sided ideals.

6.5. Lemma *Assume that L is a minimal left ideal of S. Then the following hold.*
(a) For every $s \in S$, also Ls is a minimal left ideal.
(b) L is included in every two sided ideal of S.

Proof. (a) Ls is a left ideal, by 6.2(c). Let x be an arbitrary element of Ls; by 6.4, we have to show that $(Ls)x = Ls$. Write $x = ls$ where $l \in L$. It follows that $(Ls)x = (Ls)(ls) = (L(sl))s = Ls$; here the last equality holds by 6.4 because of $sl \in L$ and minimality of L.
(b) Let I be any two sided ideal. Then $L' = L \cap I$ is nonempty because, for $l \in L$ and $i \in I$, $il \in L \cap I$. Being the intersection of two left ideals, it is a left ideal and included in L; by minimality of L, we have $L = L' \subseteq I$. □

6.6. Definition For any semigroup S, we put

$$K(S) = \bigcup \{L : L \text{ a minimal left ideal of } S\},$$

the union of all minimal left ideals of S. So $K(S)$ is non-empty iff S has at least one minimal left ideal. And in this case, it is a left ideal, being the union of a non-empty family of left ideals.

The following theorem states the central role of the left ideal $K(S)$.

6.7. Theorem *Assume that $K(S) \neq \emptyset$. Then $K(S)$ is a two sided ideal of S – in fact, the least one.*

PROOF. 6.5(b) says that $K(S)$ is included in every two sided ideal of S. To prove that is it a right ideal, we assume $s \in S$ and prove that $K(S) \cdot s \subseteq K(S)$. But $K(S) \cdot s = \bigcup\{Ls : L$ a minimal left ideal of $S\} \subseteq K(S)$ holds by 6.5(a). □

2. Abundant semigroups

To put the theory of minimal left ideals and the two sided ideal $K(S)$ to use, we want to make sure that $K(S)$ is non-empty, i.e. that there is at least one minimal left ideal. We prove here that in a compact right topological semigroup, even more is true: these semigroups are what we call abundant. In particular, βS is abundant, for every discrete semigroup S.

6.8. Definition A semigroup S is *abundant* if each of the following holds:
(a) every left ideal of S includes a minimal one.
(b) every (minimal) left ideal of S contains an idempotent element.
(In particular, $K(S)$ is non-empty, for S abundant.)

As a preparation for 6.10, let us state some easy but important properties of left ideals in a compact right topological semigroup.

6.9. Remark Assume that S is a compact right topological semigroup. Then the following hold true.
(a) Every left ideal of the form Sa, i.e. quasi-generated by a single element a, is closed in S – this holds because Sa is the image of S under right multiplication with a, a continuous map.
(b) Every left ideal I of S includes a closed one – pick $a \in I$; then Sa is a closed left ideal included in I.
(c) Every left ideal I of S contains an idempotent element – by (b), pick a closed left ideal J included in I. Then J is a closed subsemigroup of S and thus has an idempotent element by Theorem 1.21.
(d) Every minimal left ideal L of S is closed – pick $a \in L$ and note that, by 6.4, $L = Sa$.
(e) Every left ideal I of S includes a minimal one – fix a closed left ideal J included in I. As in the proof of 1.22, Zorn's lemma gives a minimal element L of the partial order $(\mathcal{I}, \subseteq)$ where $\mathcal{I}$ is the family of all closed left ideals included in J. Thus L is a minimal left ideal of S.

The main result of this section is an immediate consequence of the remark.

6.10. Corollary *Every compact right topological semigroup is abundant.*

3. Minimal left ideals and idempotents

In an arbitrary semigroup S, we defined, in 5.14, the (possibly empty) subset $E(S)$ of idempotents of S. In this section, we introduce a partial ordering relation on $E(S)$. For abundant semigroups S, it turns out that there are minimal elements in this partial order, and that they are closely connected to minimal left ideals.

6.11. Definition On the set $E(S)$ of idempotent elements of a semigroup S, we define the relation

$$e \leq f \quad \text{iff} \quad ef = fe = e.$$

This is obviously a partial ordering on $E(S)$. I.e. for $e, f, g \in E(S)$ we check that $e \leq e$; $e \leq f$ and $f \leq e$ imply that $e = f$; and $e \leq f$, $f \leq g$ imply that $e \leq g$.

A simple computation shows that $ef = e$ is equivalent to $Se \subseteq Sf$, which somewhat motivates the definition of the partial ordering $\leq$. We abbreviate the proof of the next theorem by introducing some notation.

6.12. Definition In a semigroup S, we put

$$E_{min}(S) = \{e \in E(S) : e \text{ is minimal in } (E(S), \leq)\}$$

and

$$\mathcal{L} = \{L \subseteq S : L \text{ is a minimal left ideal of } S\}.$$

We can now describe how, in abundant semigroups, minimal idempotents are connected to minimal left ideals. Recall that $K(S)$ was defined, in 6.6, to be the union of all minimal left ideals of S.

6.13. Theorem *Assume that S is an abundant semigroup and that $e \in E(S)$. Then the following hold.*
(a) If $L \subseteq Se$ is a minimal left ideal, then there is some idempotent $f \in L$ such that $f \leq e$.
(b) e is a minimal idempotent iff $e \in L$ for some minimal left ideal L, i.e. iff $e \in K(S)$.
(c) e is a minimal idempotent iff the left ideal $L = Se$ is minimal.
(d) There is some minimal idempotent f such that $f \leq e$.

PROOF. (a) By abundancy of S, pick some $\varepsilon \in L \cap E(S)$. Then $f = e\varepsilon$ works for the claim. First, $f \in L$ since L is a left ideal. Next, we note that $\varepsilon e = \varepsilon$, because $\varepsilon \in L \subseteq Se$, say $\varepsilon = se$ where $s \in S$ and thus $\varepsilon e = see = se = \varepsilon$. And f is idempotent because $f^2 = e\varepsilon e\varepsilon = e\varepsilon\varepsilon = e\varepsilon = f$. Finally we check that $ef = ee\varepsilon = e\varepsilon = f$ and $fe = e\varepsilon e = e\varepsilon = f$. So $f \leq e$.
(b) Assume that e is a minimal idempotent. By abundancy, fix some $L \in \mathcal{L}$ satisfying $L \subseteq Se$. By (a), pick $f \in L \cap E(S)$ such that $f \leq e$. Then $f = e$ by minimality of e, and thus $e \in L$. Conversely, assume that $e \in L$ where $L \in \mathcal{L}$. To prove minimality of e, let $x \in E(S)$ satisfy $x \leq e$, with the aim of showing $x = e$. Now the following hold: $Sx \subseteq Se$, by $x \leq e$ and the above remark; $L = Se = Sx$ by minimality of L, so $e = sx$ for some $s \in S$. By $x \leq e$, we have $x = ex$, and thus $x = ex = sxx = sx = e$.
(c) This follows from (b) and the observation that if $e \in L$ and $L \in \mathcal{L}$, then $L = Se$.
(d) By abundancy of S and (a), pick $L \in \mathcal{L}$ such that $L \subseteq Se$ and $f \in L \cap E(S)$ such that $f \leq e$. Then f is minimal, by (b). $\square$

6.14. Corollary *For an abundant semigroup S, the minimal idempotents of S are just those in $K(S)$, i.e. $E_{min}(S) = E(S) \cap K(S)$.*

4. More on $K(S)$

The subset $K(S)$ of an arbitrary semigroup S was defined to be the union of all minimal left ideals of S; Theorem 6.7 states that if $K(S) \neq \emptyset$, i.e. if there is a minimal left ideal, then $K(S)$ is the least two sided ideal of S. One might suspect that, by left-right symmetry, $K(S)$ is also the union of all minimal right ideals of S. A moment's reflection, however, shows that symmetry is broken by the assumption that there is a minimal *left* ideal. If we know that there is also a minimal *right* ideal, then by symmetry, we can conclude that $K(S)$ is also the union of all minimal right ideals. In fact, existence of a minimal left ideal with an idempotent in it implies that there is a minimal right ideal with an idempotent in it, as shown in the concluding result of this chapter.

The results of this section will not be used later and can be skipped by the reader. Note that the assumption of 6.15 and 6.16 (there is a minimal left ideal containing an idempotent) follows from abundance. And the notion of abundance, as defined in 6.8, breaks left-right symmetry, too.

6.15. Proposition *Assume that the semigroup S has a minimal left ideal L containing an idempotent e (so $L = Se$). Then $R = eS$ is a minimal right ideal of S (and contains e as an idempotent).*

Proof. The central idea of the proof is that the subset

$$G = eSe$$

of S is a group, under the multiplication of S. It is immediate that $G \subseteq L$, $e \in G$, G is closed under multiplication, and that $ex = xe = x$ holds for all $x \in G$.
Claim 1. For every $x \in G$, there is some $y \in G$ such that $yx = e$. – To see this, note that $x \in L$, so $L = Sx$ by minimality of L and 6.4. Now $e \in L$; say $e = sx$ where $s \in S$. Then $y = ese \in G$, and $yx = esex = esx = ee = e$.
Claim 2. For $x \in G$, there is $y \in G$ such that $xy = yx = e$, and hence G is a group. – Pick y by Claim 1, i.e. $y \in G$ and $yx = e$. To prove that $xy = e$, pick, again by Claim 1, $z \in G$ such that $zy = e$. It follows that $xy = exy = zyxy = zey = zy = e$. This finishes the proof that G is a group. We finally show that the right ideal $R = eS$ is minimal.
Claim 3. For any right ideal J included in $R = eS$, we have $R \subseteq J$. – Pick an arbitrary $t \in J$. Then $t \in eS$ and $te \in eSe = G$. By Claim 2, pick some $y \in G$ such that $tey = e$. But $t \in J$ and J is a right ideal, so $e \in J$. It follows that $R = eS \subseteq J$. □

6.16. Corollary *Assume that the semigroup S has a minimal left ideal containing an idempotent element. Then*

$$K(S) = \bigcup\{L : L \textit{ a minimal left ideal of } S\} = \bigcup\{R : R \textit{ a minimal right ideal of } S\}.$$

Proof. 6.15 guarantees that S has a minimal right ideal. Switching left and right ideals in Theorem 6.7 yields that the union of all minimal right ideals of S is the least two sided ideal of S, i.e. $K(S)$. □

Exercises

(1) We work in the semigroup $(S = {}^X X, \circ)$ of Example 1.6, for an arbitrary set X. Check which of the following subsets of S are left resp. right ideals.
(a) $I = \{f \in S : \text{ran} f \subseteq Y\}$ where Y is a non-empty subset of X
(b) $J = \{f \in S : f \text{ is not one-one}\}$
(c) $L = \{f \in S : f \text{ is not onto}\}$.

(2) Assume X is an infinite set; we consider the semigroup $(S = \mathcal{P}_f(X), \cup)$ from Example 1.4. Show that the ideals of S are the subsets I of S such that for $e \in I$, $f \in \mathcal{P}_f(X)$, and $e \subseteq f$, also $f \in I$.

(3) Given any set S, we consider S as a subset of βS as usual and put $S^* = \beta S \setminus S$, the set of all free ultrafilters on S.
(a) Prove that for the semigroup $(\omega, +)$, ω^* is a two sided ideal of $\beta\omega$.
(b) Find a semigroup S in which S^* is neither a left nor a right ideal.
(c) Try to generalize (a) to some more semigroups.

(4) Assume that I is a left ideal of a semigroup S. Prove that $\hat{I}$, the Stone set of I as defined in 3.1, is a left ideal of βS. Similarly for right respectively two sided ideals.

(5) Let e be an idempotent element of a semigroup S; so eSe is a subsemigroup of S with e as an identity. Prove that the set

$$H(e) = \{x \in eSe : \text{ there is some } y \in eSe \text{ such that } xy = yx = e\}$$

is a group (with identity e), under the multiplication of S – in fact, the largest subgroup of S having e as its identity.

(6) Assume that e is a minimal idempotent of an abundant semigroup S. Prove that, for the subgroup $H(e)$ of S defined in Exercise 5, we have $H(e) = eSe$.

(7) Assume that L is a minimal left ideal of an abundant semigroup S. Prove that L is the union of the (disjoint) subgroups $H(e) = eSe$, where $e \in E(S) \cap L$.