CHAPTER 3

βS as a topological space

In the preceding chapter, we introduced the set βS of all ultrafilters on a (non-empty) set S. Here, we will make βS a topological space which is, in fact, compact and Hausdorff. βS can be conceived, in a natural way, as a compactification of S with the discrete topology, and as such it enjoys a universal property which explains its central importance.

1. The Stone topology on βS

We assume in this chapter that S is a non-empty set. We introduce here the Stone topology on βS and prove that βS, with this topoplogy, is what we call a Boolean space; in particular, it is compact and Hausdorff.

3.1. Definition For a subset A of S, we denote the *Stone set* $\hat{A} \subseteq \beta S$ corresponding to A by

$$\hat{A} = \{p \in \beta S : A \in p\}.$$

The family

$$\mathcal{B} = \{\hat{A} : A \subseteq S\},$$

consisting of all Stone sets is the *Stone base* of βS.

We will compute over and over with Stone sets, which is facilitated by the following lemmas. Obviously, $\hat{S} = \beta S$ and $\hat{\emptyset} = \emptyset$.

3.2. Lemma *For all subsets A, B of S, the following equations hold.*

$$\widehat{A \cap B} = \hat{A} \cap \hat{B}, \quad \widehat{A \cup B} = \hat{A} \cup \hat{B}, \quad \widehat{S \setminus A} = \beta S \setminus \hat{A}.$$

Proof. The elements of the Stone sets occurring in these equations are ultrafilters p on S. Now the first equation holds since $p \in \widehat{A \cap B}$ holds iff $A \cap B \in p$, which is equivalent to $A \in p$ and $B \in p$, i.e. to $p \in \hat{A} \cap \hat{B}$. The second one follows similarly because every ultrafilter is a prime filter, and the last one by the very definition of an *ultra*filter. □

Lemma 3.2 shows that the Stone base $\mathcal{B}$ is closed under finite intersections, unions, and complementation. By closedness under intersections, there is a unique topology on βS having $\mathcal{B}$ as an open base, the Stone topology.

3.3. Definition The *Stone topology* on βS is the topology which has $\mathcal{B}$ as a base for open sets. We call βS, with the Stone topology, the *Stone-Čech compactification* of S. Cf. 3.11 and 3.12 for more explanation of this denotation.

3.4. Definition and Remark (a) By the very definition of the Stone base $\mathcal{B}$ being a base for the open sets of βS, we state that a subset U of βS is open iff it is the union of a family of Stone sets, i.e. iff there is a family $\{A_i : i \in I\}$ of subsets of S such that $U = \bigcup_{i \in I} \hat{A}_i$.
(b) By passing to complements and using the fact that the Stone base is closed under complementation, it follows that a subset Y of βS is closed iff it is the intersection of a family of Stone sets, i.e. iff there is a family $\{A_i : i \in I\}$ of subsets of S such that $Y = \bigcap_{i \in I} \hat{A}_i$.
(c) In the space βS, every point p has the family $\{\hat{A} : A \in p\}$ as a canonical neighbourhood base.

Before studying the Stone topology in detail, let us note some more simple facts on Stone sets.

3.5. Lemma *For any subsets A and B of S, the following hold:*
(a) $A = \emptyset$ iff $\hat{A} = \emptyset$
(b) $A \subseteq B$ iff $\hat{A} \subseteq \hat{B}$
(c) $A = B$ iff $\hat{A} = \hat{B}$
(d) $A = S$ iff $\hat{A} = \beta S$.

Proof. (a) is simply the statement of 2.9(b) that every nonempty subset of S is contained in an ultrafilter. (c) follows from (b), and (d) from (c). For (b), we note that $A \subseteq B$ iff $A \setminus B = \emptyset$ iff $\widehat{A \setminus B} = \hat{A} \setminus \hat{B} = \emptyset$ iff $\hat{A} \subseteq \hat{B}$. □

We begin to prove the most important properties of the space βS.

3.6. Definition For a topological space X, we define the following.
(a) A subset U of X is said to be *clopen* if it is both closed and open – i.e. if both U and $X \setminus U$ are open. E.g. every Stone set $\hat{A}$ is clopen in βS, because, by Lemma 3.2, its complement in βS is the Stone set of $S \setminus A$.
(b) X is called *zero-dimensional* if it has an open base consisting of clopen sets. It is a *Boolean space* if it is Hausdorff, compact, and zero-dimensonal.

3.7. Theorem *he space βS, with the Stone topology, is Boolean.*

Proof. The Stone base $\mathcal{B}$ shows that βS is a zero-dimensional space. To prove that it is Hausdorff, assume that p and q are two different points in βS, i.e. two different ultrafilters on S. Being a maximal filter, p cannot be a subset of q, so pick some $A \in p \setminus q$. Then $S \setminus A \in q$, and $\hat{A}$ and $\widehat{S \setminus A}$ are disjoint open neighbourhoods of p and q.
Finally assume that $\mathcal{U}$ is a family of open sets which covers βS; we have to find a finite subfamily covering βS. We may assume that the elements of $\mathcal{U}$ are basic sets, i.e. $\mathcal{U} = \{\hat{A} : A \in \mathcal{A}\}$ where $\mathcal{A}$ is a family of subsets of S. Now since $\{\hat{A} : A \in \mathcal{A}\}$ covers βS, every ultrafilter on S contains some $A \in \mathcal{A}$. Passing to complements, this means that no ultrafilter on S includes the family $\mathcal{C} = \{S \setminus A : A \in \mathcal{A}\}$. By Corollary 2.9, $\mathcal{C}$ does not have the finite intersection property, so pick a finite

subfamily $\mathcal{C}'$ of $\mathcal{C}$ satisfying $\bigcap_{C\in\mathcal{C}'} C = \emptyset$. Then $\mathcal{A}' = \{S \setminus C : C \in \mathcal{C}'\}$ is a finite subfamily of $\mathcal{A}$ satisfying $\bigcup_{A\in\mathcal{A}'} A = S$, and hence $\{\hat{A} : A \in \mathcal{A}'\}$ is a finite subcover of $\{\hat{A} : A \in \mathcal{A}\}$. □

We can use the compactness of βS to show that its clopen subsets are exactly the Stone sets.

3.8. Corollary *A subset of βS is clopen iff it is the Stone set $\hat{A}$ of some subset A of S.*

PROOF. Assume that $U \subseteq \beta S$ is clopen; fix a family $\mathcal{A}$ of subsets of S such that $U = \bigcup_{A\in\mathcal{A}} \hat{A}$. Now U is a closed and hence compact subspace of βS; thus the open cover $\{\hat{A} : A \in \mathcal{A}\}$ of U has a finite subcover. Pick $A_1, \ldots, A_n \in \mathcal{A}$ such that $U = \hat{A}_1 \cup \cdots \cup \hat{A}_n$; so U is the Stone set of $A = A_1 \cup \cdots \cup A_n$. □

2. βS as a compactification of S

In the preceding chapter, we defined the canonical injection e from S into βS mapping every element s of S to the fixed ultrafilter $\dot{s}$. We will prove that the pair $(\beta S, e)$ is what topologists call a compactification of S. Let us first note a few simple but important facts on Stone sets and specific points in βS.

3.9. Remark (a) For $s \in S$ and $A \subseteq S$, clearly

$$s \in A \text{ iff } A \in \dot{s} \text{ iff } \dot{s} \in \hat{A}.$$

Thus if A is infinite, then so is $\hat{A}$.
(b) A point x of a topological space X is called *isolated* if the set $\{x\}$ happens to be open in X, i.e. if $\{x\}$ is a neighbourhood of x.
We claim that the isolated points of βS are exactly the fixed ultrafilters $\dot{s}$, for $s \in S$. First, if p is a free ultrafilter on S, then every set A in p is infinite, by 2.13 and so is $\hat{A}$; thus every neighourhood of p is infinite and p is non-isolated. On the other hand, let $p = \dot{s}$ where $s \in S$; let $A \subseteq S$ be the singleton $\{s\}$. The Stone set $\hat{A}$ is the smallest neighbourhood of p, and $\hat{A} = \{\dot{s}\}$: by finiteness of A, $\hat{A}$ contains no free ultrafilter, and the only fixed ultrafilter in $\hat{A}$ is $\dot{s}$, by (a).
(c) Thus the canonical map e defined by $e(s) = \dot{s}$ is a bijection from S onto the set of isolated points of βS.
(d) Additionally, the set $\{\dot{s} : s \in S\}$ of isolated points is dense in βS because every non-empty Stone set $\hat{A} \subseteq \beta S$ contains some point $\dot{s}$, by (a).

3.10. Definition (a) A map $f : X \to Y$ between two topological spaces X and Y is an *embedding* if it is a homeomorphism from X onto its image $f[X]$ under f; i.e. if it is one-one, continuous, and for every subset U of X, U is open in X iff $f[U]$ is open in $f[X]$ (where $f[X]$ is equipped with the subspace topology inherited from Y).
(b) For a topological space X, a *compactification of* X is a pair (Y, f) where Y is a compact Hausdorff space, $f : X \to Y$ is an embedding, and the range $f[X]$ of f is a dense subspace of Y.

3.11. Theorem *Consider the set S as a topological space with the discrete topology. Then the pair $(\beta S, e)$ is a compactification of S.*

PROOF. We are left with proving that e is a homeomorphism from the discrete space S onto its image $e[S] = \{\dot{s} : s \in S\}$, i.e. that $e[S]$ is discrete, with the subspace topology from βS. But every point $e(s) = \dot{s}$ of $e[S]$ is isolated in βS, by 3.9, hence in $e[S]$. □

Identifying every point $s \in S$ with its image $\dot{s}$ under the map e, we shall, beginning from Chapter 4, view S as a dense subspace of the compact space βS. Moreover we know that S is the set of isolated points of βS.

3. The universal property of βS

We now prove a characteristic extension property of the compactification $(\beta S, e)$ of S which will frequently be used when dealing with βS.

3.12. Theorem *Assume X is a compact Hausdorff space and $f : S \to X$ is an arbitrary map. Then there is a unique continuous map $\tilde{f} : \beta S \to X$ such that $\tilde{f} \circ e = f$, the* Stone-Čech extension *of f.*
Thus if we identify S with the dense subspace of βS consisting of all isolated points, i.e. we think about e as being the inclusion map from S to βS, then $\tilde{f}$ is the unique continuous extension of f to βS.

PROOF. For uniqueness, just note that $e[S]$ is a dense subset of βS; thus every map from $e[S]$ into a Hausdorff space has at most one continuous extension.
For existence, we rely on the notion of a p-limit introduced in 2.17. Consider the family $(f(s))_{s \in S}$ of points in X, indexed by S. Since X is compact Hausdorff, we can define, for $p \in \beta S$,

$$\tilde{f}(p) = p - lim_{s \in S} f(s).$$

The map $\tilde{f}$ thus defined satisfies $\tilde{f} \circ e = f$ because for $a \in S$ and $p = \dot{a}$, we have $p - lim_{s \in S} f(s) = f(a)$, by Example 2.18.
We are left with showing that $\tilde{f}$ is continuous. So consider some $p \in \beta S$ and $x = \tilde{f}(p)$; let U be a neighbourhood of x in X with the aim of finding a neighbourhood $\hat{A}$ of p in βS which is mapped by $\tilde{f}$ into U. Pick a neighbourhood V of x with its closure included in U. Since $x = p - lim_{s \in S} f(s) \in V$, there is some $A \in p$ such that $\{f(s) : s \in A\} \subseteq V$. This set A works, since for every $q \in \hat{A}$, we conclude from $A \in q$ and Theorem 2.19 (a) that $\tilde{f}(q) = q - lim_{s \in S} f(s) \in cl\{f(s) : s \in A\} \subseteq cl(V) \subseteq U$. □

Let us note that our construction of the Stone-Čech compactification βS of a discrete space S is a very special case of a more general fact of set theoretic topology: given a completely regular topological space X, there is a (unique) compactification (Z, e) of X with the universal property described in 3.12. I.e.for every continuous map $f : X \to Y$ from X into a compact Hausdorff space Y, there is a unique continuous map $\tilde{f} : Z \to Y$ satisfying $\tilde{f} \circ e = f$. The pair (Z, e) is then called the Stone-Čech compactification of X.

The universal property 3.12 of βS allows for a more natural explanation of Theorem 2.22. Given a sequence $(x_s)_{s \in S}$ of points in a compact Hausdorff space X, we consider the function f from S into X defined by $f(s) = x_s$ and its Stone-Čech extension $\tilde{f} : \beta S \to X$. The image Y of βS under $\tilde{f}$ is closed in X, by compactness of

βS and continuity of $\tilde{f}$, and it includes $\{x_s : s \in S\}$. Moreover S is dense in βS and hence its image $\{x_s : s \in S\}$ under f is dense in Y, so Y is the closure of $\{x_s : s \in S\}$ in X. And by the construction of $\tilde{f}$ in 3.12, $Y = \{p - \lim_{s \in S} x_s : p \in \beta S\}$.

4. The relationship between $A \subseteq S$ and $\hat{A} \subseteq \beta S$

In this section, we make two additional statements on how a subset A of S is related to its corresponding Stone set $\hat{A}$, a clopen subset of βS. Recall that e denotes the canonical embedding from S into βS.

3.13. Proposition *For any $A \subseteq S$, $\hat{A}$ is the closure (in βS) of the set $e[A] = \{\dot{s} : s \in A\}$, and $\hat{A} \cap \{\dot{s} : s \in S\} = \{\dot{s} : s \in A\}$. I.e. if we identify every $s \in S$ with $e(s) = \dot{s}$ and S with the dense set of isolated points of βS, we obtain that $\hat{A}$ is the closure of A and $\hat{A} \cap S = A$.*

Proof. The second statement has already been noted in part (a) of Remark 3.9. The first one is proved like the density of $e[S]$ in βS in 3.9: for a non-empty open subset U of $\hat{A}$, say $U = \hat{B}$ where $B \subseteq A$, pick some $s \in B$; so $s \in A$ and $\dot{s} \in \hat{B}$. ☐

For the next proposition, note that we consider S and hence every subset A of S with the discrete topology, so the Stone-Čech compactification βA of A is well defined. On the other hand, the preceding proposition says that also $(\hat{A}, e \restriction A)$ is a compactification of A.

3.14. Proposition *For any $A \subseteq S$, $\hat{A}$ is canonically homeomorphic to βA.*

Proof. The points of $\hat{A}$ are the ultrafilters on S containing A and the points of βA are the ultrafilters on A. There are obvious maps $f : \hat{A} \to \beta A$ and $g : \beta A \to \hat{A}$ defined by $f(p) = p \cap \mathcal{P}(A)$ and $g(q) = \{B \subseteq S : B \cap A \in q\}$; they are both continuous and inverses of each other. ☐

Exercises

(1) Assume that Y and Z are disjoint closed subsets of βS. Prove that Y and Z are separated by basic sets, i.e. there are disjoint subsets A, B of S such that $Y \subseteq \hat{A}$ and $Z \subseteq \widehat{B}$.

(2) Prove that, in the space βS, the closure of every open subset is clopen. (This very special property of βS is called extreme disconnectedness.)

(3) Let A, and, for $i \in I$, A_i be subsets of S. Prove that $\hat{A} = \bigcup_{i \in I} \widehat{A_i}$ iff $A_i \subseteq A$ holds for all $i \in I$ and there is a finite subset J of I such that $A = \bigcup_{i \in J} A_i$. Similarly, $\hat{A} = \bigcap_{i \in I} \widehat{A_i}$ iff $A \subseteq A_i$ holds for all $i \in I$ and there is some finite $J \subseteq I$ such that $A = \bigcap_{i \in J} A_i$.
This can be proved as the compactness statement of Theorem **??** or directly concluded from it.

(4) Prove that in the space βS, the p-limit of the sequence $(\dot{s})_{s \in S}$ is the point p.

(5) Prove that for a free ultrafilter p on S, the point p of βS does not have a countable neighbourhood base.

(6) Assume that $f : S \to T$ is a map between the discrete spaces S and T. We consider f as a mapping from S into the compact Hausdorff space βT and thus obtain its Stone-Čech extension $\tilde{f} : \beta S \to \beta T$. So for $p \in \beta S$, $\tilde{f}(p)$ is an ultrafilter on T. Prove that a subset B of T is in $\tilde{f}(p)$ iff its preimage $f^{-1}[B]$ under f is in p.

(7) Give a more detailed proof of Proposition 3.14: for $A \subseteq S$, the Stone-Čech compactification βA of the discrete space A is homeomorphic to the Stone set $\hat{A}$.

(8) We call a compactification (C, g) of a discrete space S *universal* if it has the property proved for $(\beta S, e)$ in Theorem 3.12, i.e. for any compact Hausdorff space X and any map $f : S \to X$, there is a unique continuous map $k : C \to X$ such that $k \circ g = f$.
Prove that any two universal compactifications (C, g) and (D, h) of S are homeomorphic, more precisely there is a unique homeomorphism $k : C \to D$ such that $k \circ g = h$.

(9) (For model theorists.) Let $\mathcal{L}$ be a first order language. We denote the set of all sentences (i.e. closed formulas) of $\mathcal{L}$ by F and the set of all maximally consistent $\mathcal{L}$-theories by X. For $\mathcal{A}$ an $\mathcal{L}$-structure, $Th(\mathcal{A}) = \{\alpha \in F : \mathcal{A} \models \alpha\}$, the first order theory of $\mathcal{A}$, is maximally consistent, and

$$X = \{Th(\mathcal{A}) : \mathcal{A} \text{ an } \mathcal{L}\text{-structure}\}.$$

(a) For $\alpha \in F$, define $\hat{\alpha}$ to be the subset $\{T \in X : \alpha \in T\}$ of X. Prove that X is a Boolean space under the topology with base $\mathcal{B} = \{\hat{\alpha} : \alpha \in F\}$.
(b) Assume that for every i in a set I, $\mathcal{A}_i$ is an $\mathcal{L}$-structure and $T_i = Th(\mathcal{A}_i)$. Let p be an ultrafilter on I, $\mathcal{A}$ the ultraproduct of the family $(\mathcal{A}_i)_{i \in I}$ with respect to p, and $T = Th(\mathcal{A})$. Prove that T is the p-limit of the sequence $(T_i)_{i \in I}$, in the space X.