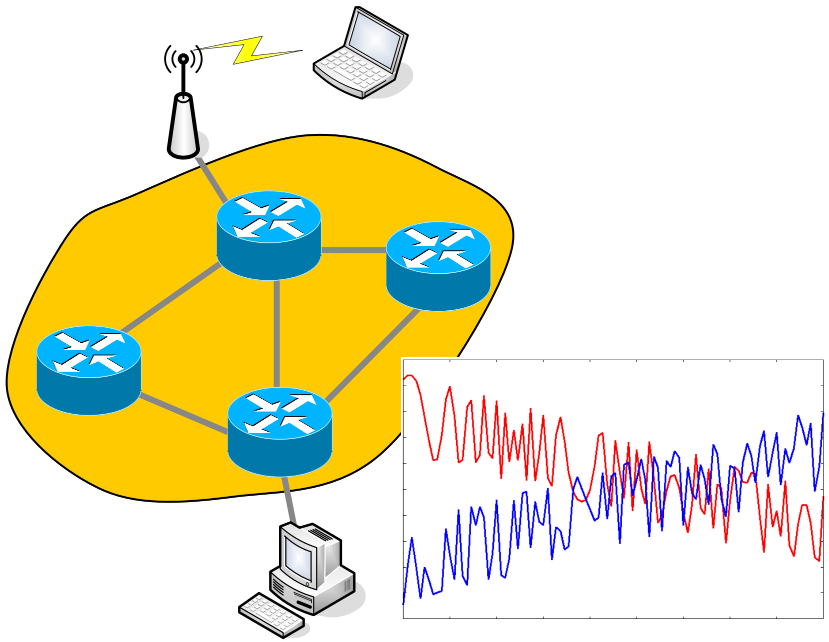
Chapter 12

Comparison and Evaluation of Alternative System Designs



Contents

- For two-system comparisons
  - Independent sampling
  - Correlated sampling (common random numbers)
- For multiple system comparisons
  - Bonferroni approach: confidence-interval estimation, screening, and selecting the best
- Metamodels

Purpose

- Purpose: comparison of alternative system designs.
- Approach: discuss a few of many statistical methods that can be used to compare two or more system designs.
- Statistical analysis is needed to discover whether observed differences are due to:
  - Differences in design or
  - The random fluctuation inherent in the models

Comparison of Two System Designs

Comparison of Two System Designs

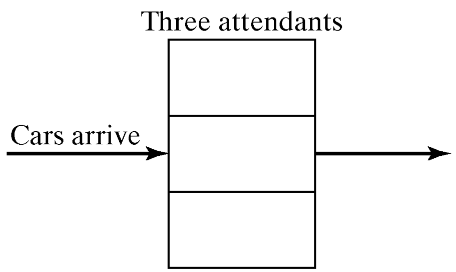
- Goal: compare two possible configurations of a system
  - Two possible ordering policies in a supply-chain system, two possible scheduling rules in a job shop
  - Two routing protocols in a network
  - Two different congestion control algorithms on the transport layer
  - Two MAC protocols

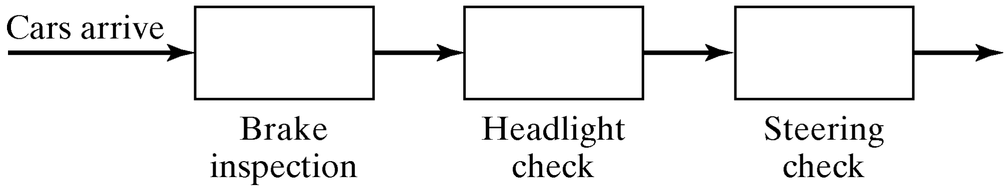
- Approach: the method of replications is used to analyze the output data

- The mean performance measure for system i
  - Denoted by θ_i , $i = 1,2,\dots$
- To obtain point and interval estimates for the difference in mean performance, namely $\theta_1 - \theta_2$

Comparison of Two System Designs

- Vehicle-safety inspection example:
  - The station performs 3 jobs: (1) brake check, (2) headlight check, and (3) steering check.
  - Vehicles arrival: Poisson with rate = 9.5/hour.
  - Present system:
    - Three stalls in parallel (one attendant makes all 3 inspections at each stall).
    - Service times for the 3 jobs: normally distributed with means 6.5, 6.0 and 5.5 minutes, respectively.
  - Alternative system:
    - Each attendant specializes in a single task, each vehicle will pass through three work stations in series
    - Mean service times for each job decreases by 10% (5.85, 5.4, and 4.95 minutes).
  - Performance measure: mean response time per vehicle (total time from vehicle arrival to its departure).





Comparison of Two System Designs

- From replication r of system i, the analyst obtains an estimate Y_{ir} of the mean performance measure θ_i

- Assuming that the estimators Y_{ir} are (at least approx.) unbiased:

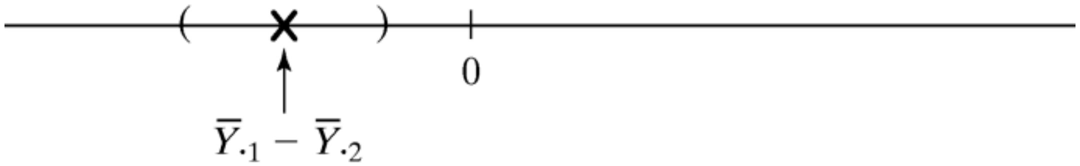
$$\theta_1 = E(Y_{1r}), \qquad r = 1, \ldots, R_1$$

$$\theta_2 = E(Y_{2r}), \qquad r = 1, \ldots, R_2$$

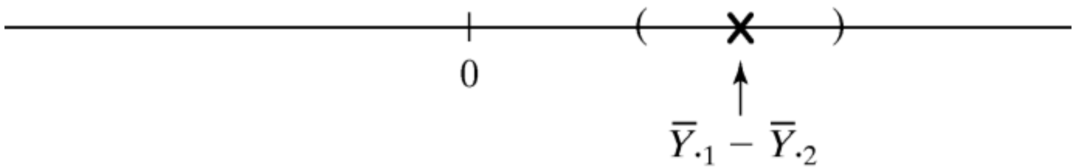
- Goal:
  Compute a confidence interval for $\theta_1 - \theta_2$ to compare the two system designs

Comparison of Two System Designs

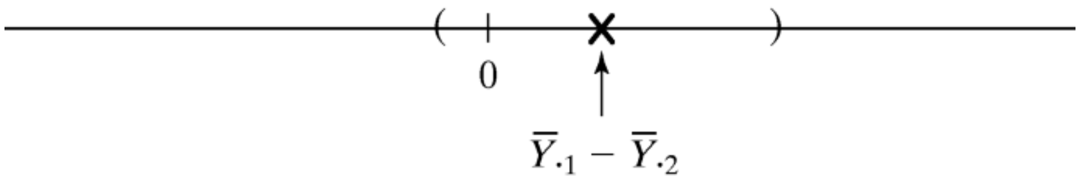
- If CI is totally to the left of 0, strong evidence for the hypothesis that $\theta_1 - \theta_2 < 0$ ($\theta_1 < \theta_2$)



- If CI is totally to the right of 0, strong evidence for the hypothesis that $\theta_1 - \theta_2 > 0$ ($\theta_1 > \theta_2$)



- If CI contains 0, no strong statistical evidence that one system is better than the other



> If enough additional data were collected (i.e., R_i increased), the CI would most likely shift, and definitely shrink in length, until conclusion of $\theta_1 < \theta_2$ or $\theta_1 > \theta_2$ would be drawn.

Comparison of Two System Designs

- In this chapter:
  - A two-sided 100(1-α)% CI for $\theta_1 - \theta_2$ always takes the form of:

$$\left(\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}\right) \pm t_{\alpha/2,\upsilon} \cdot \text{s.e.}(\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2})$$

Sample mean for system i

Degree of freedom

Standard error of the estimator

  - All three techniques assume that the basic data Y_{ir} are approximately normally distributed.

Comparison of Two System Designs

- Statistically significant versus practically significant
  - Statistical significance: is the observed difference $\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}$ larger than the variability in $\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}$?
  - Practical significance: is the true difference $\theta_1 - \theta_2$ large enough to matter for the decision we need to make?

  - Confidence intervals do not answer the question of practical significance directly, instead, they bound the true difference within the range:

$$\left(\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}\right) - t_{\frac{\alpha}{2},\upsilon}\, s.e.(\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}) \le \theta_1 - \theta_2 \le \left(\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}\right) + t_{\frac{\alpha}{2},\upsilon}\, s.e.(\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2})$$

  - Whether a difference within these bounds is practically significant depends on the particular problem.

Comparison of Two System Designs

Independent Sampling

Independent Sampling with Equal Variances

- Different and independent random number streams are used to simulate the two systems
  - All observations of simulated system 1 are statistically independent of all the observations of simulated system 2.

- The variance of the sample mean $\bar{Y}_{\bullet i}$ is:

$$V\left(\bar{Y}_{\bullet i}\right) = \frac{V\left(Y_{\bullet i}\right)}{R_i} = \frac{\sigma_i^2}{R_i}, \qquad i = 1,2$$

- For independent samples:

$$V\left(\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}\right) = V\left(\bar{Y}_{\bullet 1}\right) + V\left(\bar{Y}_{\bullet 2}\right) = \frac{\sigma_1^2}{R_1} + \frac{\sigma_2^2}{R_2}$$

Independent Sampling with Equal Variances

- If it is reasonable to assume that $\sigma^2_1 = \sigma^2_2$ (approx.) or if $R_1 = R_2$, a two-sample-t confidence-interval approach can be used:
  - The point estimate of the mean performance difference is: $\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}$
  - The sample variance for system i is:

$$S_i^2 = \frac{1}{R_i - 1}\sum_{r=1}^{R_i}\left(Y_{ri} - \bar{Y}_{\bullet i}\right)^2 = \frac{1}{R_i - 1}\sum_{r=1}^{R_i} Y_{ri}^{\;2} - R_i \bar{Y}_{\bullet i}^{\;2}$$

  - The pooled estimate of σ^2 is:

$$S_p^2 = \frac{(R_1 - 1)S_1^2 + (R_2 - 1)S_2^2}{R_1 + R_2 - 2}, \qquad \text{where } \upsilon = R_1 + R_2\text{-}2 \text{ degrees of freedom}$$

  - CI is given by:

$$\left(\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}\right) \pm t_{\alpha/2,\upsilon}\, s.e.(\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2})$$

  - Standard error:

$$s.e.\left(\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}\right) = S_p\sqrt{\frac{1}{R_1} + \frac{1}{R_2}}$$

Independent Sampling with Unequal Variances

- If the assumption of equal variances cannot safely be made, an approximate $100(1-\alpha)\%$ CI can be computed as:

$$s.e.\left(\bar{Y}_{.1} - \bar{Y}_{.2}\right) = \sqrt{\frac{S_1^2}{R_1} + \frac{S_2^2}{R_2}}$$

  - With degrees of freedom:

$$\upsilon = \frac{\left(\dfrac{S_1^2}{R_1} + \dfrac{S_2^2}{R_2}\right)^2}{\dfrac{\left(\dfrac{S_1^2}{R_1}\right)^2}{R_1 - 1} + \dfrac{\left(\dfrac{S_2^2}{R_2}\right)^2}{R_2 - 1}}, \quad \text{round to an interger}$$

  - In this case, the minimum number of replications $R_1 > 7$ and $R_2 > 7$ is recommended.

Comparison of Two System Designs

Common Random Numbers (CRN)

Common Random Numbers (CRN)

- For each replication, the same random numbers are used to simulate both systems ➡ $R_1=R_2=R$.
  - For each replication r, the two estimates, Y_{r1} and Y_{r2}, are correlated.
  - However, independent streams of random numbers are used on different replications, so the pairs (Y_{r1}, Y_{s2}) are mutually independent for $r \neq s$.

- Purpose: induce positive correlation between $\bar{Y}_{\bullet 1}, \bar{Y}_{\bullet 2}$ (for each r) to reduce variance in the point estimator of $\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}$.

$$V\left(\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}\right) = V\left(\bar{Y}_{\bullet 1}\right) + V\left(\bar{Y}_{\bullet 2}\right) - 2\operatorname{cov}\left(\bar{Y}_{\bullet 1}, \bar{Y}_{\bullet 2}\right)$$

$$= \frac{\sigma_1^2}{R} + \frac{\sigma_2^2}{R} - \frac{2\rho_{12}\sigma_1\sigma_2}{R}$$

Correlation:
ρ_{12} is positive

Common Random Numbers (CRN)

- Compare variance from independent sampling with variance from CRN:

$$V_{CRN} = V_{IND} - \frac{2\rho_{12}\sigma_1\sigma_2}{R}$$

- Variance of $\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}$ arising from CRN is less than that of independent sampling (with $R_1 = R_2$).

Common Random Numbers (CRN)

- The estimator based on CRN is more precise, leading to a shorter confidence interval for the difference.
- Sample variance of the differences $\bar{D} = \bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}$

$$S_D^2 = \frac{1}{R-1}\sum_{r=1}^{R}\left(\bar{D}_r - \bar{D}\right)^2 = \frac{1}{R-1}\left(\sum_{r=1}^{R} D_r^2 - R\bar{D}^2\right)$$

where $D_r = Y_{r1}\text{-}Y_{r2}$ and $\bar{D} = \frac{1}{R}\sum_{r=1}^{R} D_r$, with degress of freedom $\upsilon = R\text{-}1$

- Standard error: $s.e.(\bar{D}) = s.e.\left(\bar{Y}_{.1} - \bar{Y}_{.2}\right) = \frac{S_D}{\sqrt{R}}$

Common Random Numbers (CRN)

- It is never enough to simply use the same seed for the random-number generator(s):

  - The random numbers must be synchronized: each random number used in one model for some purpose should be used for the same purpose in the other model.

  - Example: if the i-*th* random number is used to generate a service time at work station 2 for the 5-*th* arrival in model 1, the i-*th* random number should be used for the very same purpose in model 2.

Common Random Numbers (CRN): Example

- Vehicle inspection example:
  - 4 input random variables:
    - A_n interarrival time between vehicle n and vehicle $n+1$,
    - $S_n^{(i)}$ inspection time for task i for vehicle n in model 1 ($i=1,2,3$; refers to brake, headlight and steering task, respectively).

  - When using CRN:
    - Same values should be generated for $A_1, A_2, A_3, \ldots$ in both models.
    - However, mean service time for model 2 is 10% less.
    - Two possible approaches to obtain correlated service times:
      - Let $S_n^{(i)}$ be the service times generated for model 1, use:

$$S_n^{(i)} - 0.1E[S_n^{(i)}]$$

      - Let $Z_n^{(i)}$ as the standard normal variate, σ = 0.5 *minutes*, use:

$$E[S_n^{(i)}] + \sigma Z_n^{(i)}$$

    - For synchronized runs: the service times for a vehicle were generated at the instant of arrival and stored as its attribute and used as needed.

Common Random Numbers (CRN): Example

- Each replication run of 16 hours

Model 2 with independent random numbers

Model 1

Model 2 with common random numbers without synchronisation

Model 2 with common random numbers with synchronisation

	Average Response Time for Model				*Observed Differences*	
Replication	*1*	*2I*	*2C**	*2C*	$D_{1,2C^*}$	$D_{1,2C}$
1	29.59	51.62	56.47	29.55	−26.88	0.04
2	23.49	51.91	33.34	24.26	−9.85	−0.77
3	25.68	45.27	35.82	26.03	−10.14	−0.35
4	41.09	30.85	34.29	42.64	6.80	−1.55
5	33.84	56.15	39.07	32.45	−5.23	1.39
6	39.57	28.82	32.07	37.91	7.50	1.66
7	37.04	41.30	51.64	36.48	−14.60	0.56
8	40.20	73.06	41.41	41.24	−1.21	−1.04
9	61.82	23.00	48.29	60.59	13.53	1.23
10	44.00	28.44	22.44	41.49	21.56	2.51
Sample mean	37.63	43.04			−1.85	0.37
Sample variance	118.90	244.33			208.94	1.74
Standard error	6.03				4.57	0.42

Common Random Numbers (CRN): Example

- Compare the two systems using independent sampling and CRN where $R = R_1 = R_2 = 10$:
  - Independent sampling: $\bar{Y}_{.1} - \bar{Y}_{.2} = -5.4$ minutes

    with $\upsilon = 17$, $t_{0.05,17} = 2.11$, $S_1^2 = 118.9$ and $S_2^2 = 244.3$, CI: $-18.1 \le \theta_1 - \theta_2 \le 7.3$

  - CRN without synchronization: $\bar{Y}_{.1} - \bar{Y}_{.2} = -1.9$ minutes

    with $\upsilon = 9$, $t_{0.05,9} = 2.26$, $S_D^2 = 208.9$, CI: $-12.3 \le \theta_1 - \theta_2 \le 8.5$

  - CRN with synchronization: $\bar{Y}_{.1} - \bar{Y}_{.2} = 0.4$ minutes

    with $\upsilon = 9$, $t_{0.05,9} = 2.26$, $S_D^2 = 1.7$, CI: $-0.50 \le \theta_1 - \theta_2 \le 1.30$

CRN with Specified Precision

- Goal: The error in our estimate of $\theta_1 - \theta_2$ to be less than ε
- Approach: determine the # of replications R such that the half-width of CI:

$$H = t_{\alpha/2,\upsilon} s.e.\left(\bar{Y}_{\bullet 1} - \bar{Y}_{\bullet 2}\right) \le \varepsilon$$

- Vehicle inspection example (cont.):
  - $R_0 = 10$, complete synchronization of random numbers yield 95% CI: 0.4 ± 0.9 minutes
  - Suppose ε = 0.5 minutes for practical significance, we know R is the smallest integer satisfying $R \geq R_0$ and:

$$R \ge \left(\frac{t_{\alpha/2,R-1} S_D}{\varepsilon} \right)^2$$

  - Since $t_{\alpha/2,R-1} \le t_{\alpha/2,R_0-1}$, a conservation estimate of R is: $R \ge \left(\frac{t_{\alpha/2,R_0-1} S_D}{\varepsilon} \right)^2$
  - Hence, 35 replications are needed (25 additional).

Comparison of Several System Designs

Comparison of Several System Designs

- To compare K alternative system designs
  - Based on some specific performance measure, θ_i, of system i, for $i = 1, 2, ..., K$
- Procedures are classified as:
  - Fixed-sample-size procedures: predetermined sample size is used to draw inferences via hypothesis tests of confidence intervals
  - Sequential sampling (multistage): more and more data are collected until an estimator with a prespecified precision is achieved or until one of several alternative hypotheses is selected
- Some goals/approaches of system comparison:
  - Estimation of each parameter θ
  - Comparison of each performance measure θ_i to a control θ_1
  - All pair wise comparisons θ_i - θ_j for $i \neq j$
  - Selection of the best θ_i

Bonferroni Approach

- To make statements about several parameters simultaneously, where all statements are true simultaneously.
- Bonferroni inequality:

$$P(\text{all statements } S_i \text{ are true}, i = 1, \ldots, C) \geq 1 - \sum_{j=1}^{C} \alpha_j = 1 - \alpha_E$$

Overall error probability, provides an upper bound on the probability of a false conclusion

  - The smaller α_j is, the wider the *j-th* confidence interval will be.

- Major advantage: inequality holds whether models are run with independent sampling or CRN
- Major disadvantage: width of each individual interval increases as the number of comparisons increases.

Bonferroni Approach

- Should be used only for a small number of comparisons
  - Practical upper limit: about 20 comparisons

- There are 3 possible applications:
  1. Individual CI's: Construct a 100(1- α_j)% CI for parameter θ_i, where number of comparisons $= K$.

  2. Comparison to an existing system: Construct a 100(1−α_j)% CI for parameter $\theta_i - \theta_1$ (i = 2,3, …, K), number of comparisons $= K - 1$.

  3. All pairwise: For any 2 different system designs, construct a 100(1−α_j)% CI for parameter θ_i-θ_j.
     Hence, total number of comparisons $= K(K-1)/2$.

Comparison of Several System Designs

Bonferroni Approach to Selecting the Best

Bonferroni Approach to Selecting the Best

- Among K system designs, to find the best system
  - "Best" = the maximum expected performance, where the *i-th* design has expected performance θ_i .

- Focus on parameters: $\theta_i - \max_{j \neq i}\{\theta_j\} \quad \text{for } i = 1, 2, ..., K$
  - If system design i is the best, it is the difference in performance between the best and the second best.
  - If system design i is not the best, it is the difference between system i and the best.

- Goal: the probability of selecting the best system is at least $1-\alpha$, whenever $\theta_i - \max_{j \neq i}\{\theta_j\} \geq \varepsilon$
  - Hence, both the probability of correct selection *1*-α, and the practically significant difference ε, are under our control.

- A two-stage simulation procedure

Bonferroni Approach to Selecting the Best

- First stage
  - Obtain R0 replications from each system
  - Delete (screen out) the statistically inferior systems
  - If only one system survives, stop!

- Second stage
  - More than one system survived
  - Do additional replications to select the best

Metamodeling

Metamodeling

- Goal: describe the relationship between variables and the output response.
- The simulation output response variable, Y, is related to k independent variables $x_1, x_2, \ldots, x_k$ (the design variables).
- The true relationship between variables Y and x is represented by a (complex) simulation model.
- Approximate the relationship by a simpler mathematical function called a metamodel, some metamodel forms:
  - Linear regression.
  - Multiple linear regression.

Simple Linear Regression

- Suppose the true relationship between Y and x is assumed to be linear, the expected value of Y for a given x is:

$$E(Y \mid x) = \beta_0 + \beta_1 x$$

where β_0 is the intercept on the Y axis, and β_1 is the slope.

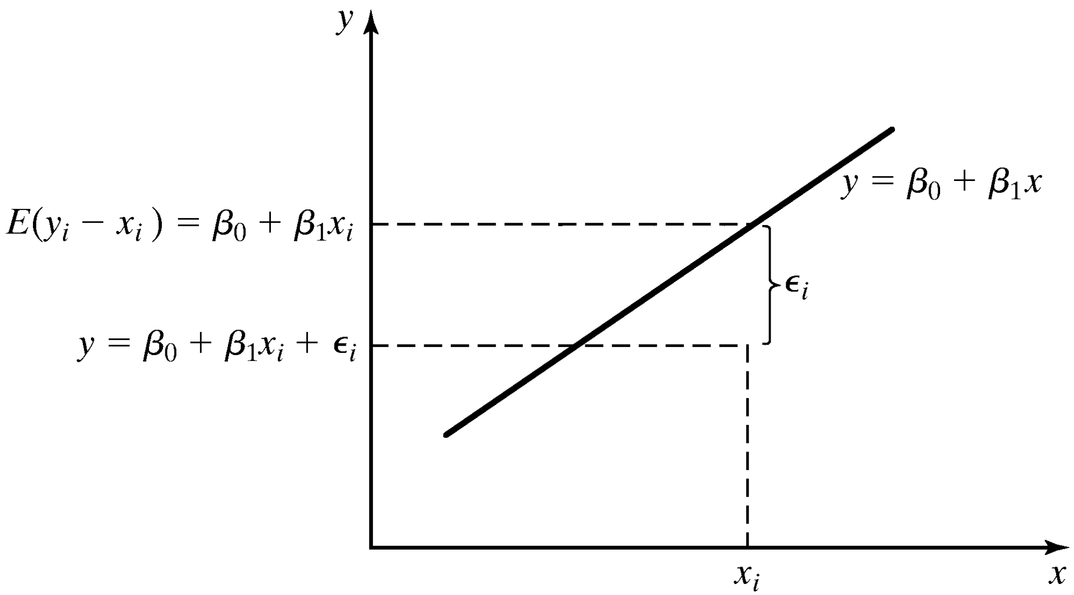
- Each observation of Y can be described by the model:

$$Y = \beta_0 + \beta_1 x + \varepsilon$$

where ε is the random error with mean zero and constant variance σ^2

Simple Linear Regression

- Suppose there are n pairs of observations, the method of least squares is commonly used to estimate β_0 and β_1.
  - The sum of squares of the deviation between the observations and the regression line is minimized.



Simple Linear Regression

- The individual observation can be written as:

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

where $\varepsilon_1, \varepsilon_2 \ldots$ are assumed to be uncorrelated random variables

- Rewrite:

$$Y_i = \beta_0^{'} + \beta_1 (x_i - \bar{x}) + \varepsilon_i$$

$$\text{where } \beta_0^{'} = \beta_0 + \beta_1 \bar{x} \quad \text{and} \quad \bar{x} = \sum_{i=1}^{n} x_i / n$$

- The least-square function (the sum of squares of the deviations):

$$L = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left(Y_i - \beta_0 - \beta_1 x_i\right)^2 = \sum_{i=1}^{n} \left[Y_i - \beta_0^{'} - \beta_1 (x_i - x)\right]^2$$

- To minimize L, find $\partial L / \partial \beta_0^{'}$ and $\partial L / \partial \beta_1$, set each to zero, and solve for:

$$\hat{\beta}_0^{'} = \bar{Y} = \sum_{i=1}^{n} \frac{Y_i}{n} \quad \text{and} \quad \hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{\sum_{i=1}^{n} Y_i (x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

S_{xy} corrected sum of cross products of x and Y

S_{xx} corrected sum of squares of x

Test for Significance of Regression

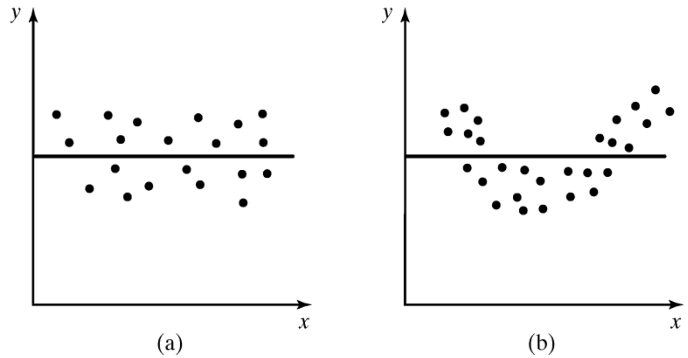
- The adequacy of a simple linear relationship should be tested prior to using the model.

  - Testing whether the order of the model tentatively assumed is correct, commonly called the “lack-of-fit” test.

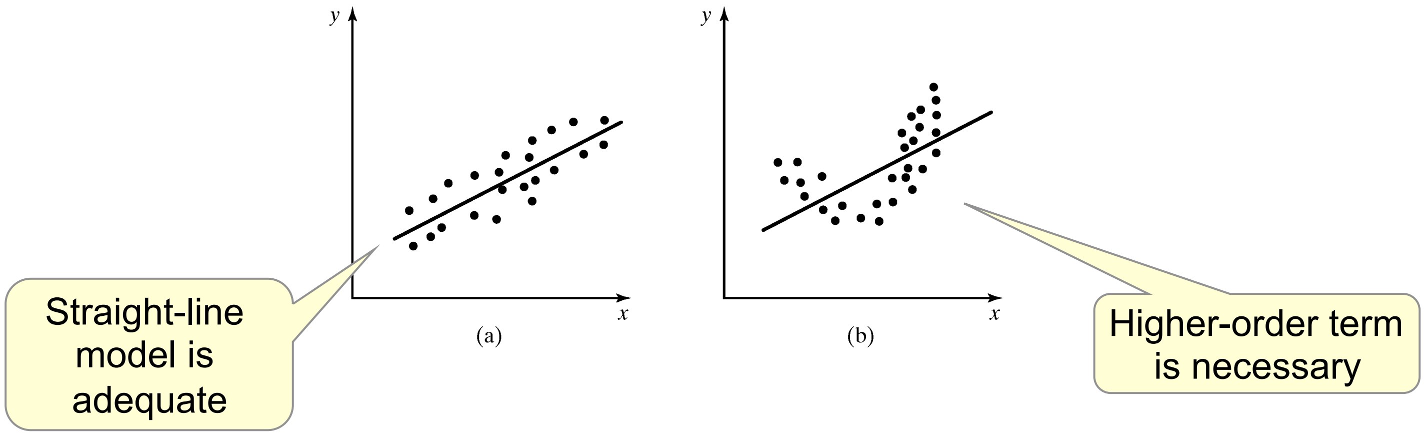
  - The adequacy of the assumptions that errors are (normally and independent) $NID(0,\sigma^2)$ can and should be checked by residual analysis.

Test for Significance of Regression

- Hypothesis testing: $H_0 : \beta_1 = 0$ and $H_1 : \beta_1 \neq 0$
  - Failure to reject H_0 indicates no linear relationship between x and Y.



  - If H_0 is rejected, implies that x can explain the variability in Y, but there may be in higher-order terms.



Test for Significance of Regression

- The appropriate test statistics:

$$t_0 = \frac{\hat{\beta}_1}{\sqrt{MS_E / S_{xx}}}$$

  - The mean squared error is:

$$MS_E = \sum_{i=1}^{n} \frac{e_i^2}{n-2} = \frac{S_{yy} - \hat{\beta}_1 S_{xy}}{n-2}$$

which is an unbiased estimator of $\sigma^2 = V(\varepsilon_i)$

  - t_0 has the t-distribution with n-2 degrees of freedom.
  - Reject H_0 if $|t_0| > t_{\alpha/2,\, n-2}$

Multiple Linear Regression

- Suppose simulation output Y has several independent variables (decision variables).
- The possible relationship forms are:

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_m x_m + \varepsilon$$

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x^2 + \varepsilon$$

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \varepsilon$$

Random-Number Assignment for Regression

- Independent sampling:
  - Assign a different seed or stream to different design points.
  - Guarantees that the responses Y from different design points will be significantly independent.

- CRN:
  - Use the same random number seeds or streams for all of the design points.
  - A fairer comparison among design points (subjected to the same experimental conditions)
  - Typically reduces variance of estimators of slope parameters, but increases variance of intercept parameter

Optimization via Simulation

Optimization via Simulation

- Optimization usually deals with problems with certainty, but in stochastic discrete-event simulation, the result of any simulation run is a random variable.

- Let $x_1, x_2, \ldots, x_m$ be the m controllable design variables and $Y(x_1, x_2, \ldots, x_m)$ be the observed simulation output performance on one run:

- To optimize $Y(x_1, x_2, \ldots, x_m)$ with respect to $x_1, x_2, \ldots, x_m$ is to maximize or minimize the mathematical expectation (long-run average) of performance

$$E[Y(x_1, x_2, \ldots, x_m)]$$

Optimization via Simulation

- Example: select the material handling system that has the best chance of costing less than $\$D$ to purchase and operate.
  - Objective: maximize $Pr(Y(x_1,x_2,\ldots,x_m) \le D)$.
- Define a new performance measure:
  - Maximize $E(Y'(x_1,x_2,\ldots,x_m))$ *instead*

$$Y'(x_1,x_2,\ldots x_m) = \begin{cases} 1, & \text{if } Y(x_1,x_2,\ldots x_m) \le D \\ 0, & \text{otherwise} \end{cases}$$

Summary

- Basic introduction to comparative evaluation of alternative system design:
  - Emphasized comparisons based on confidence intervals.
  - Discussed the differences and implementation of independent sampling and common random numbers.
  - Introduced concept of metamodels.