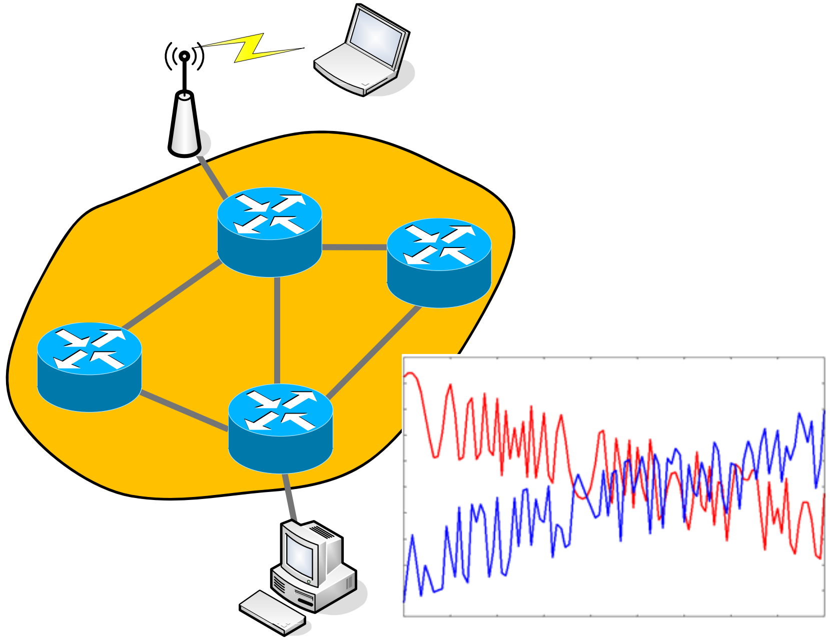
Chapter 7

Random-Variate Generation



Contents

Purpose & Overview

- Develop understanding of generating samples from a specified distribution as input to a simulation model.

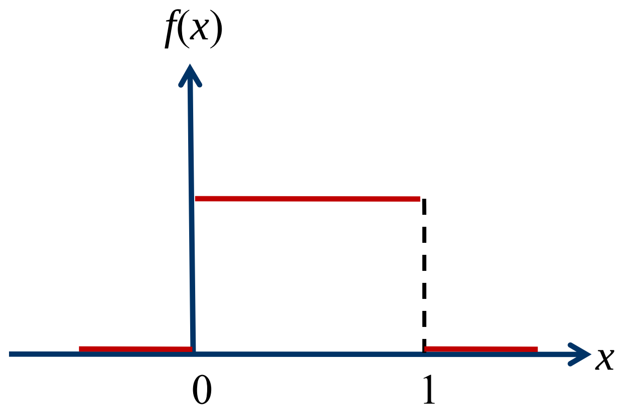
- Illustrate some widely-used techniques for generating random variates:
  - Inverse-transform technique
  - Acceptance-rejection technique
  - Special properties

Preparation

- It is assumed that a source of uniform [0,1] random numbers exists.
  - Linear Congruential Method (LCM)

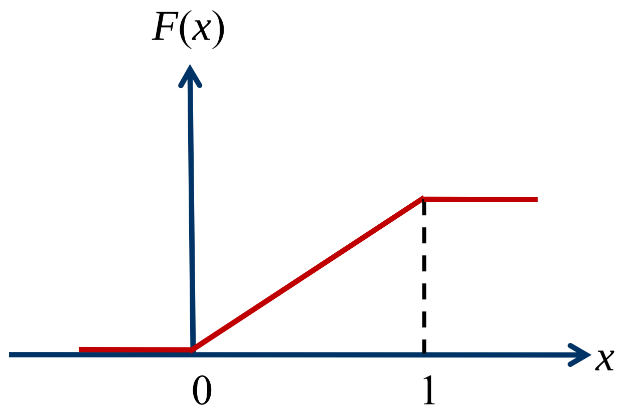
- Random numbers $R, R_1, R_2, \ldots$ with
  - PDF

$$f_R(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



  - CDF

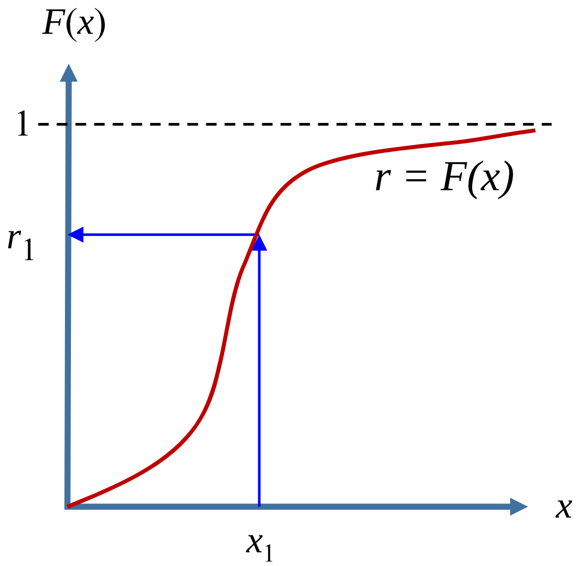
$$F_R(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

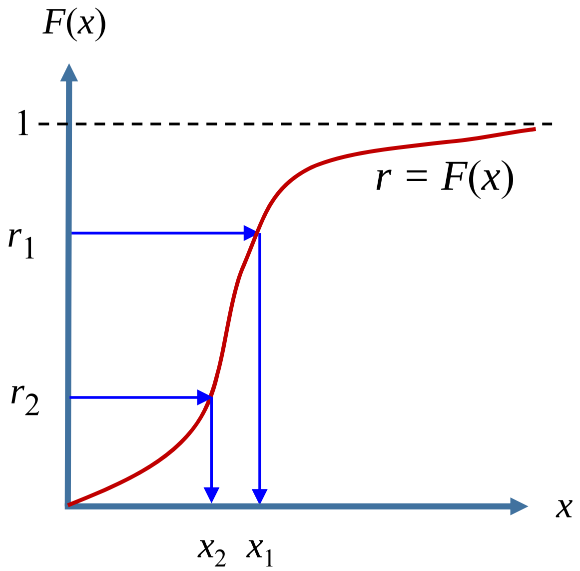


Inverse-transform Technique

Inverse-transform Technique

- The concept:
  - For CDF function: $r = F(x)$
  - Generate r from uniform $(0,1)$, a.k.a $U(0,1)$
  - Find x, $x = F^{-1}(r)$





Inverse-transform Technique

- The inverse-transform technique can be used in principle for any distribution.
- Most useful when the CDF $F(x)$ has an **inverse** $F^{-1}(x)$ which is easy to compute.

- Required steps
  1. Compute the CDF of the desired random variable X
  2. Set $F(X) = R$ on the range of X
  3. Solve the equation $F(X) = R$ for X in terms of R
  4. Generate uniform random numbers $R_1, R_2, R_3, \ldots$ and compute the desired random variate by $X_i = F^{-1}(R_i)$

Inverse-transform Technique: Example

- Exponential Distribution
  - PDF

$$f(x) = \lambda e^{-\lambda x}$$

  - CDF

$$F(x) = 1 - e^{-\lambda x}$$

- Simplification

$$X = -\frac{\ln(R)}{\lambda}$$

- Since R and $(1\text{-}R)$ are uniformly distributed on [0,1]

- To generate $X_1, X_2, X_3 \ldots$

$$1 - e^{-\lambda X} = R$$

$$e^{-\lambda X} = 1 - R$$

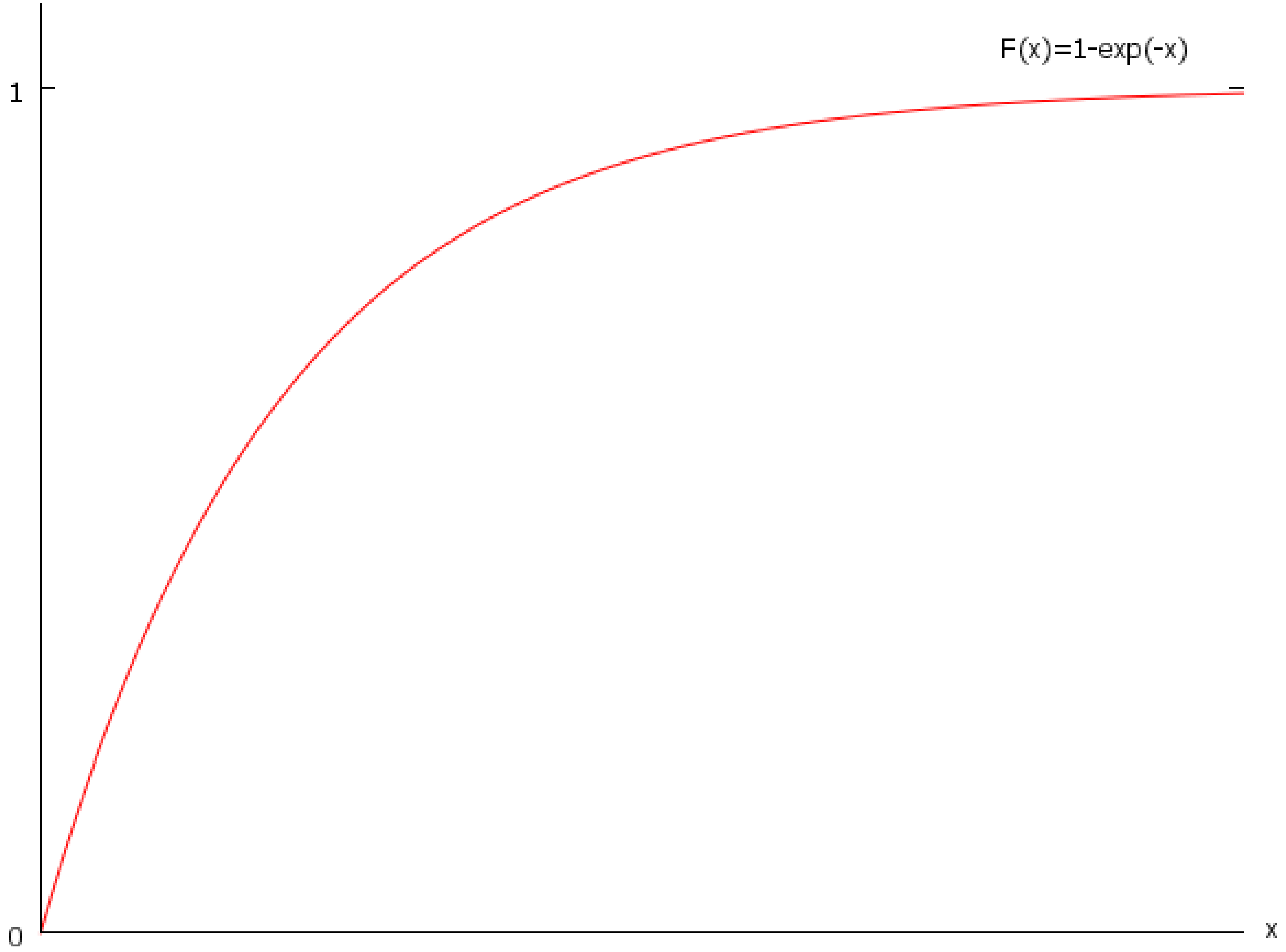
$$-\lambda X = \ln(1-R)$$

$$X = \frac{\ln(1-R)}{-\lambda}$$

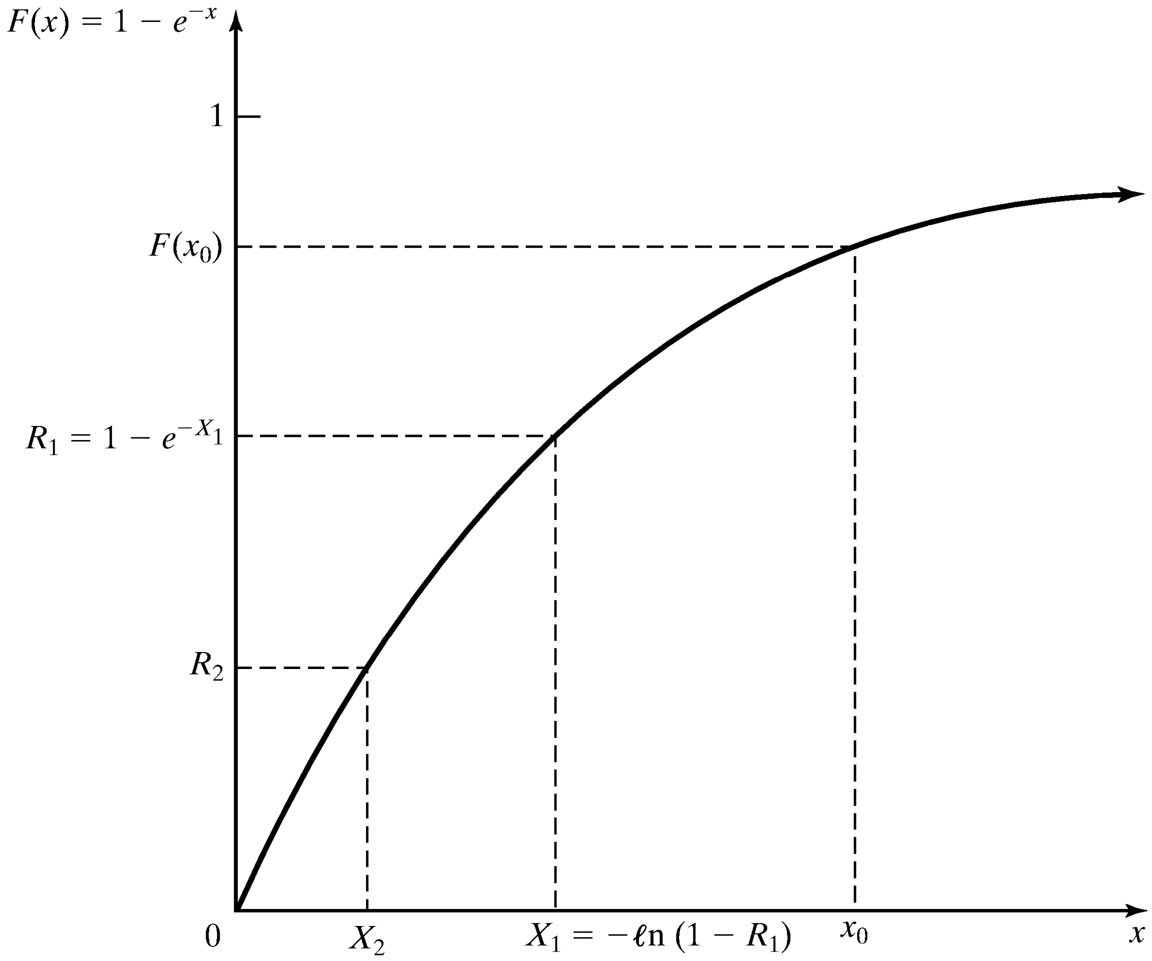
$$X = -\frac{\ln(1-R)}{\lambda}$$

$$X = F^{-1}(R)$$

Inverse-transform Technique: Example



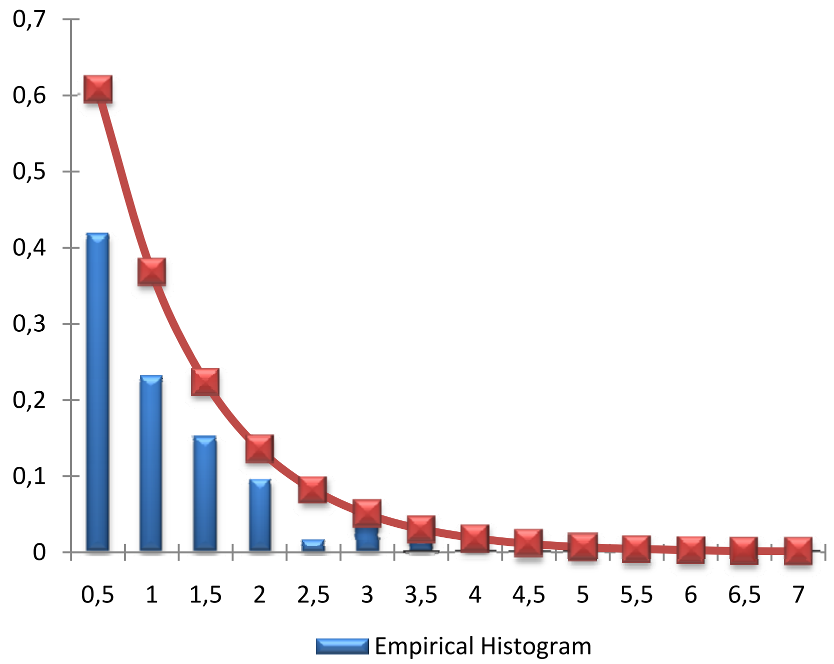
Inverse-transform Technique: Example

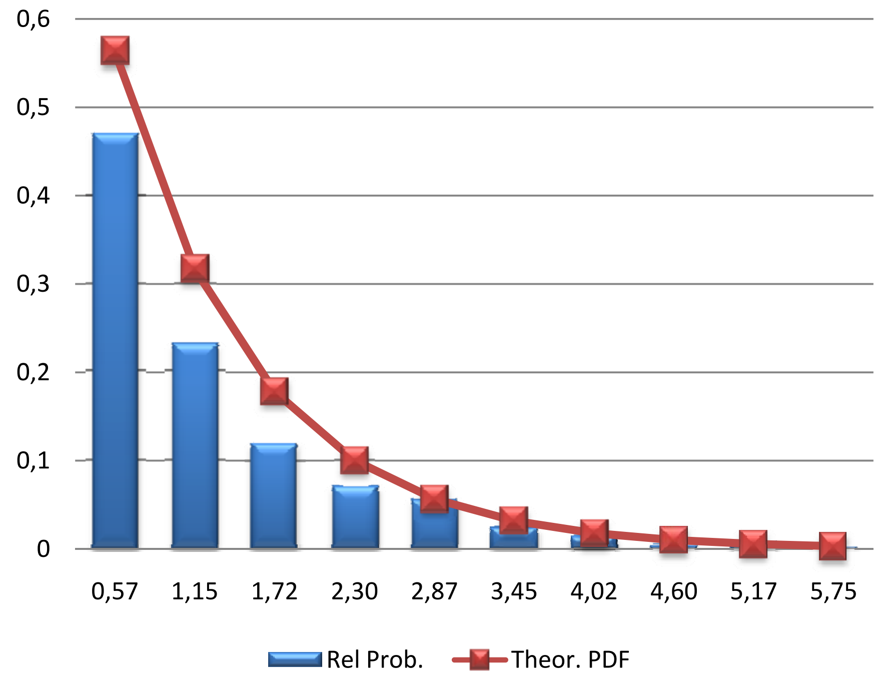


Inverse-transform technique for exp(λ = 1)

Inverse-transform Technique: Example

- Example:
  - Generate 200 or 500 variates X_i with distribution $\exp(\lambda = 1)$
  - Generate 200 or 500 R_s with $U(0,1)$, the histogram of X_s becomes:

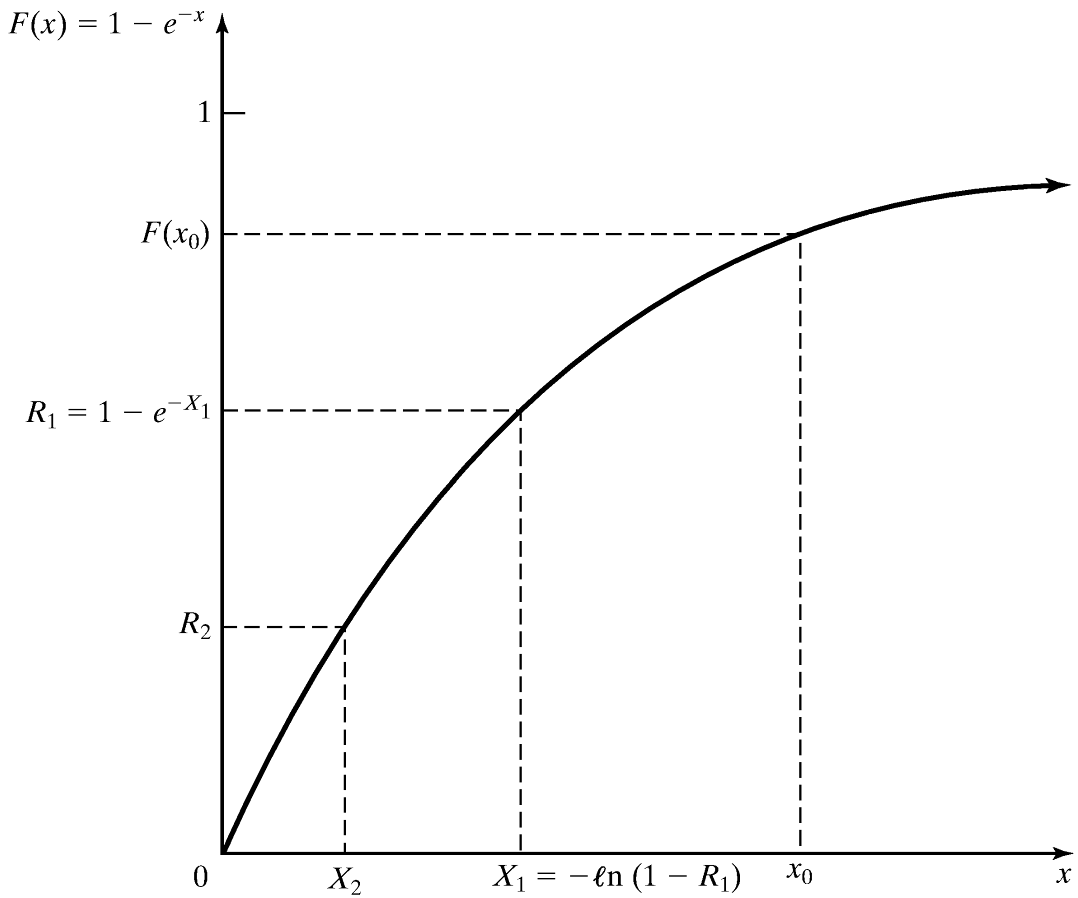




Inverse-transform Technique

- Check: Does the random variable X_1 have the desired distribution?

$$P(X_1 \le x_0) = P(R_1 \le F(x_0)) = F(x_0)$$



Inverse-transform Technique: Other Distributions

- Examples of other distributions for which inverse CDF works are:
  - Uniform distribution
  - Weibull distribution
  - Triangular distribution

Inverse-transform Technique: Uniform Distribution

- Random variable X uniformly distributed over $[a, b]$

$$F(X) = R$$

$$\frac{X - a}{b - a} = R$$

$$X - a = R(b - a)$$

$$X = a + R(b - a)$$

Inverse-transform Technique: Weibull Distribution

- The Weibull Distribution is described by
  - PDF

$$f(x) = \frac{\beta}{\alpha^\beta} x^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta}$$

  - CDF

$$F(X) = 1 - e^{-\left(\frac{x}{\alpha}\right)^\beta}$$

- The variate is

$$F(X) = R$$

$$1 - e^{-\left(\frac{X}{\alpha}\right)^\beta} = R$$

$$e^{-\left(\frac{X}{\alpha}\right)^\beta} = 1 - R$$

$$-\left(\frac{X}{\alpha}\right)^\beta = \ln(1-R)$$

$$\frac{X^\beta}{\alpha^\beta} = -\ln(1-R)$$

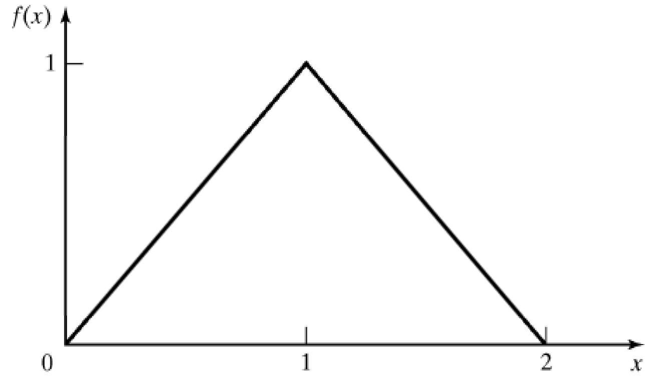
$$X^\beta = -\alpha^\beta \cdot \ln(1-R)$$

$$X = \sqrt[\beta]{-\alpha^\beta \cdot \ln(1-R)}$$

$$X = \alpha \cdot \sqrt[\beta]{-\ln(1-R)}$$

Inverse-transform Technique: Triangular Distribution

- The CDF of a Triangular Distribution with endpoints (0, 2) is given by



$$F(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{x^2}{2} & 0 < x \le 1 \\ 1 - \dfrac{(2-x)^2}{2} & 1 < x \le 2 \\ 1 & x > 2 \end{cases}$$

$$R(X) = \begin{cases} \dfrac{X^2}{2} & 0 \le X \le 1 \\ 1 - \dfrac{(2-X)^2}{2} & 1 \le X \le 2 \end{cases}$$

- X is generated by

$$X = \begin{cases} \sqrt{2R} & 0 \le R \le \frac{1}{2} \\ 2 - \sqrt{2(1-R)} & \frac{1}{2} < R \le 1 \end{cases}$$

Inverse-transform Technique: Empirical Continuous Distributions

- When theoretical distributions are not applicable
- To collect empirical data:
  - Resample the observed data
  - Interpolate between observed data points to fill in the gaps

Inverse-transform Technique: Empirical Continuous Distributions

- For a small sample set (size n):
  - Arrange the data from smallest to largest

$$\mathrm{x}_{(1)} \le \mathrm{x}_{(2)} \le \ldots \le \mathrm{x}_{(n)}$$

  - Set $x_{(0)}=0$
  - Assign the probability $1/n$ to each interval $\mathrm{x}_{(i-1)} \le \mathrm{x} \le \mathrm{x}_{(i)} \quad i = 1,2,\ldots,n$
  - The slope of each line segment is defined as

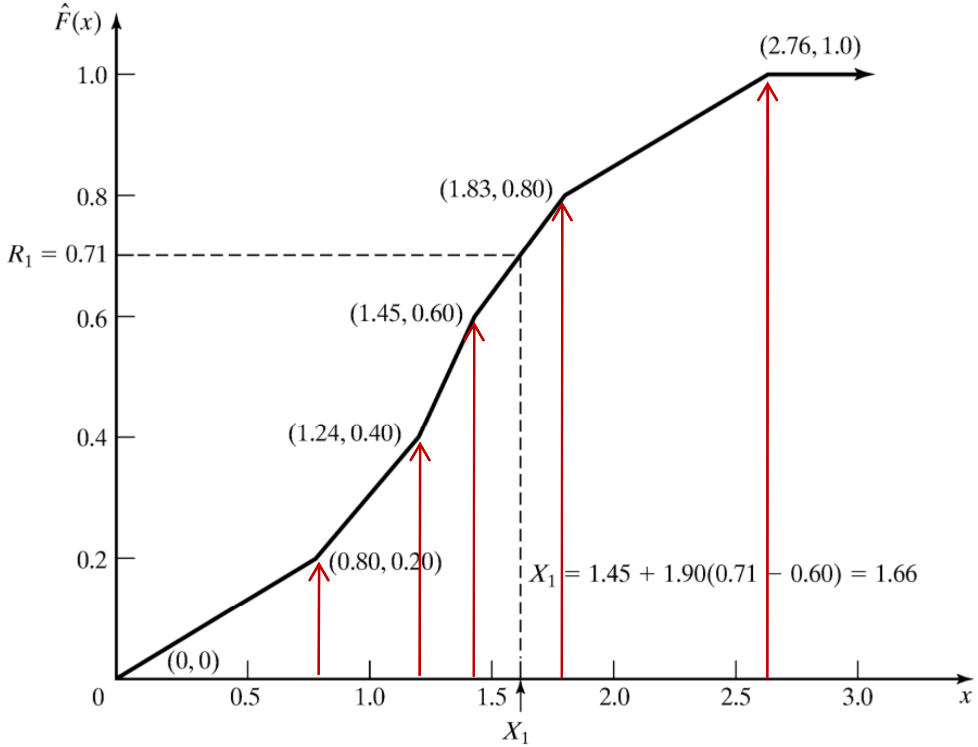
$$a_i = \frac{x_{(i)} - x_{(i-1)}}{\frac{i}{n} - \frac{(i-1)}{n}} = \frac{x_{(i)} - x_{(i-1)}}{\frac{1}{n}}$$

  - The inverse CDF is given by

$$X = \hat{F}^{-1}(R) = x_{(i-1)} + a_i \left(R - \frac{(i-1)}{n} \right) \quad \text{when} \quad \frac{(i-1)}{n} < R \le \frac{i}{n}$$

Inverse-transform Technique: Empirical Continuous Distributions

i	Interval	PDF	CDF	Slope a_i
1	0.0 < x ≤ 0.8	0.2	0.2	4.00
2	0.8 < x ≤ 1.24	0.2	0.4	2.20
3	1.24 < x ≤ 1.45	0.2	0.6	1.05
4	1.45 < x ≤ 1.83	0.2	0.8	1.90
5	1.83 < x ≤2.76	0.2	1.0	4.65



$$R_1 = 0.71$$

$$X_1 = x_{(4-1)} + a_4(R_1 - (4-1)/n)$$

$$= 1.45 + 1.90(0.71 - 0.6)$$

$$= 1.66$$

Inverse-transform Technique: Empirical Continuous Distributions

- What happens for large samples of data
  - Several hundreds or tens of thousand
- First summarize the data into a frequency distribution with smaller number of intervals
- Afterwards, fit continuous empirical CDF to the frequency distribution
- Slight modifications
  - Slope

$$a_i = \frac{x_{(i)} - x_{(i-1)}}{c_i - c_{i-1}}$$

c_i cumulative probability of the first i intervals

  - The inverse CDF is given by

$$X = \hat{F}^{-1}(R) = x_{(i-1)} + a_i\left(R - c_{i-1}\right) \quad \text{when} \quad c_{i-1} < R \le c_i$$

Inverse-transform Technique: Empirical Continuous Distributions

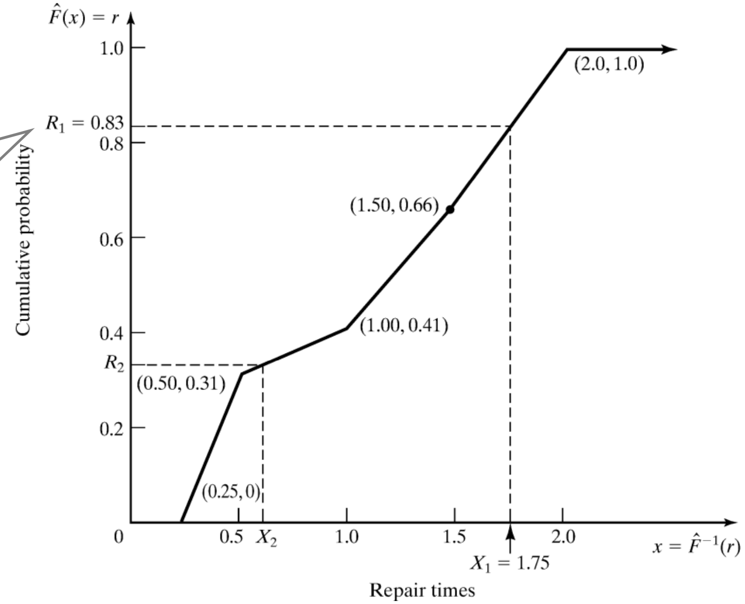
- Example: Suppose the data collected for 100 broken-widget repair times are:

Interval (Hours)	*Frequency*	*Relative Frequency*	*Cumulative Frequency, c_i*	*Slope, a_i*
$0.25 \le x \le 0.5$	31	0.31	0.31	0.81
$0.5 \le x \le 1.0$	10	0.10	0.41	5.00
$1.0 \le x \le 1.5$	25	0.25	0.66	2.00
$1.5 \le x \le 2.0$	34	0.34	1.00	1.47

Consider $R_1 = 0.83$:

$c_3 = 0.66 < R_1 < c_4 = 1.00$

$$X_1 = x_{(4-1)} + a_4(R_1 - c_{(4-1)})$$
$$= 1.5 + 1.47(0.83-0.66)$$
$$= 1.75$$



Inverse-transform Technique: Empirical Continuous Distributions

- Problems with empirical distributions
  - The data in the previous example is restricted in the range $0.25 \le X \le 2.0$
  - The underlying distribution might have a wider range
  - Thus, try to find a theoretical distribution

- Hints for building empirical distributions based on frequency tables
  - It is recommended to use relatively short intervals
    - Number of bins increase
  - This will result in a more accurate estimate

Inverse-transform Technique: Continuous Distributions

- A number of continuous distributions do not have a closed form expression for their CDF, e.g.
  - Normal $$F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right)dt$$
  - Gamma
  - Beta
- The presented method does not work for these distributions
- Solution
  - Approximate the CDF or numerically integrate the CDF
- Problem
  - Computationally slow

Inverse-transform Technique: Discrete Distribution

- All discrete distributions can be generated via inverse-transform technique
- Method: numerically, table-lookup procedure, algebraically, or a formula
- Examples of application:
  - Empirical
  - Discrete uniform
  - Geometric

Inverse-transform Technique: Discrete Distribution

- Example: Suppose the number of shipments, x, on the loading dock of a company is either 0, 1, or 2
  - Data - Probability distribution:

x	$P(x)$	$F(x)$
0	0.50	0.50
1	0.30	0.80
2	0.20	1.00

- The inverse-transform technique as table-lookup procedure

$$F(x_{i-1}) = r_{i-1} < R \leq r_i = F(x_i)$$

  - Set $X = x_i$

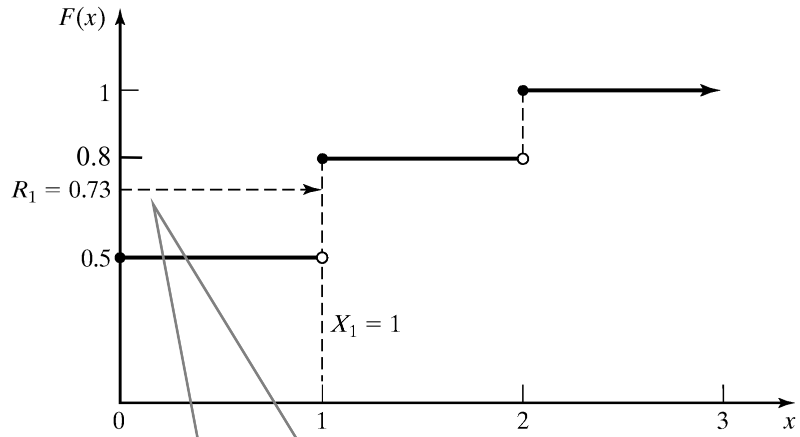
Inverse-transform Technique: Discrete Distribution

Method - **Given R, the generation scheme becomes:**

$$x = \begin{cases} 0, & R \le 0.5 \\ 1, & 0.5 < R \le 0.8 \\ 2, & 0.8 < R \le 1.0 \end{cases}$$

Table for generating the discrete variate X

i	Input r_i	Output x_i
1	0.5	0
2	0.8	1
3	1.0	2



Consider $R_1 = 0.73$:
$F(x_{i-1}) < R \le F(x_i)$
$F(x_0) < 0.73 \le F(x_1)$
Hence, $X_1 = 1$

Acceptance-Rejection Technique

Acceptance-Rejection Technique

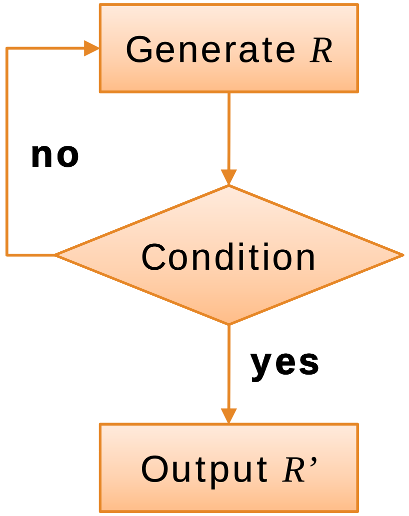
- Useful particularly when inverse CDF does not exist in closed form
  - Thinning
- Illustration: To generate random variates, $X \sim U(1/4,1)$

Procedure:

Step 1. Generate $R \sim U(0,1)$

Step 2. If $R \geq$ ¼, accept $X{=}R$.

Step 3. If $R <$ ¼, reject R, return to Step 1



- R does not have the desired distribution, but R conditioned (R') on the event $\{R \geq ¼\}$ does.
- Efficiency: Depends heavily on the ability to minimize the number of rejections.

Acceptance-Rejection Technique: Poisson Distribution

- Probability mass function of a Poisson Distribution

$$P(N=n)=\frac{\alpha^n}{n!}e^{-\alpha}$$

- Exactly n arrivals during one time unit

$$A_1+A_2+\cdots+A_n \le 1 < A_1+A_2+\cdots+A_n+A_{n+1}$$

- Since interarrival times are exponentially distributed we can set

$$A_i=\frac{-\ln(R_i)}{\alpha}$$

  - Well known, we derived this generator in the beginning of the class

Acceptance-Rejection Technique: Poisson Distribution

- Substitute the sum by

$$\sum_{i=1}^{n} \frac{-\ln(R_i)}{\alpha} \le 1 < \sum_{i=1}^{n+1} \frac{-\ln(R_i)}{\alpha}$$

- Simplify by
  - multiply by -α, which reverses the inequality sign
  - sum of logs is the log of a product

$$\sum_{i=1}^{n} \ln(R_i) \ge -\alpha > \sum_{i=1}^{n+1} \ln(R_i)$$

$$\ln \prod_{i=1}^{n} R_i \ge -\alpha > \ln \prod_{i=1}^{n+1} R_i$$

- Simplify by $e^{\ln(x)} = x$

$$\prod_{i=1}^{n} R_i \ge e^{-\alpha} > \prod_{i=1}^{n+1} R_i$$

Acceptance-Rejection Technique: Poisson Distribution

- Procedure of generating a Poisson random variate N is as follows
  1. Set $n=0$, $P=1$
  2. Generate a random number R_{n+1}, and replace P by $P \times R_{n+1}$
  3. If $P < \exp(-\alpha)$, then accept $N=n$
     - Otherwise, reject the current n, increase n by one, and return to step 2.

Acceptance-Rejection Technique: Poisson Distribution

- Example: Generate three Poisson variates with mean α=0.2
  - exp(-0.2) = 0.8187
- Variate 1
  - Step 1: Set $n = 0, P = 1$
  - Step 2: $R1 = 0.4357, P = 1 \text{ x } 0.4357$
  - Step 3: Since $P = 0.4357 < \exp(-\ 0.2)$, accept $N = 0$
- Variate 2
  - Step 1: Set $n = 0, P = 1$
  - Step 2: $R1 = 0.4146, P = 1 \text{ x } 0.4146$
  - Step 3: Since $P = 0.4146 < \exp(-0.2)$, accept $N = 0$
- Variate 3
  - Step 1: Set $n = 0, P = 1$
  - Step 2: $R1 = 0.8353, P = 1 \text{ x } 0.8353$
  - Step 3: Since $P = 0.8353 > \exp(-0.2)$, reject $n = 0$ and return to Step 2 with $n = 1$
  - Step 2: $R2 = 0.9952, P = 0.8353 \text{ x } 0.9952 = 0.8313$
  - Step 3: Since $P = 0.8313 > \exp(-0.2)$, reject $n = 1$ and return to Step 2 with $n = 2$
  - Step 2: $R3 = 0.8004, P = 0.8313 \text{ x } 0.8004 = 0.6654$
  - Step 3: Since $P = 0.6654 < \exp(-0.2)$, accept $N = 2$

Acceptance-Rejection Technique: Poisson Distribution

- It took five random numbers to generate three Poisson variates
- In long run, the generation of Poisson variates requires some overhead!

N	R_{n+1}	P	Accept/Reject		Result
0	0.4357	0.4357	$P < \exp(-\alpha)$	Accept	N=0
0	0.4146	0.4146	$P < \exp(-\alpha)$	Accept	N=0
0	0.8353	0.8353	$P \geq \exp(-\alpha)$	Reject	
1	0.9952	0.8313	$P \geq \exp(-\alpha)$	Reject	
2	0.8004	0.6654	$P < \exp(-\alpha)$	Accept	N=2

Special Properties

Special Properties

- Based on features of particular family of probability distributions
- For example:
  - Direct Transformation for normal and lognormal distributions
  - Convolution

Direct Transformation

- Approach for $N(0,1)$
  - PDF

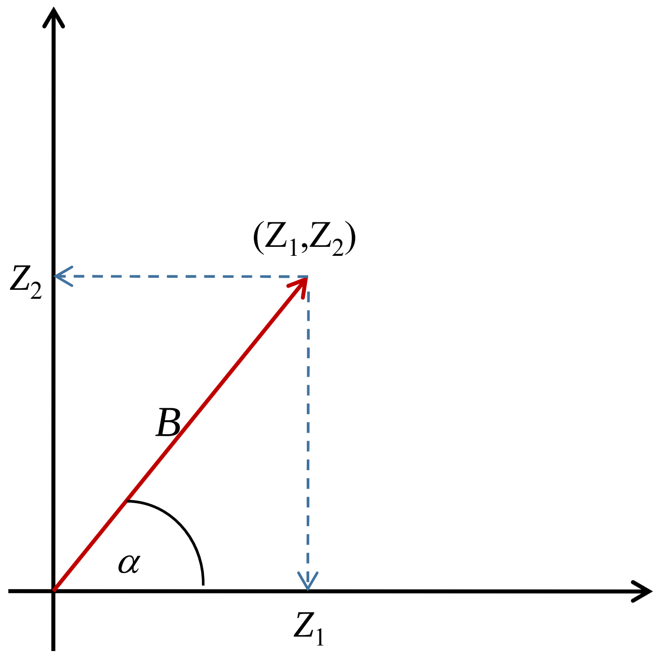
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

  - CDF, No closed form available

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

Direct Transformation

- Approach for $N(0,1)$
  - Consider two standard normal random variables, Z_1 and Z_2, plotted as a point in the plane:
  - In polar coordinates:
    - $Z_1 = B\cos(\alpha)$
    - $Z_2 = B\sin(\alpha)$



Direct Transformation

- Chi-square distribution
  - Given k independent $N(0, 1)$ random variables $X_1, X_2, \ldots, X_k$, then the sum is according to the Chi-square distribution

$$\chi_k^2 = \sum_{i=1}^{k} X_i^2$$

  - PDF

$$f(x,k) = \frac{1}{\Gamma\left(\frac{k}{2}\right) 2^{\frac{k}{2}}} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}$$

Direct Transformation

- The following relationships are known
  - $B^2 = Z^2_1 + Z^2_2 \sim \chi^2$ distribution with 2 degrees of freedom = $\exp(\lambda = 1/2)$.
  - Hence:

$$B = \sqrt{-2\ln R}$$

  - The radius B and angle α are mutually independent.

$$\boxed{\begin{aligned} Z_1 &= \sqrt{-2\ln R_1}\cos(2\pi R_2) \\ Z_2 &= \sqrt{-2\ln R_1}\sin(2\pi R_2) \end{aligned}}$$

Direct Transformation

- Approach for $N(\mu, \sigma^2)$:
  - Generate $Z_i \sim N(0,1)$

$$X_i = \mu + \sigma Z_i$$

- Approach for Lognormal(μ,σ^2):
  - Generate $X \sim N(\mu,\sigma^2)$

$$Y_i = e^{X_i}$$

Direct Transformation: Example

- Let $R_1 = 0.1758$ and R_2=0.1489
- Two standard normal random variates are generated as follows:

$$Z_1 = \sqrt{-2\ln(0.1758)}\cos(2\pi 0.1489) = 1.11$$
$$Z_2 = \sqrt{-2\ln(0.1758)}\sin(2\pi 0.1489) = 1.50$$

- To obtain normal variates X_i with mean μ=10 and variance $\sigma^2 = 4$

$$X_1 = 10 + 2 \cdot 1.11 = 12.22$$
$$X_2 = 10 + 2 \cdot 1.50 = 13.00$$

Convolution

- Convolution
  - The sum of independent random variables

- Can be applied to obtain
  - Erlang variates
  - Binomial variates

Convolution

- Erlang Distribution
  - Erlang random variable X with parameters (k, θ) can be depicted as the sum of k independent exponential random variables X_i, $i = 1, \ldots, k$ each having mean $1/(k\theta)$

$$
\begin{aligned}
X &= \sum_{i=1}^{k} X_i \\
&= \sum_{i=1}^{k} -\frac{1}{k\theta}\ln(R_i) \\
&= -\frac{1}{k\theta}\ln\left(\prod_{i=1}^{k} R_i\right)
\end{aligned}
$$

Summary

- Principles of random-variate generation via
  - Inverse-transform technique
  - Acceptance-rejection technique
  - Special properties

- Important for generating continuous and discrete distributions