A Top-down Approach to Combining Logics

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Abstract: The mechanization and automation of combination of logics, expressive ontologies and notions of context are prominent current challenge problems. I propose to approach these challenge topics from the perspective of classical higher-order logic. From this perspective these topics are closely related and a common, uniform solution appears in reach.

1 MOTIVATION

The mechanization and automation of (A) combination of logics, (B) context and (C) expressive ontologies are prominent current challenge problems. Their solution is of significant interest to computer scientists, artificial intelligence researchers, computational linguists and philosophers. Application areas include, for example, logic-based knowledge representation and reasoning, multi-agent systems, the semantic web, and computational social choice.

I propose to approach these challenge topics from an higher-order logic (HOL) perspective. From this perspective these topics are closely related and a uniform solution appears in reach. Moreover, off-the-shelf higher-order automated theorem provers (HOL-ATPs) can be readily employed for the automation of reasoning with respect to these challenge topics.

(A) Combining Logics Researchers from various disciplines have developed and studied a wide range of classical and non-classical logics (Woods and Gabbay, 2004). These developments were often targeted at desirable properties that previous logics did not adequately address. Examples include intuitionistic logics, deontic logics, epistemic logics, many-valued logics, relevant logic, linear logic, etc.

Philosophers and logicians have recently developed an increased interest in combining logics, and in general theories of logical systems; new schools and new journals have emerged. The interest is in tools and techniques for modeling and analyzing integrations, extensions, embeddings, and translations of logics, and for studying commonalities and deviations of logics, e.g., with respect to fundamental theorems or computational properties.

Computer scientists and artificial intelligence researchers in contrast have a strong practical interest in combinations of logics and in developing effective reasoning systems for them. This is because complex and real-world reasoning tasks often require reasoning about beliefs, obligations, actions and change and a host of other tasks that cannot be adequately modeled in simpler logics.

However, building automated reasoning systems that support combinations of logics is a very demanding endeavor. One option is to develop a specific system for each particular logic combination in question. Doing this for all relevant and interesting combinations is hardly feasible. In fact, there is a strong discrepancy between the number of combined reasoning systems that have been sketched on paper, and the number of (non-trivial) combined reasoning systems that have actually been implemented.

A second option is to develop flexible, plug-and-play frameworks for various logics and their combinations. Notable developments in this direction include: Logic Workbench (LWB), LoTREC, Tableaux Workbench (TWB), FaCT, ileanCoP, and the translation based MSPASS system.[1]

[1] System websites: LWB: http://www.lwb.unibe.ch/, LoTREC: http://www.irit.fr/Lotrec/, TWB: http://twb.rsise.anu.edu.au/, FaCT: http://www.cs.man.ac.uk/~horrocks/FaCT/, ileanCoP: http://www.leancop.de/ileancop/, MSPASS: http://www.cs.man.ac.uk/~schmidt/mspass/. Systems overview: http://www.cs.man.ac.uk/~schmidt/tools/

However, these systems are mainly restricted to comparably inexpressive propositional logics and in particular their support for flexible logic combinations is still very limited. For example, none of these systems supports quantified modal logics (QMMLs) or quantified conditional logics (QCLs) and their combinations. Only for first-order quantified monomodal logics some specialist provers do exist, such as MleanCoP, MleanTAP, MleanSeP, GQML, and f2p+MSPASS[2], but none of these systems currently supports logic combinations.

(B) Context The study of notions of context has a long history in philosophy, linguistics, and artificial intelligence. In artificial intelligence, a main motivation has been to resolve the problem of generality of computer programs as identified by McCarthy (McCarthy, 1987). The generality aspect of context scrutinizes flexible combinations (nestings) of contexts in combination with rich context descriptions.

Giunchiglia (Giunchiglia, 1993) additionally emphasizes the locality aspect and the need for structured representations of knowledge. The locality aspect is particularly important for large knowledge bases; the challenge is to effectively identify and access information that is relevant within a given reasoning context.

Different approaches to formalizing and mechanizing context have been proposed in the last decades; many of these are outlined in overview articles (Akman and Surav, 1996; de Paiva, 2003; Serafini and Bouquet, 2004) or in special issues of journals (Lehmann et al., 2012).

McCarthy (McCarthy, 1993) has pioneered the modeling of contexts as first class objects (in first-order logic) and he introduced the predicate *ist*. For example, in his approach the expression *ist(context_of("Ben's Knowledge"),likes(Sue,Bill))* encodes that proposition Sue likes Bill is true in the context of Ben's knowledge. A main motivation of McCarthy's approach actually is to avoid modal logics (here for the modeling of Ben's knowledge). His line of research has been followed by a number of researchers, including, for example, Guha (who has put contexts into Cyc), Buvac and Mason (Buvac et al., 1995; Guha, 1991). Also Giunchiglia and Serafini (Giunchiglia and Serafini, 1994) avoid modal logics and propose the use of so called multilanguage systems. They show various equivalence results to common modal logics, but they also discuss several properties of multilanguage systems not supported in modal logics.

All of the above approaches avoid a higher-order perspective on context. My position is complementary and I argue that a solid higher-order perspective on context can be very valuable for various reasons. On the theory side the twist between formalisms based on modal logic and formalisms based on first-order logic seems to dissolve, since both modal logics (and other non-classical logics) and first-order logics are just natural fragments of HOL. Moreover, modal (and other) contexts can be elegantly combined and nested in HOL, so that a flexible solution to McCarthy's generality problem appears feasible.

Giunchiglia's locality aspect can also be addressed. The means for this is relevance filtering (Meng and Paulson, 2009; Pease et al., 2010; Hoder and Voronkov, 2011).

(C) Expressive Ontologies. Expressive ontologies such as the Suggested Upper Merged Ontology SUMO (Pease, 2011) or CYC (Ramachandran et al., 2005) contain a small but significant number of higher-order representations (Benzmüller and Pease, 2012). They are particularly employed for modeling contexts, including temporal, epistemic, or doxastic contexts. In SUMO, for example, a statement like *(loves Bill Mary)* can be restricted to the year 2009 by wrapping it (at subterm level) into respective context information: *(holdsDuring (YearFn 2009) (loves Bill Mary))*. Similarly, the statement can be put into an epistemic or doxastic context: *(knows/believes Ben (loves Bill Mary))*. Moreover, contexts can be flexibly combined and the embedded formulas may be complex: *(believes Bill (knows Ben (forall (?X) ((woman ?X) => (loves Bill ?X)))))*. The close relation to McCarthy's approach is obvious. A crucial requirement for challenge (C) thus is to support flexible context reasoning in combination with other first-order and even higher-order reasoning aspects, and in combination with relevance filtering in large knowledge bases.

The Proposed Solution The challenges (A), (B) and (C) are addressed from a fresh, analytical perspective. The starting point is HOL, that is, classical higher-order logic (Church, 1940). Instead of synthesizing new logic combinations from source logics bottom-up as typically done in other approaches, the approach works top-down: HOL is decomposed into its embedded logic fragments (respectively their compositions). Recent work has shown that these HOL fragments comprise prominent non-classical logics such as propositional (normal) modal logics, intuitionistic logic, security logics, conditional logics and

[2] See `http://www.cs.uni-potsdam.de/ti/iltp/qmltp/systems.html` for more information on these systems.

logics for time and space (Benzmüller and Paulson, 2008, 2010, 2012; Benzmüller, 2011; Benzmüller and Genovese, 2011; Benzmüller at al., 2012). These fragments also comprise first-order and even higher-order extensions of non-classical logics, for which only little practical automation support has been available so far. Most importantly, however, combinations of embedded logics can be elegantly achieved in this approach.

The HOL approach bridges between the Tarski view of logics (for 'metalogic' HOL) and the Kripke view (for the embedded source logics) and exploits the fact that well known translations of logics, such as the relational translation (Ohlbach, 1991), can be easily formalized in HOL. This way HOL-ATP systems can be uniformly applied to reason *within* and also *about* embedded logics and their combinations.

2 HOL AND ITS AUTOMATION

Classical higher-order logic HOL (Andrews, 2002; Church, 1940) is built on top of the simply typed λ-calculus. The set $\mathcal{T}$ of simple types is usually freely generated from a set of basic types $\{o,\iota\}$ (where o is the type of Booleans and ι is the type of individuals) using the function type constructor $\to$. Instead of $\{o,\iota\}$ a set of base types $\{o,\iota,\mu\}$ is used, providing an additional base type μ (the type of possible worlds).

The simple type theory language HOL is defined by (α, β, $o \in \mathcal{T}$):

$$s,t ::= p_\alpha \mid X_\alpha \mid (\lambda X_\alpha \mathbin{\cdot} s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \mid (\neg_{o\to o}\, s_o)_o \mid$$
$$(s_o \vee_{o\to o\to o} t_o)_o \mid (s_\alpha =_{\alpha\to\alpha\to o} t_\alpha)_o \mid (\Pi_{(\alpha\to o)\to o}\, s_{\alpha\to o})_o$$

p_α denotes typed constants and X_α typed variables (distinct from p_α). Complex typed terms are constructed via abstraction and application. The logical connectives of choice are $\neg_{o\to o}$, $\vee_{o\to o\to o}$, $=_{\alpha\to\alpha\to o}$ and $\Pi_{(\alpha\to o)\to o}$ (for each type α). From these connectives, other logical connectives can be defined in the usual way. Often binder notation $\forall X_\alpha \mathbin{\cdot} s$ is used for $\Pi_{(\alpha\to o)\to o}(\lambda X_\alpha \mathbin{\cdot} s_o)$.

The semantics of HOL is well understood and thoroughly documented in the literature (Andrews, 1972; Henkin, 1950; Benzmüller et al., 2004).

When choosing Henkin semantics (Andrews, 1972; Henkin, 1950) then Gödel's incompleteness results are circumvented and a framework is achieved that obeys important theoretical properties such as recursive axiomatizability, compactness, and countable models. Theoretically the expressive power of HOL with Henkin semantics corresponds to that of multi-sorted first-order logic enriched by infinitely many comprehension axioms. In some applications, however, HOL is better suited in practice both for elegant modeling and for proof automation — this is a main hypothesis of my work.

Automation of HOL has been pioneered by the work of Andrews on resolution in type theory (Andrews, 1971), by Huet's pre-unification algorithm (Huet, 1975) and his constrained resolution calculus (Huet, 1973), and by Jensen and Pietroswski's (Pietrzykowski and Jensen, 1972) work. More recently extensionality and equality reasoning in HOL has been studied (Brown, 2007; Benzmüller et al., 2004; Benzmüller, 2002; Benzmüller, 1999). The TPS system[3] which is based on a higher-order mating calculus, is a pioneering ATP system for HOL.

The automation of HOL has recently made strong progress. This has been fostered by the recent extension of the successful TPTP infrastructure for first-order logic (Sutcliffe, 2009) to higher-order logic, called TPTP THF0 (Sutcliffe and Benzmüller, 2010; Benzmüller et al., 2008).

Meanwhile several higher-order provers and model finders accept the THF0 language as input. These systems are available online via the SystemOnTPTP tool (Sutcliffe, 2007), through which they can be easily employed avoiding local installations. These THF0 compliant systems currently include four HOL-ATPs (TPS, LEO-II, Isabelle, and Satallax) and three HOL (counter-)model finders (Refute, Nitpick, and Satallax).[4]

The progress in automating HOL is measurable in terms of the improvement rates achieved in the yearly THF0 CASC competitions[5]: In 2010 the winner LEO-II performed 56% better than the 2009 champion TPS, the 2011 winner Satallax was 21% better than the 2010 champion LEO-II, and in 2012 Isabelle-HOL was 35% better than 2011 winner Satallax.

3 THE PROPOSED APPROACH

To illustrate the approach the embedding of QMML in HOL is sketched next. The idea is simple: QMML formulas are lifted in HOL to predi-

[3] `http://gtps.math.cmu.edu/tps.html`

[4] The system websites are: TPS: `http://gtps.math.cmu.edu/tps.html`, LEO-II: `http://www.leoprover.org`, Satallax: `http://www.ps.uni-saarland.de/~cebrown/satallax/`, Isabelle: `http://isabelle.in.tum.de/`. Refute/Nitpick: `http://www4.in.tum.de/~blanchet/nitpick.html`. SystemOnTPTP: `http://www.cs.miami.edu/~tptp/cgi-bin/SystemOnTPTP`

[5] `http://www.cs.miami.edu/~tptp/CASC/`

cates over possible worlds, that is, HOL terms of type $\mu \to o$, where μ is the reserved base type denoting the set of possible worlds. Modal operators such as $\neg$, $\vee$, $\Box$, and even quantification over individuals $\forall^{ind}$ and propositions $\forall^{prop}$ can then be elegantly defined as abbreviations of proper HOL terms, for example, $\vee = \lambda\phi_{\mu\to o}.\ \lambda\psi_{\mu\to o}.\lambda W_{\mu}.\ \phi W \vee \psi W$ and $\Box = \lambda R_{\iota\to\iota\to o}.\ \lambda\phi_{\iota\to o}.\ \lambda W_{\iota}.\forall V_{\iota}.\neg R W V \vee \phi V$.[6]

Similarly, the notion of validity of QMML formulas can be explicitly defined as an abbreviation in HOL: valid $= \lambda\phi_{\mu\to o}.\forall W_{\mu}.\ \phi W$. Thus, QMML proof problems can be formulated in HOL in its original syntax, for example, valid$\,\Box_r \exists^{prop} P_{\mu\to o}.\,P$.[7] Using rewriting or definition expanding, such proof problems are reduced to corresponding statements containing only the basic connectives of HOL; no external, error-prone transformation mechanism (as e.g. needed in the first-order based MSPASS approach) is required for this. For the trivial example formula this leads to $\forall W_{\mu}.\forall Y_{\mu}.\neg r W Y \ \vee \neg\forall P_{\mu\to o}.\neg(P Y)$. This formula is obviously valid in Henkin semantics (put $P = \lambda Y_{\mu}.\top$), and it can be effectively proved in a fraction of a second by the above HOL provers.

Specific modal logics can now be easily axiomatized, for example, to model $\Box_r$ as an S5 modality the axioms (M) valid$\forall^{prop}\phi.\,\Box_r \phi \supset \phi$ and (5) valid$\forall^{prop}\phi.\,\Diamond_r \phi \supset \Box_r \Diamond_r \phi$ are postulated. Alternatively, corresponding semantic properties like (reflexive r and (serial r) could be stated.[8]

Similar semantic embeddings already exist e.g. for intuitionistic logics, for access control logics, for logics for spatial reasoning, for propositional conditional logics and for quantified conditional logics (Benzmüller and Paulson, 2008, 2010, 2012; Benzmüller, 2011; Benzmüller and Genovese, 2011; Benzmüller at al., 2012): in each case the logical connectives of the source logic are equated to specific λ-terms in HOL. All these logics are thus natural fragments of HOL. The embedding of conditional logics is particularly interesting since its selection function semantics (Stalnaker, 1968), which is what we have studied, can be seen as a higher-order extension of Kripke semantics for modal logics and cannot be naturally embedded into first-order logic.

Figure 1 illustrates how logic combinations can be achieved in the HOL approach (Benzmüller, 2011). A small epistemic puzzle is presented, and Baldoni's formalization in QMML (Baldoni, 1998) is adapted to the HOL framework.

The formalization employs a 4-dimensional quantified modal logic, combining the four epistemic modalities $\Box_{\mathrm{a}}$, $\Box_{\mathrm{b}}$, and $\Box_{\mathrm{c}}$, and $\Box_{\mathrm{fool}}$. The accessibility relations associated with these box operators are a, b, c, and fool. They are all of type $\mu \to \mu \to o$, hence, they all range over the same world type μ. In this sense the particular notion of a logic combination employed here is related to that of a fusion (Thomason, 1984). In order to model the example as a product (Segerberg, 1973) of four logics, different world types $\mu_1, \ldots, \mu_4$ could be introduced, and the modal connectives could be copied for each of those. Moreover, axioms could be postulated to model the desired product properties.

An important observation concerns the bridge rules in Figure 1 (axioms 6, 7 and 8); they express mutual relations in the scenario between the local epistemic contexts of the agents a, b, c and fool. Such bridge rules, which are often crucial in the modeling of logic combinations, can be directly expressed as axioms in the HOL approach. In traditional bottom-up approaches to combining logic, however, this is often not possible due the lack of expressiveness and axiom schemata or new calculus rules are needed instead. This clearly poses a challenge, in particular, for flexible proof automation.

In ongoing work I also study semantic embeddings of the OWL2-full ontology language and the Dolce ontology in HOL. The embedding is easy and straightforward in both cases. This also implies that the HOL approach is potentially suited as a flexible reasoning framework for semantic web applications that require combinations of OWL2-full with other, more expressive logics. Note that hardly any implemented provers exist to date for such kind of logic combinations.

4 SUMMARY

I have outlined a plug-and-play environment for reasoning within and about combinations of logics. This HOL based approach even supports combinations of very expressive logics (such as QMMLs and QCLs) that are hardly supported to date. At the same time the approach enables the integration of existing specialist reasoners, if available, for single embedded logics (or logic combinations) and it is capable of cooperating with them. This unique combination has the potential to significantly advance the state-of-the art for the combinations of logics challenge in practice. This challenge is timely and relevant for various ar-

[6]Note how the definiens of $\Box$ abstracts over accessibility relations R. Via function application to concrete accessibility relations r multiple $\Box_r$ operators are obtained.

[7]$\Box_r$ stands for the application of $\Box$ to relation r.

[8]In the HOL approach it is even possible to effectively prove the correspondence of semantic properties and their corresponding axioms (Benzmüller, 2011).

(Wise Men Puzzle) *Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked to each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white. (How could he do this?)*

We introduce epistemic modalities $\Box_{\mathrm{a}}$, $\Box_{\mathrm{b}}$, and $\Box_{\mathrm{c}}$, for the three wise men, and $\Box_{\mathrm{fool}}$, for encoding common knowledge. The entire problem encoding consists now of the following axioms for $X,Y,Z \in \{a,b,c\}$ and $X \neq Y \neq Z$:

$$\text{valid } \Box_{\mathrm{fool}}\,((\mathrm{ws\ a}) \vee (\mathrm{ws\ b}) \vee (\mathrm{ws\ c})) \quad (1)$$
$$\text{valid } \Box_{\mathrm{fool}}\,((\mathrm{ws}\ X) \supset \Box_Y\,(\mathrm{ws}\ X)) \quad (2)$$
$$\text{valid } \Box_{\mathrm{fool}}\,(\neg\,(\mathrm{ws}\ X) \supset \Box_Y\,\neg\,(\mathrm{ws}\ X)) \quad (3)$$
$$\text{valid } \forall^{\mathrm{prop}}\phi\text{.}\ \Box_{\mathrm{fool}}\,\phi \supset \phi \quad (4)$$
$$\text{valid } \forall^{\mathrm{prop}}\phi\text{.}\ \Box_{\mathrm{fool}}\,\phi \supset \Box_{\mathrm{fool}}\,\Box_{\mathrm{fool}}\,\phi \quad (5)$$
$$\text{valid } \forall^{\mathrm{prop}}\phi\text{.}\ \Box_{\mathrm{fool}}\,\phi \supset \Box_{\mathrm{X}}\,\phi \quad (6)$$
$$\text{valid } \forall^{\mathrm{prop}}\phi\text{.}\ \neg\,\Box_{\mathrm{X}}\,\phi \supset \Box_{\mathrm{Y}}\,\neg\,\Box_{\mathrm{X}}\,\phi \quad (7)$$
$$\text{valid } \forall^{\mathrm{prop}}\phi\text{.}\ \Box_{\mathrm{X}}\,\phi \supset \Box_{\mathrm{Y}}\,\Box_{\mathrm{X}}\,\phi \quad (8)$$
$$\text{valid } \neg\,\Box_{\mathrm{a}}\,(\mathrm{ws\ a}) \quad (9)$$
$$\text{valid } \neg\,\Box_{\mathrm{b}}\,(\mathrm{ws\ b}) \quad (10)$$

Axiom (1) says that a, b, or c must have a white spot and that this information is known to everybody. Axioms (2) and (3) express that it is generally known that if someone has a white spot (or not) then the others see and hence know this. Common knowledge $\Box_{\mathrm{fool}}$ is axiomatized as an S4 modality in axioms (4) and (5). For $\Box_{\mathrm{a}}$, $\Box_{\mathrm{b}}$, and $\Box_{\mathrm{c}}$ it is sufficient to consider K modalities. The relation between those and common knowledge ($\Box_{\mathrm{fool}}$ modality) is axiomatized in inclusion axioms (6). Axioms (7) and (8) encode that whenever a wise man does (not) know something the others know that he does (not) know this. Axioms (9) and (10) say that a and b do not know whether they have a white spot. Finally, the conjecture valid $\Box_{\mathrm{c}}\,(\mathrm{ws\ c})$ states that that c knows he has a white spot. To solve the puzzle we thus want to prove $(1),\ldots,(10) \models \text{valid } \Box_{\mathrm{c}}\,(\mathrm{ws\ c})$; the fastest HOL-ATPs can solve this problems in a fraction of a second. For more details see (Benzmüller, 2011).

Figure 1: The wise men puzzle in the HOL based approach; adapting Baldoni's formalization in QMML (Baldoni, 1998)

eas, including artificial intelligence, formal methods, computer security, and the semantic web.

From a philosophical perspective a framework with some universal logic characteristics is provided. This framework not only supports the reasoning *within* logic combinations but also *about* their metalogical properties. To the best of my knowledge no other such framework exists to date.

Another aim of my work is to attack the longstanding preconception that HOL-ATP is not feasible in practice, and that it is not suited for applications. This preconception has had a strong influence on the development of logic based approaches in various research areas since the beginning of the last century: HOL-ATP has usually been avoided due to its allowedly quite unfavorable worst case complexity. However, worst case complexity alone not necessarily provides a good basis for judging about a particular approaches competitiveness and effectiveness in specific application domains. In fact, modern HOL-ATPs such as Satallax, LEO-II and Isabelle do integrate specialist reasoners such as state of the art SAT solvers or first-order ATPs, and unsurprisingly they can be quite competitive for these logic fragments. At the same time they offer support for the automation of more expressive formalizations whenever needed.

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