The Ontological Modal Collapse as a Collapse of the Square of Opposition

Christoph Benzmüller and Bruno Woltzenlogel Paleo

Abstract. The *modal collapse* that afflicts Gödel's modal ontological argument for God's existence is discussed from the perspective of the modal square of opposition.

Mathematics Subject Classification (2000). Prim. 03B15; Sec. 68T15.

Keywords. Modal Logics, Higher-Order Logics, Ontological Argument, Interactive and Automated Theorem Proving.

1. Introduction

Attempts to prove the existence (or non-existence) of God by means of abstract, ontological arguments are an old tradition in western philosophy, with contributions by several prominent philosophers, including St. Anselm of Canterbury, Descartes and Leibniz. Kurt Gödel and Dana Scott studied and improved this argument, bringing it to a mathematically more precise form, as a chain of axioms, lemmas and theorems in a second-order modal logic [18, 26], shown in Fig. 1.

Gödel defines God as a being who possesses all *positive* properties and states a few reasonable (but debatable) axioms that such properties should satisfy. The overall idea of Gödel's proof is in the tradition of Anselm's argument, who defined God as an entity of which nothing greater can be conceived. Anselm argued that existence in the actual world would make such an assumed being even greater (more perfect), hence, by definition, God must exist. However, for Anselm existence was treated as a predicate and the possibility of God's existence was assumed as granted. These issues were criticized by Kant and Leibniz, respectively, and they were addressed in the work of Gödel.

This work has been supported by the German National Research Foundation (DFG) under grant BE 2501/9-1,2.

A1 Either a property or its negation is positive, but not both:

$$\forall\varphi[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

A2 A property necessarily implied by a positive property is positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

T1 Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

D1 A *God-like* being possesses all positive properties:

$$G(x) \equiv \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$$

A3 The property of being God-like is positive:

$$P(G)$$

C Possibly, a God-like being exists:

$$\Diamond\exists x G(x)$$

A4 Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \rightarrow \Box\, P(\varphi)]$$

D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:

$$\varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$$

T2 Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G\ ess\ x]$$

D3 *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$NE(x) \equiv \forall\varphi[\varphi\ ess\ x \rightarrow \Box\exists y\varphi(y)]$$

A5 Necessary existence is a positive property:

$$P(NE)$$

L1 If a god-like being exists, then necessarily a god-like being exists:

$$\exists x G(x) \rightarrow \Box\exists y G(y)$$

L2 If possibly a god-like being exists, then necessarily a god-like being exists:

$$\Diamond\exists x G(x) \rightarrow \Box\exists y G(y)$$

T3 Necessarily, a God-like being exists:

$$\Box\exists x G(x)$$

FIGURE 1. Scott's version of Gödel's ontological argument [26].

Nevertheless, Gödel's work still leaves room for criticism. In particular, his axioms are so strong that, when assuming unrestricted comprehension principles[1], they entail a *modal collapse* [27, 28]: everything that is the case is so necessarily.

[1] A possible direction to remedy modal collapse, as studied e.g. by Koons [21] is to impose restrictions on the domain of properties.

There has been an impressive body of recent and ongoing work (cf. [28, 16, 2, 1, 11, 17, 19, 20, 15] and the references therein) proposing solutions for the modal collapse. The goal of this article is to discuss the modal collapse from the point of view of the modal square of opposition. Ontological arguments typically rely on an inversion of the normal direction of entailment in the modal square of opposition for one particular proposition (i.e. God's existence), and the modal collapse shows that this inversion in fact occurs for all propositions, resulting in a total collapse of the modal square of opposition.

2. A Collapse of the Modal Square

A crucial step of most ontological arguments is the claim that if God's existence is possible, then it is necessary. This is Lemma **L2** in Gödel's proof. In the modal square of opposition (Fig. 2), this is an unusual situation in which the **I** corner must imply and entail the **A** corner, in the particular case when ϕ is $\exists x G(x)$. Gödel's proof shows that his axioms are indeed strong enough to invert the direction of entailment for this choice of ϕ. This observation, however, immediately leads to the question whether the axioms are eventually even strong enough to enable the inverted entailment for any arbitrary sentence ϕ. That is essentially the question asked by Sobel [27], and his proof of the modal collapse (**MC**, cf. Fig. 3) provides an affirmative answer. It is possible to show that this form of the modal collapse entails (in modal logic **K**) a collapse of the modal square (**MCs**), causing the subcontraries to entail (and even imply) their respective contraries. Normally, as shown in Fig. 2, in the modal square of opposition only the other direction of entailment holds: the contraries entail their subcontraries, assuming the *modal existential import* **ExImp** [14].

Moreover, in any modal logic where the axiom **T** holds (i.e. where the accessibility relation is reflexive), even a total collapse of the modalities (**MCt**) is entailed by **MC**. Interestingly, under this stronger form of modal collapse, the contraries entail their subcontraries even without the existential import.

Although Gödel's axioms lead to modal collapse, there are several variants (e.g. [2, 1, 11]) that are known to be immune to it. This means there must be at least one proposition ϕ such that the implication $\phi \rightarrow \Box\phi$ (from now on abbreviated as *collapse*(ϕ)) is not valid under the axioms and definitions used by the variant. But if the variant is sufficiently similar to Gödel's argument, also deriving Lemmas **L1** and **L2**, then *collapse*($\exists x G(x)$) must be valid. Therefore, one may wonder how strong is their immunity to the modal collapse: is there any other proposition ϕ for which *collapse*(ϕ) is also valid?

For Anderson's emendation [2], for example, a form of the modal collapse (**A:MC**), restricted to positive properties applied to god-like beings, can be derived. The proof, under the modal logic **K**, depends only on Anderson's alternative definition of god-like being (**A:D1**). This class of propositions for which the collapse occurs is tight: weaker restrictions (**A:MC1** and **A:MC2**), which could lead

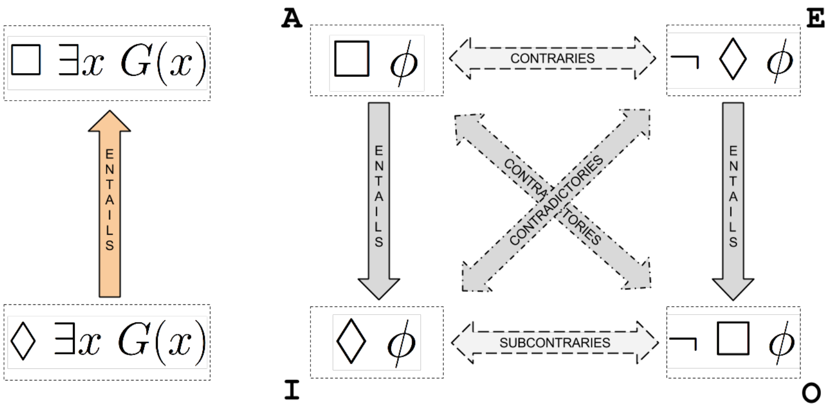

FIGURE 2. Modal Square of Opposition.

MC	Everything that is the case is so necessarily: $\forall\phi[\phi \to \Box\phi]$
MCs	Everything that is possible is necessary: $\forall\phi[\Diamond\phi \to \Box\phi]$
T	Everything that is necessary is the case: $\forall\phi[\Box\phi \to \phi]$
ExImp	(Modal Existential Import): $\Diamond\top$
AI	Everything that is necessary is possible: $\forall\phi[\Box\phi \to \Diamond\phi]$
MCt	Modalities collapse completely: $\forall\phi[(\phi \leftrightarrow \Box\phi) \wedge (\Diamond\phi \leftrightarrow \Box\phi)]$

FIGURE 3. Modal Collapse

to larger classes, are counter-satisfiable. These results hold under both constant and varying domain quantification, with possibilist and actualist quantifiers.

In any modal logic at least as strong as **K**, and even without relying on axioms specific to ontological arguments, it is easy to see (and even easier to check with an automated theorem prover) the following facts about classes of collapsing propositions:

1. Valid propositions are collapsing: if ϕ is valid, then *collapse*(ϕ) is valid.
2. The class of collapsing propositions is closed under logical equivalence: if *collapse*(ϕ) is valid and $\phi \leftrightarrow \phi'$ is valid, then *collapse*(ϕ') is valid.
3. The class of collapsing propositions is not generally closed under equi-validity: even if *collapse*(ϕ) is valid and ϕ and ϕ' are equi-valid, *collapse*(ϕ') may not be valid.

A:D1 A *God-like* being necessarily possesses those and only those properties that are positive:

$$G_A(x) \equiv \forall\varphi[P(\varphi) \leftrightarrow \Box\varphi(x)]$$

A:MC The modal collapse happens for any positive properties applied to any god-like being:

$$\forall\varphi\forall x[(P(\varphi) \wedge G_A(x)) \to \mathit{collapse}(\varphi(x))]$$

A:MC1 The modal collapse does *not* happen for positive properties applied to arbitrary individuals (*counter-satisfiable*):

$$\forall\varphi\forall x[P(\varphi) \to \mathit{collapse}(\varphi(x))]$$

A:MC2 The modal collapse does *not* happen for an arbitrary properties applied to a god-like being (*counter-satisfiable*):

$$\forall\varphi\forall x[G_A(x) \to \mathit{collapse}(\varphi(x))]$$

Figure 4. Restricted Collapse for Anderson's Emendation [2]

4. The class of collapsing propositions is not generally closed under implication: even if $\mathit{collapse}(\phi)$ is valid and $\phi \to \phi'$ is valid, $\mathit{collapse}(\phi')$ may not be valid.

An easy corollary of the second fact above is that any ontological argument relying on Lemmas **L1** and **L2** will necessarily lead to a modal collapse for all propositions that are logically equivalent to God's existence. The third and fourth facts indicate that characterizations of larger classes of propositions for which the modal collapse holds require using axioms specific to the variant of the ontological argument under consideration, as in the case of **A:MC**.

3. Final Remarks

All results announced in this note have been obtained experimentally using interactive and automated theorem provers and model finders [9, 13, 22, 10, 12]. The source codes of the experiments, as well as the resulting proofs and counter-models, are available in `github.com/FormalTheology/GoedelGod/` in the files `ModalCollapse.thy` and `ModalSquareOfOpposition.thy` inside the folder `Formalizations/Isabelle/Meta` as well as in files inside the folder `Formalizations/Isabelle/Anderson`.

The technique enabling these experiments is the embedding of quantified modal logics into higher-order logics [8, 7, 3], for which automated theorem provers exist. This technique has already been successfully employed in the verification and reconstruction of Gödel's proof [5, 4, 24], and a detailed mathematical description is available in [6].

The modal collapse is an interesting example of philosophical controversy and dispute, to which we can apply Leibniz's idea of a *calculus ratiocinator* brought to reality in the form of contemporary automated theorem provers. A significant advantage provided by the use of computers is that all parameters (e.g. modal logic, domain conditions, semantics) under which the announced results hold must be explicitly specified in the source code. Consequently, the danger of misunderstandings is reduced. Current technology is increasingly ready to be embraced by those willing to practice computer-assisted theoretical philosophy [23, 25].

Ongoing and future work includes the computer-assisted study of the modal collapse in other variants of the ontological argument (e.g. [11, 17]). Furthermore, our experiments in Isabelle revealed a weakness of the current integration of the HOL-ATPs LEO-II and Satallax via Sledgehammer: most of the problems in our study solved by the two HOL-ATPs were still too hard to be reconstructed and verified by Isabelle's internal prover Metis. This points to relevant future work regarding the integration of HOL-ATPs in Isabelle.

Acknowledgment

We would like to thank Paul Weingartner, André Fuhrmann and Melvin Fitting for several discussions about Gödel's ontological argument.

Note about authorship

Alphabetic order has been used for the authors' names. The extent and kind of contribution of each author cannot be inferred from the order.

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Christoph Benzmüller
Department of Mathematics and Computer Science
Arnimallee 7
Room 115
14195 Berlin
Germany
e-mail: c.benzmueller@gmail.com

Bruno Woltzenlogel Paleo
Favoritenstraße 9
Room HA0402
1040 Wien
Austria
e-mail: bruno.wp@gmail.com