

# Åqvist’s Dyadic Deontic Logic **E** in HOL<sup>\*</sup>

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**Abstract.** We devise a shallow semantical embedding of Åqvist’s dyadic deontic logic **E** in classical higher-order logic. This embedding is encoded in Isabelle/HOL, which turns this system into a proof assistant for deontic logic reasoning. The experiments with this environment provide evidence that this logic *implementation* fruitfully enables interactive and automated reasoning at the meta-level and the object-level.

**Keywords:** Dyadic deontic logic **E** · Preference models · Classical higher-order logic · Semantic embedding · Automated reasoning.

## 1 Introduction

Normative notions such as obligation and permission are the subject of deontic logics [18] and conditional obligations are addressed in so-called *dyadic deontic logic*. A landmark and historically important dyadic deontic logic has been proposed by B. Hansson [21] and Åqvist [3]. This dyadic deontic logic comes with a preference models semantics [24], in which a binary preference relation ranks the possible worlds in terms of betterness. The framework is immune to the well known paradoxes of *contrary-to-duty* (CTD) reasoning such as Chisholm [14]’s paradox. The class of all preference models, in which no specific constraints are put on the betterness relation, has a known axiomatic characterisation, given by Åqvist’s system **E**. (See Parent [24].)

When applied as a meta-logical tool, *simple type theory* [15], aka classical higher-order logic (HOL), can help to better understand semantical issues (of embedded object logics). The syntax and semantics of HOL are well understood [7] and there exist automated proof tools for it; examples include Isabelle/HOL [23] and LEO-II [11]. As mentioned in the *Handbook of Deontic Logic and Normative Systems* [19], the development of computational techniques in deontic logic is still in its infancy.

In this paper we devise an *embedding* of **E** in HOL. This embedding utilizes the *shallow semantical embedding* approach that has been put forward by

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Benzmüller (cf. [6] and references therein) as a pragmatical solution towards universal logic reasoning. This approach uses classical higher-order logic as (universal) meta-logic to specify, in a shallow way, the syntax and semantics of various object logics, in our case system **E**. The embedding has been encoded in Isabelle/HOL to enable experiments in deontic reasoning.

Related work [9] has developed an analogous shallow semantical embedding for the dyadic deontic logic proposed by Carmo and Jones [13]. A core difference concerns the notion of semantics employed in both works, which leads to different semantical embeddings. Instead of the semantics based on preference models as employed by Hansson and Åqvist, a neighborhood semantics is employed by Carmo and Jones. Based on the embeddings provided in [9] and here, further empirical studies are planned in order to compare these rival formalisations in particular regarding their practical reasoning performance and regarding their suitability to address relevant challenges (such as CTD reasoning) in practical applications.

Deep semantical embeddings of non-classical logics have been studied extensively in the related literature (examples are given in [17, 16]). The emphasis in these works typically is on interactive proofs of meta-logical properties. While meta-logical studies are also in reach for the methods presented here (see e.g. [8, 20]), the primary motivation, at least for this paper, is different. Our interest is in proof automation at object level, i.e. proof automation of Åqvist’s system **E**. In other words, we are interested in practical normative reasoning applications of system **E** in which a high degree of automation (at object level) is required. Moreover, we are interested not only in the ‘propositional’ system **E**, but also in quantified extensions of it. For this, we plan to accordingly adapt the achievements of previous work (see e.g. [10, 4]). Making deep semantical embeddings scale for quantified non-classical logics, on the contrary, seems more challenging and less promising regarding proof automation.

The article is structured as follows: Sec. 2 describes system **E** and Sec. 3 introduces HOL. The semantical embedding of **E** in HOL is then devised and studied in Sec. 4. This section also shows the faithfulness (viz. soundness and completeness) of the embedding. Sec. 5 discusses the implementation in Isabelle/HOL [23]. Sec. 6 concludes the paper.

## 2 Dyadic Deontic Logic **E**

The language of **E** is obtained by adding the following operators to the syntax of propositional logic:  $\Box$  (for necessity);  $\Diamond$  (for possibility); and  $\bigcirc(-/-)$  (for conditional obligation).  $\bigcirc(\psi/\phi)$  is read “If  $\phi$ , then  $\psi$  is obligatory”. The set of well-formed formulas is defined in the straightforward way. Iteration of the modal and deontic operators is permitted, and so are “mixed” formulas, e.g.,  $\bigcirc(q/p)\wedge p$ . We put  $\top =_{df} \neg q \vee q$ , for some propositional symbol  $q$ , and  $\perp =_{df} \neg\top$ .

A preference model is a structure  $M = \langle S, \succeq, V \rangle$  where

- $S$  is a non-empty set of items called possible worlds;

- $\succeq \subseteq S \times S$  (intuitively,  $\succeq$  is a betterness or comparative goodness relation; “ $s \succeq t$ ” can be read as “world  $s$  is at least as good as world  $t$ ”);
- $V$  is a function assigning to each atomic sentence a set of worlds (i.e  $V(q) \subseteq S$ ).

Given a preference model  $M = \langle S, \succeq, V \rangle$  and a world  $s \in W$ , we define the satisfaction relation  $M, s \models \varphi$  (read as “world  $s$  satisfies  $\varphi$  in model  $M$ ”) by induction on the structure of  $\varphi$  as described below. Standard deontic logic (SDL) is based on two class of states: good/bad (or green/red). In preference models we allow gradations between good and bad states. The closer a state is to ideality, the better. Intuitively, the evaluation rule for the dyadic obligation operator puts  $\bigcirc(\psi/\phi)$  true whenever all the best  $\phi$ -worlds are  $\psi$ -worlds. We define  $V^M(\varphi) = \{s \in S \mid M, s \models \varphi\}$  and  $\text{opt}_{\succeq}(V(\varphi)) = \{s \in V(\varphi) \mid \forall t(t \models \varphi \rightarrow s \succeq t)\}$ . Whenever the model  $M$  is obvious from context, we write  $V(\varphi)$  instead of  $V^M(\varphi)$ .

$$\begin{aligned}
 M, s \models p & \text{ if and only if } s \in V(p) \\
 M, s \models \neg\varphi & \text{ if and only if } M, s \not\models \varphi \text{ (that is, not } M, s \models \varphi) \\
 M, s \models \varphi \vee \psi & \text{ if and only if } M, s \models \varphi \text{ or } M, s \models \psi \\
 M, s \models \Box\varphi & \text{ if and only if } V(\varphi) = S \\
 M, s \models \bigcirc(\psi/\varphi) & \text{ if and only if } \text{opt}_{\succeq}(V(\varphi)) \subseteq V(\psi)
 \end{aligned}$$

As usual, a formula  $\varphi$  is valid in a preference model  $M = \langle S, \succeq, V \rangle$  (notation:  $M \models \varphi$ ) if and only if, for all worlds  $s \in S$ ,  $M, s \models \varphi$ . A formula  $\varphi$  is valid (notation:  $\models^{\mathbf{E}} \varphi$ ) if and only if it is valid in every preference model. The notions of semantic consequence and satisfiability in a model are defined as usual.

System **E** is defined by the following axioms and rules:

All truth functional tautologies	(PL)
S5-schemata for $\Box$ and $\Diamond$	(S5)
$\bigcirc(\psi_1 \rightarrow \psi_2/\varphi) \rightarrow (\bigcirc(\psi_1/\varphi) \rightarrow \bigcirc(\psi_2/\varphi))$	(COK)
$\bigcirc(\psi/\varphi) \rightarrow \Box \bigcirc(\psi/\varphi)$	(Abs)
$\Box\psi \rightarrow \bigcirc(\varphi/\psi)$	(Nec)
$\Box(\varphi_1 \leftrightarrow \varphi_2) \rightarrow (\bigcirc(\psi/\varphi_1) \leftrightarrow \bigcirc(\psi/\varphi_2))$	(Ext)
$\bigcirc(\varphi/\varphi)$	(Id)
$\bigcirc(\psi/\varphi_1 \wedge \varphi_2) \rightarrow \bigcirc(\varphi_2 \rightarrow \psi/\varphi_1)$	(Sh)
If $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$ then $\vdash \psi$	(MP)
If $\vdash \varphi$ then $\vdash \Box\varphi$	(N)

The notions of theoremhood, deducibility and consistency are defined as usual.

**Theorem 1.** *System **E** is (strongly) sound and complete with respect to:*

- the class of all preference models
- the class of preference models in which  $\succeq$  is reflexive

- the class of preference models in which  $\succeq$  is total (for all  $s, t \in S$ ,  $s \succeq t$  or  $t \succeq s$ )

*Proof.* See Parent [24].

### 3 Classical Higher-order Logic

In this section we introduce classical higher-order logic (HOL). The presentation, which has partly been adapted from [5], is rather detailed in order to keep the article sufficiently self-contained.

#### 3.1 Syntax of HOL

To define the syntax of HOL, we first introduce the set  $T$  of *simple types*. We assume that  $T$  is freely generated from a set of *basic types*  $BT \supseteq \{o, i\}$  using the function type constructor  $\rightarrow$ . Type  $o$  denotes the (bivalent) set of Booleans, and  $i$  a non-empty set of individuals.

For the definition of HOL, we start out with a family of denumerable sets of typed constant symbols  $(C_\alpha)_{\alpha \in T}$ , called the HOL *signature*, and a family of denumerable sets of typed variable symbols  $(V_\alpha)_{\alpha \in T}$ .<sup>3</sup> We employ Church-style typing, where each term  $t_\alpha$  explicitly encodes its type information in subscript  $\alpha$ .

The *language of HOL* is given as the smallest set of terms obeying the following conditions.

- Every typed constant symbol  $c_\alpha \in C_\alpha$  is a HOL term of type  $\alpha$ .
- Every typed variable symbol  $X_\alpha \in V_\alpha$  is a HOL term of type  $\alpha$ .
- If  $s_{\alpha \rightarrow \beta}$  and  $t_\alpha$  are HOL terms of types  $\alpha \rightarrow \beta$  and  $\alpha$ , respectively, then  $(s_{\alpha \rightarrow \beta} t_\alpha)_\beta$ , called *application*, is an HOL term of type  $\beta$ .
- If  $X_\alpha \in V_\alpha$  is a typed variable symbol and  $s_\beta$  is an HOL term of type  $\beta$ , then  $(\lambda X_\alpha s_\beta)_{\alpha \rightarrow \beta}$ , called *abstraction*, is an HOL term of type  $\alpha \rightarrow \beta$ .

The above definition encompasses the simply typed  $\lambda$ -calculus. In order to extend this base framework into logic HOL we simply ensure that the signature  $(C_\alpha)_{\alpha \in T}$  provides a sufficient selection of primitive logical connectives. Without loss of generality, we here assume the following *primitive logical connectives* to be part of the signature:  $\neg_{o \rightarrow o} \in C_{o \rightarrow o}$ ,  $\forall_{o \rightarrow o \rightarrow o} \in C_{o \rightarrow o \rightarrow o}$ ,  $\Pi_{(\alpha \rightarrow o) \rightarrow o} \in C_{(\alpha \rightarrow o) \rightarrow o}$  and  $=_{\alpha \rightarrow \alpha \rightarrow \alpha} \in C_{\alpha \rightarrow \alpha \rightarrow \alpha}$ , abbreviated as  $=^\alpha$ . The symbols  $\Pi_{(\alpha \rightarrow o) \rightarrow o}$  and  $=_{\alpha \rightarrow \alpha \rightarrow \alpha}$  are generally assumed for each type  $\alpha \in T$ . The denotation of the primitive logical connectives is fixed below according to their intended meaning. *Binder notation*  $\forall X_\alpha s_o$  is used as an abbreviation for  $(\Pi_{(\alpha \rightarrow o) \rightarrow o}(\lambda X_\alpha s_o))$ . Universal quantification in HOL is thus modeled with the help of the logical constants  $\Pi_{(\alpha \rightarrow o) \rightarrow o}$  to be used in combination with lambda-abstraction. That is, the only binding mechanism provided in HOL is lambda-abstraction.

<sup>3</sup> For example in Sec. 4 we will assume constant symbol  $r$ , with type  $i \rightarrow i \rightarrow o$  as part of the signature.

HOL is a logic of terms in the sense that the *formulas of HOL* are given as the terms of type  $o$ . In addition to the primitive logical connectives selected above, we could assume *choice operators*  $\epsilon_{(\alpha \rightarrow o) \rightarrow \alpha} \in C_{(\alpha \rightarrow o) \rightarrow \alpha}$  (for each type  $\alpha$ ) in the signature. We are not pursuing this here.

Type information as well as brackets may be omitted if obvious from the context, and we may also use infix notation to improve readability. For example, we may write  $(s \vee t)$  instead of  $((\vee_{o \rightarrow o \rightarrow o} s_o) t_o)$ .

From the selected set of primitive connectives, other logical connectives can be introduced as abbreviations.<sup>4</sup> For example, we may define  $s \wedge t := \neg(\neg s \vee \neg t)$ ,  $s \rightarrow t := \neg s \vee t$ ,  $s \longleftrightarrow t := (s \rightarrow t) \wedge (t \rightarrow s)$ ,  $\top := (\lambda X_i X) = (\lambda X_i X)$ ,  $\perp := \neg \top$  and  $\exists X_\alpha s := \neg \forall X_\alpha \neg s$ .

The notions of *free variables*,  $\alpha$ -*conversion*,  $\beta\eta$ -*equality* (denoted as  $=_{\beta\eta}$ ) and *substitution* of a term  $s_\alpha$  for a variable  $X_\alpha$  in a term  $t_\beta$  (denoted as  $[s/X]t$ ) are defined as usual.

### 3.2 Semantics of HOL

The semantics of HOL is well understood and thoroughly documented. The introduction provided next focuses on the aspects as needed for this article. For more details we refer to the previously mentioned literature [7].

The semantics of choice for the remainder is Henkin semantics, i.e., we work with Henkin's general models [22]. Henkin models (and standard models) are introduced next. We start out with introducing frame structures.

A *frame*  $D$  is a collection  $\{D_\alpha\}_{\alpha \in \mathbb{T}}$  of nonempty sets  $D_\alpha$ , such that  $D_o = \{T, F\}$  (for truth and falsehood). The  $D_{\alpha \rightarrow \beta}$  are collections of functions mapping  $D_\alpha$  into  $D_\beta$ .

A *model* for HOL is a tuple  $M = \langle D, I \rangle$ , where  $D$  is a frame, and  $I$  is a family of typed interpretation functions mapping constant symbols  $p_\alpha \in C_\alpha$  to appropriate elements of  $D_\alpha$ , called the *denotation of  $p_\alpha$* . The logical connectives  $\neg$ ,  $\vee$ ,  $\exists$  and  $=$  are always given their expected, standard denotations:<sup>5</sup>

- $I(\neg_{o \rightarrow o}) = \text{not} \in D_{o \rightarrow o}$  such that  $\text{not}(T) = F$  and  $\text{not}(F) = T$ .

<sup>4</sup> As demonstrated by Andrews [2], we could in fact start out with only primitive equality in the signature (for all types  $\alpha$ ) and introduce all other logical connectives as abbreviations based on it. Alternatively, we could remove primitive equality from the above signature, since equality can be defined in HOL from these other logical connectives by exploiting Leibniz' principle, expressing that two objects are equal if they share the same properties. *Leibniz equality*  $\doteq^\alpha$  at type  $\alpha$  is thus defined as  $s_\alpha \doteq^\alpha t_\alpha := \forall P_{\alpha \rightarrow o} (Ps \longleftrightarrow Pt)$ . The motivation for the redundant signature as selected here is to stay close to the choices taken in implemented theorem provers such as LEO-II and Leo-III and also to theory paper [7], which is recommended for further details.

<sup>5</sup> Since  $=_{\alpha \rightarrow \alpha \rightarrow o}$  (for all types  $\alpha$ ) is in the signature, it is ensured that the domains  $D_{\alpha \rightarrow \alpha \rightarrow o}$  contain the respective identity relations. This addresses an issue discovered by Andrews [1]: if such identity relations did not exist in the  $D_{\alpha \rightarrow \alpha \rightarrow o}$ , then Leibniz equality in Henkin semantics might not denote as intended.

- $I(\vee_{o \rightarrow o \rightarrow o}) = or \in D_{o \rightarrow o \rightarrow o}$  such that  $or(a, b) = T$  iff  $(a = T \text{ or } b = T)$ .
- $I(=\alpha \rightarrow \alpha \rightarrow o) = id \in D_{\alpha \rightarrow \alpha \rightarrow o}$  such that for all  $a, b \in D_\alpha$ ,  $id(a, b) = T$  iff  $a$  is identical to  $b$ .
- $I(\Pi_{(\alpha \rightarrow o) \rightarrow o}) = all \in D_{(\alpha \rightarrow o) \rightarrow o}$  such that for all  $s \in D_{\alpha \rightarrow o}$ ,  $all(s) = T$  iff  $s(a) = T$  for all  $a \in D_\alpha$ ; i.e.,  $s$  is the set of all objects of type  $\alpha$ .

Variable assignments are a technical aid for the subsequent definition of an interpretation function  $\|\cdot\|^{M,g}$  for HOL terms. This interpretation function is parametric over a model  $M$  and a variable assignment  $g$ .

A *variable assignment*  $g$  maps variables  $X_\alpha$  to elements in  $D_\alpha$ .  $g[d/W]$  denotes the assignment that is identical to  $g$ , except for variable  $W$ , which is now mapped to  $d$ .

The *denotation*  $\|s_\alpha\|^{M,g}$  of an HOL term  $s_\alpha$  on a model  $M = \langle D, I \rangle$  under assignment  $g$  is an element  $d \in D_\alpha$  defined in the following way:

$$\begin{aligned} \|p_\alpha\|^{M,g} &= I(p_\alpha) \\ \|X_\alpha\|^{M,g} &= g(X_\alpha) \\ \|(\mathit{s}_{\alpha \rightarrow \beta} \mathit{t}_\alpha)_\beta\|^{M,g} &= \|\mathit{s}_{\alpha \rightarrow \beta}\|^{M,g}(\|\mathit{t}_\alpha\|^{M,g}) \\ \|(\lambda X_\alpha \mathit{s}_\beta)_{\alpha \rightarrow \beta}\|^{M,g} &= \text{the function } f \text{ from } D_\alpha \text{ to } D_\beta \text{ such that} \\ & f(d) = \|\mathit{s}_\beta\|^{M,g[d/X_\alpha]} \text{ for all } d \in D_\alpha \end{aligned}$$

A model  $M = \langle D, I \rangle$  is called a *standard model* if and only if for all  $\alpha, \beta \in T$  we have  $D_{\alpha \rightarrow \beta} = \{f \mid f : D_\alpha \rightarrow D_\beta\}$ . In a *Henkin model* (*general model*) function spaces are not necessarily full. Instead it is only required that for all  $\alpha, \beta \in T$ ,  $D_{\alpha \rightarrow \beta} \subseteq \{f \mid f : D_\alpha \rightarrow D_\beta\}$ . However, it is required that the valuation function  $\|\cdot\|^{M,g}$  from above is total, so that every term denotes. Note that this requirement, which is called *Denotatpflicht*, ensures that the function domains  $D_{\alpha \rightarrow \beta}$  never become too sparse, that is, the denotations of the lambda-abstractions as devised above are always contained in them.

**Corollary 1.** *For any Henkin model  $M = \langle D, I \rangle$  and variable assignment  $g$ :*

1.  $\|(\neg_{o \rightarrow o} s_o)_o\|^{M,g} = T$  iff  $\|s_o\|^{M,g} = F$ .
2.  $\|((\vee_{o \rightarrow o \rightarrow o} s_o) t_o)_o\|^{M,g} = T$  iff  $\|s_o\|^{M,g} = T$  or  $\|t_o\|^{M,g} = T$ .
3.  $\|((\wedge_{o \rightarrow o \rightarrow o} s_o) t_o)_o\|^{M,g} = T$  iff  $\|s_o\|^{M,g} = T$  and  $\|t_o\|^{M,g} = T$ .
4.  $\|((\rightarrow_{o \rightarrow o \rightarrow o} s_o) t_o)_o\|^{M,g} = T$  iff (if  $\|s_o\|^{M,g} = T$  then  $\|t_o\|^{M,g} = T$ ).
5.  $\|((\leftarrow_{o \rightarrow o \rightarrow o} s_o) t_o)_o\|^{M,g} = T$  iff ( $\|s_o\|^{M,g} = T$  iff  $\|t_o\|^{M,g} = T$ ).
6.  $\|\top\|^{M,g} = T$ .
7.  $\|\perp\|^{M,g} = F$ .
8.  $\|(\forall X_\alpha s_o)_o\|^{M,g} = T$  iff for all  $d \in D_\alpha$  we have  $\|s_o\|^{M,g[d/X_\alpha]} = T$ .
9.  $\|(\exists X_\alpha s_o)_o\|^{M,g} = T$  iff there exists  $d \in D_\alpha$  such that  $\|s_o\|^{M,g[d/X_\alpha]} = T$ .

*Proof.* We leave the proof as an exercise to the reader.

An HOL formula  $s_o$  is *true* in a Henkin model  $M$  under assignment  $g$  if and only if  $\|s_o\|^{M,g} = T$ ; this is also expressed by writing that  $M, g \models^{\text{HOL}} s_o$ . An HOL formula  $s_o$  is called *valid* in  $M$ , which is expressed by writing that  $M \models^{\text{HOL}} s_o$ , if and only if  $M, g \models^{\text{HOL}} s_o$  for all assignments  $g$ . Moreover, a formula  $s_o$  is called *valid*, expressed by writing that  $\models^{\text{HOL}} s_o$ , if and only if  $s_o$  is valid in all Henkin models  $M$ .

## 4 Embedding **E** into HOL

This section, the core contribution of this article, presents a shallow semantical embedding of system **E** in HOL and proves its soundness and completeness. In contrast to a deep logical embedding, in which the syntactical structure and the semantics of logic  $L$  would be formalized in full detail (using e.g. structural induction and recursion), only the core differences in the semantics of both system **E** and meta-logic HOL are explicitly encoded here. In a certain sense we show, that system **E** can in fact be identified and handled as a natural fragment of HOL.

### 4.1 Semantical Embedding

The formulas of **E** are identified in our semantical embedding with certain HOL terms (predicates) of type  $i \rightarrow o$ . They can be applied to terms of type  $i$ , which are assumed to denote possible worlds. That is, the HOL type  $i$  is now identified with a (non-empty) set of worlds. Type  $i \rightarrow o$  is abbreviated as  $\tau$  in the remainder. The HOL signature is assumed to contain the constant symbol  $r_{i \rightarrow \tau}$ . Moreover, for each propositional symbol  $p^j$  of **E**, the HOL signature must contain the corresponding constant symbol  $p_\tau^j$ . Without loss of generality, we assume that besides those symbols and the primitive logical connectives of HOL, no other constant symbols are given in the signature of HOL.

The mapping  $[\cdot]$  translates a formula  $\varphi$  of **E** into a formula  $[\varphi]$  of HOL of type  $\tau$ . The mapping is defined recursively:

$$\begin{aligned} [p^j] &= p_\tau^j \\ [\neg s] &= \neg_\tau [s] \\ [s \vee t] &= \vee_{\tau \rightarrow \tau \rightarrow \tau} [s] [t] \\ [\Box s] &= \Box_{\tau \rightarrow \tau} [s] \\ [\bigcirc(t/s)] &= \bigcirc_{\tau \rightarrow \tau \rightarrow \tau} [s] [t] \end{aligned}$$

$\neg_\tau, \vee_{\tau \rightarrow \tau \rightarrow \tau}, \Box_{\tau \rightarrow \tau}, \bigcirc_{\tau \rightarrow \tau \rightarrow \tau}$  abbreviate the following formulas of HOL:

$$\begin{aligned} \neg_{\tau \rightarrow \tau} &= \lambda A_\tau \lambda X_i \neg(A X) \\ \vee_{\tau \rightarrow \tau \rightarrow \tau} &= \lambda A_\tau \lambda B_\tau \lambda X_i (A X \vee B X) \\ \Box_{\tau \rightarrow \tau} &= \lambda A_\tau \lambda X_i \forall Y_i (A Y) \\ \bigcirc_{\tau \rightarrow \tau \rightarrow \tau} &= \lambda A_\tau \lambda B_\tau \lambda X_i \forall W_i ((\lambda V_i (A V \wedge (\forall Y_i (A Y \rightarrow r_{i \rightarrow \tau} V Y)))) W \rightarrow B W)^6 \end{aligned}$$

Analyzing the truth of a translated formula  $[s]$  in a world represented by term  $w_i$  corresponds to evaluating the application  $([s] w_i)$ . In line with previous work [10], we define  $\text{vld}_{\tau \rightarrow o} = \lambda A_\tau \forall S_i (A S)$ . With this definition, validity of a formula  $s$  in **E** corresponds to the validity of the formula  $(\text{vld } [s])$  in HOL, and vice versa.

<sup>6</sup> If  $\text{opt}_{\geq}(A)$  is taken as a abbreviation for  $\lambda V_i (A V \wedge (\forall Y_i (A Y \rightarrow r_{i \rightarrow \tau} V Y)))$ , then this can be simplified to  $\bigcirc_{\tau \rightarrow \tau \rightarrow \tau} = \lambda A_\tau \lambda B_\tau \lambda X_i (\text{opt}_{\geq}(A) \subseteq B)$ .

## 4.2 Soundness and Completeness

To prove the soundness and completeness, that is, faithfulness, of the above embedding, a mapping from preference models into Henkin models is employed.

**Definition 1 (Preference model  $\Rightarrow$  Henkin model).** Let  $M = \langle S, \succeq, V \rangle$  be a preference model. Let  $p^1, \dots, p^m \in PV$ , for  $m \geq 1$  be propositional symbols and  $[p^j] = p^j$  for  $j = 1, \dots, m$ . A Henkin model  $H^M = \langle \{D_\alpha\}_{\alpha \in T}, I \rangle$  for  $M$  is defined as follows:  $D_i$  is chosen as the set of possible worlds  $S$  and all other sets  $D_{\alpha \rightarrow \beta}$  are chosen as (not necessarily full) sets of functions from  $D_\alpha$  to  $D_\beta$ . For all  $D_{\alpha \rightarrow \beta}$  the rule that every term  $t_{\alpha \rightarrow \beta}$  must have a denotation in  $D_{\alpha \rightarrow \beta}$  must be obeyed, in particular, it is required that  $D_\tau$  and  $D_{i \rightarrow \tau}$  contain the elements  $I p_\tau^j$  and  $I r_{i \rightarrow \tau}$ . Interpretation  $I$  is constructed as follows:

1. For  $1 \leq i \leq m$ ,  $I p_\tau^j \in D_\tau$  is chosen such that  $I p_\tau^j(s) = T$  if  $s \in V(p^j)$  in  $M$  and  $I p_\tau^j(s) = F$  otherwise.
2.  $I r_{i \rightarrow \tau} \in D_{i \rightarrow \tau}$  is chosen such that  $I r_{i \rightarrow \tau}(s, u) = T$  if  $s \succeq u$  in  $M$  and  $I r_{i \rightarrow \tau}(s, u) = F$  otherwise.

Since we assume that there are no other symbols (besides the  $r$ , the  $p^j$  and the primitive logical connectives) in the signature of  $HOL$ ,  $I$  is a total function. Moreover, the above construction guarantees that  $H^M$  is a Henkin model:  $\langle D, I \rangle$  is a frame, and the choice of  $I$  in combination with the Denotatpflicht ensures that for arbitrary assignments  $g$ ,  $\|\cdot\|^{H^M, g}$  is a total evaluation function.

**Lemma 1.** Let  $H^M$  be a Henkin model for a preference model  $M$ . For all formula  $\delta$  of  $\mathbf{E}$ , all assignment  $g$  and world  $s$  it holds:

$$M, s \models \delta \text{ if and only if } \|\llbracket \delta \rrbracket S_i\|^{H^M, g[s/S_i]} = T$$

*Proof.* See appendix.

**Lemma 2 (Henkin model  $\Rightarrow$  Preference model).** For every Henkin model  $H = \langle \{D_\alpha\}_{\alpha \in T}, I \rangle$  there exists a corresponding preference model  $M$ . Corresponding here means that for all formula  $\delta$  of  $\mathbf{E}$  and for all assignment  $g$  and world  $s$ ,  $\|\llbracket \delta \rrbracket S_i\|^{H, g[s/S_i]} = T$  if and only if  $M, s \models \delta$ .

*Proof.* Suppose that  $H = \langle \{D_\alpha\}_{\alpha \in T}, I \rangle$  is a Henkin model. Without loss of generality, we can assume that the domains of  $H$  are denumerable [22]. We construct the corresponding preference model  $M$  as follows:

- $S = D_i$ .
- $s \succeq u$  for  $s, u \in S$  iff  $I r_{i \rightarrow \tau}(s, u) = T$ .
- $s \in V(p^j)$  iff  $I p_\tau^j(s) = T$  for all  $p^j$ .

Moreover, the above construction ensures that  $H$  is a Henkin model for  $M$ . Hence, Lemma 1 applies. This ensures that for all formulas  $\delta$  of  $\mathbf{E}$ , for all assignment  $g$  and all world  $s$  we have  $\|\llbracket \delta \rrbracket S_i\|^{H, g[s/S_i]} = T$  if and only if  $M, s \models \delta$ .



**Theorem 2 (Soundness and Completeness of the Embedding).**

$$\models^{\mathbf{E}} \varphi \text{ if and only if } \models^{\text{HOL}} \text{vld} \lfloor \varphi \rfloor$$

*Proof.* (Soundness,  $\leftarrow$ ) The proof is by contraposition. Assume  $\not\models^{\mathbf{E}} \varphi$ , i.e., there is a preference model  $M = \langle S, \succeq, V \rangle$ , and a world  $s \in S$ , such that  $M, s \not\models \varphi$ . By Lemma 1 for an arbitrary assignment  $g$  it holds that  $\|\lfloor \varphi \rfloor S_i\|^{H^M, g[s/S_i]} = F$  in Henkin model  $H^M = \langle \{D_\alpha\}_{\alpha \in T}, I \rangle$ . Thus, by definition of  $\|\cdot\|$ , it holds that  $\|\forall S_i(\lfloor \varphi \rfloor S_i)\|^{H^M, g} = \|\text{vld} \lfloor \varphi \rfloor\|^{H^M, g} = F$ . Hence,  $H^M \not\models^{\text{HOL}} \text{vld} \lfloor \varphi \rfloor$ . By definition  $\not\models^{\text{HOL}} \text{vld} \lfloor \varphi \rfloor$ .

(Completeness,  $\rightarrow$ ) The proof is again by contraposition. Assume  $\not\models^{\text{HOL}} \text{vld} \lfloor \varphi \rfloor$ , i.e., there is a Henkin model  $H = \langle \{D_\alpha\}_{\alpha \in T}, I \rangle$  and an assignment  $g$  such that  $\|\text{vld} \lfloor \varphi \rfloor\|^{H, g} = F$ . By Lemma 2, there is a preference model  $M$  such that  $M \not\models \varphi$ . Hence,  $\not\models^{\mathbf{E}} \varphi$ .

## 5 Implementation in Isabelle/HOL

The semantical embedding as devised in Sec. 4 has been implemented in the higher-order proof assistant Isabelle/HOL [23]. Figure 1 displays the respective encoding. Figure 2 applies this encoding to Chisholm's paradox (cf. [14]), which involves the following four statements:

1. It ought to be that a certain man go to help his neighbors;
2. It ought to be that if he goes he tells them he is coming;
3. If he does not go, he ought not to tell them he is coming;
4. He does not go.

These statements can be given a consistent formalisation in DDL see Fig. 2. This is confirmed by the model finder Nitpick [12] integrated with Isabelle/HOL. Nitpick computes an intuitive, small model for the scenario consisting of one possible world  $i_1$ . The actual world is  $i_1$  also. Preference relation  $r$  is interpreted in this model as  $r = \emptyset$ .

In the actual world the man doesn't go to help his neighbors and doesn't tell them that he is coming. That is,  $V(\neg go) = V(\neg tell) = \{i_1\}$ . Also, we have  $op(V(\top)) = \emptyset$ . So,  $i_1 \models \bigcirc(go/\top)$  by the evaluation rule for  $\bigcirc$ . Similarly,  $op(V(go)) = op(V(\neg go)) = \emptyset$  implies  $i_1 \models \bigcirc(tell/go)$  and  $i_1 \models \bigcirc(\neg tell/\neg go)$ .

## 6 Conclusion

A shallow semantical embedding of Åqvist's dyadic deontic logic **E** in classical higher-order logic has been presented, and shown to be faithful (sound and complete). The works presented here and in [9] provide the theoretical foundation for the implementation and automation of dyadic deontic logic within existing theorem provers and proof assistants for HOL. We do not provide new logics. Instead, we provide an empirical infrastructure for assessing practical aspects of an ambitious, state-of-the-art deontic logics; this has not been done

before. An interesting and relevant aspect of the approach is that (based on the ideas of previous work [10, 4]) quantified extensions of system **E** and [9] can easily be implemented and studied in the framework and experimentally assessed. There is much room for future work. For example, experiments could investigate whether the provided implementation already supports non-trivial applications in practical normative reasoning, or whether further improvements are required. Moreover, we could employ our implementation to systematically study some meta-logical properties of dyadic deontic logic system **E** within Isabelle/HOL.

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## Appendix (Proof for Lemma 1)

In the proof we implicitly employ curring and uncuring, and we associate sets with their characteristic functions. Throughout the proof whenever possible we omit types in order to avoid making the notation too cumbersome. The proof of lemma 1 is by induction on the structure of  $\delta$ . We start with the case where  $\delta$  is  $p^j$ . We have  $\| [p^j] S \|^{H^M, g[s/S_i]} = T$

$$\begin{aligned}
 &\Leftrightarrow \| [p^j] S \|^{H^M, g[s/S_i]} = T \\
 &\Leftrightarrow I p^j_\tau(s) = T \\
 &\Leftrightarrow s \in V(p^j) \quad (\text{by definition of } H^M) \\
 &\Leftrightarrow M, s \models p^j
 \end{aligned}$$

For the inductive cases we make the hypothesis that the claim holds for sentences  $\delta'$  shorter than  $\delta$ :

**Inductive hypothesis:** For all assignment  $g$  and state  $s$ ,

$$\| [\delta'] S \|^{H^M, g[s/S_i]} = T \text{ if and only if } M, s \models \delta'$$

We consider each inductive case in turn:

(a)  $\delta = \varphi \vee \psi$ . In this case:

$$\begin{aligned}
 &\| [\varphi \vee \psi] S \|^{H^M, g[s/S_i]} = T \\
 \Leftrightarrow &\| ([\varphi] \vee_{\tau \rightarrow \tau} [\psi]) S \|^{H^M, g[s/S_i]} = T \\
 \Leftrightarrow &\| ([\varphi] S) \vee ([\psi] S) \|^{H^M, g[s/S_i]} = T \quad ((([\varphi] \vee_{\tau \rightarrow \tau} [\psi]) S] =_{\beta\eta} (([\varphi] S) \vee ([\psi] S)))
 \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \|\llbracket \varphi \rrbracket S\|^{H^M, g[s/S_i]} = T \text{ or } \|\llbracket \psi \rrbracket S\|^{H^M, g[s/S_i]} = T \\
&\Leftrightarrow M, s \models \varphi \text{ or } M, s \models \psi \quad (\text{by induction hypothesis}) \\
&\Leftrightarrow M, s \models \varphi \vee \psi
\end{aligned}$$

(b)  $\delta = \neg\varphi$ . In this case:

$$\begin{aligned}
&\|\llbracket \neg\varphi \rrbracket S\|^{H^M, g[s/S_i]} = T \\
&\Leftrightarrow \|\llbracket (\neg_{\tau \rightarrow \tau} \varphi) \rrbracket S\|^{H^M, g[s/S_i]} = T \\
&\Leftrightarrow \|\llbracket \neg(\llbracket \varphi \rrbracket S) \rrbracket S\|^{H^M, g[s/S_i]} = T \quad ((\neg_{\tau \rightarrow \tau} \llbracket \varphi \rrbracket S) =_{\beta\eta} \neg(\llbracket \varphi \rrbracket S)) \\
&\Leftrightarrow \|\llbracket \varphi \rrbracket S\|^{H^M, g[s/S_i]} = F \\
&\Leftrightarrow M, s \not\models \varphi \quad (\text{by induction hypothesis}) \\
&\Leftrightarrow M, s \models \neg\varphi
\end{aligned}$$

(c)  $\delta = \Box\varphi$ . We have the following chain of equivalences:

$$\begin{aligned}
&\|\llbracket \Box\varphi \rrbracket S\|^{H^M, g[s/S_i]} = T \\
&\Leftrightarrow \|\llbracket (\lambda X \forall Y (\llbracket \varphi \rrbracket Y)) S \rrbracket S\|^{H^M, g[s/S_i]} = T \\
&\Leftrightarrow \|\llbracket \forall Y (\llbracket \varphi \rrbracket Y) \rrbracket S\|^{H^M, g[s/S_i]} = T \\
&\Leftrightarrow \text{For all } a \in D_i \text{ we have } \|\llbracket \varphi \rrbracket Y\|^{H^M, g[s/S_i][a/Y_i]} = T \\
&\Leftrightarrow \text{For all } a \in D_i \text{ we have } \|\llbracket \varphi \rrbracket Y\|^{H^M, g[a/Y_i]} = T \quad (S \notin \text{free}(\llbracket \varphi \rrbracket)) \\
&\Leftrightarrow \text{For all } a \in S \text{ we have } M, a \models \varphi \quad (\text{by induction hypothesis}) \\
&\Leftrightarrow M, s \models \Box\varphi
\end{aligned}$$

(d)  $\delta = \bigcirc(\psi/\varphi)$ . We have the following chain of equivalences:

$$\begin{aligned}
&\|\llbracket \bigcirc(\psi/\varphi) \rrbracket S\|^{H^M, g[s/S_i]} = T \\
&\Leftrightarrow \|\llbracket (\lambda X \forall W ((\lambda V (\llbracket \varphi \rrbracket V \wedge (\forall Y (\llbracket \varphi \rrbracket Y \rightarrow r V Y)))) W \rightarrow \llbracket \psi \rrbracket W)) S \rrbracket S\|^{H^M, g[s/S_i]} = T \\
&\Leftrightarrow \|\llbracket \forall W ((\lambda V (\llbracket \varphi \rrbracket V \wedge (\forall Y (\llbracket \varphi \rrbracket Y \rightarrow r V Y)))) W \rightarrow \llbracket \psi \rrbracket W) \rrbracket S\|^{H^M, g[s/S_i]} = T \\
&\Leftrightarrow \text{For all } u \in D_i \text{ we have:} \\
&\quad \|\llbracket (\lambda V (\llbracket \varphi \rrbracket V \wedge (\forall Y (\llbracket \varphi \rrbracket Y \rightarrow r V Y)))) W \rrbracket W\|^{H^M, g[s/S_i][u/W_i]} = T \\
&\Leftrightarrow \text{For all } u \in D_i \text{ we have:} \\
&\quad \text{If } \|\llbracket (\lambda V (\llbracket \varphi \rrbracket V \wedge (\forall Y (\llbracket \varphi \rrbracket Y \rightarrow r V Y)))) W \rrbracket W\|^{H^M, g[s/S_i][u/W_i]} = T, \\
&\quad \text{then } \|\llbracket \psi \rrbracket W\|^{H^M, g[s/S_i][u/W_i]} = T \\
&\Leftrightarrow \text{For all } u \in D_i \text{ we have:} \\
&\quad \text{If } \|\llbracket \varphi \rrbracket W\|^{H^M, g[s/S_i][u/W_i]} = T \text{ and} \\
&\quad \|\llbracket \forall Y (\llbracket \varphi \rrbracket Y \rightarrow r W Y) \rrbracket W\|^{H^M, g[s/S_i][u/W_i]} = T, \\
&\quad \text{then } \|\llbracket \psi \rrbracket W\|^{H^M, g[s/S_i][u/W_i]} = T \\
&\Leftrightarrow \text{For all } u \in D_i \text{ we have:} \\
&\quad \text{If } \|\llbracket \varphi \rrbracket W\|^{H^M, g[s/S_i][u/W_i]} = T \text{ and} \\
&\quad \text{for all } t \in D_i \text{ we have } \|\llbracket \forall Y (\llbracket \varphi \rrbracket Y \rightarrow r W Y) \rrbracket W\|^{H^M, g[s/S_i][u/W_i][t/Y_i]} = T, \\
&\quad \text{then } \|\llbracket \psi \rrbracket W\|^{H^M, g[s/S_i][u/W_i]} = T \\
&\Leftrightarrow \text{For all } u \in D_i \text{ we have:} \\
&\quad \text{If } \|\llbracket \varphi \rrbracket W\|^{H^M, g[s/S_i][u/W_i]} = T \text{ and} \\
&\quad \text{for all } t \in D_i \text{ we have } \|\llbracket \varphi \rrbracket Y\|^{H^M, g[s/S_i][u/W_i][t/Y_i]} = T \text{ implies } Ir_{i \rightarrow \tau}(u, t) = T, \\
&\quad \text{then } \|\llbracket \psi \rrbracket W\|^{H^M, g[s/S_i][u/W_i]} = T
\end{aligned}$$

- $\Leftrightarrow$  For all  $u \in D_i$  we have:  
 If  $u \in V(\varphi)$  and  
 for all  $t \in D_i$  we have  $t \in V(\varphi)$  implies  $u \succeq t$ ,  
 then  $u \in V(\psi)$  (**see the justification \***)  
 $\Leftrightarrow \text{opt}_{\succeq}(V(\varphi)) \subseteq V(\psi)$   
 $\Leftrightarrow M, s \models \bigcirc(\psi/\varphi)$

**Justification \*:** What we need to show is:  $\llbracket \varphi \rrbracket^{H^M, g[s/S_i]}$  is identified with  $V(\varphi)$  (analogously  $\psi$ ). By induction hypothesis, for all assignment  $g$  and state  $s$ , we have  $\llbracket \varphi \rrbracket S \llbracket \varphi \rrbracket^{H^M, g[s/S_i]} = T$  if and only if  $M, s \models \varphi$ . Expanding the details of this equivalence we have: for all assignment  $g$  and state  $s$

$$\begin{aligned}
 &\Leftrightarrow s \in \llbracket \varphi \rrbracket^{H^M, g[s/S_i]} \quad (\text{functions to type } o \text{ are associated with sets}) \\
 &\Leftrightarrow \llbracket \varphi \rrbracket^{H^M, g[s/S_i]}(s) = T \\
 &\Leftrightarrow \llbracket \varphi \rrbracket^{H^M, g[s/S_i]} \llbracket S \rrbracket^{H^M, g[s/S_i]} = T \\
 &\Leftrightarrow \llbracket \varphi \rrbracket S \llbracket \varphi \rrbracket^{H^M, g[s/S_i]} = T \\
 &\Leftrightarrow M, s \models \varphi \\
 &\Leftrightarrow s \in V(\varphi)
 \end{aligned}$$

Hence,  $s \in \llbracket \varphi \rrbracket^{H^M, g[s/S_i]}$  if and only if  $s \in V(\varphi)$ .

By extensionality we thus know that  $\llbracket \varphi \rrbracket^{H^M, g[s/S_i]}$  is identified with  $V(\varphi)$ . Moreover, since  $H^M$  obeys the Denotatpflicht we know that  $V(\varphi) \in D_\tau$ .

```

1 theory DDLE imports Main
2 begin
3 typedef i -- "type for possible worlds"
4 type_synonym σ = "(i⇒bool)"
5 consts aw::i (* actual world *)
6
7 abbreviation(input) mtrue  :: "σ" ("T") where "T ≡ λw. True"
8 abbreviation(input) mfalse :: "σ" ("⊥") where "⊥ ≡ λw. False"
9 abbreviation(input) mnnot  :: "σ⇒σ" ("¬"[52]53) where "¬φ ≡ λw. ¬φ(w)"
10 abbreviation(input) mand   :: "σ⇒σ⇒σ" (infixr"∧"51) where "φ∧ψ ≡ λw. φ(w)∧ψ(w)"
11 abbreviation(input) mor    :: "σ⇒σ⇒σ" (infixr"∨"50) where "φ∨ψ ≡ λw. φ(w)∨ψ(w)"
12 abbreviation(input) mimp   :: "σ⇒σ⇒σ" (infixr"→"49) where "φ→ψ ≡ λw. φ(w)→ψ(w)"
13 abbreviation(input) mequ   :: "σ⇒σ⇒σ" (infixr"↔"48) where "φ↔ψ ≡ λw. φ(w)↔ψ(w)"
14
15 abbreviation(input) mbox  :: "σ⇒σ" ("□") where "□ ≡ λφ w. ∀v. φ(v)"
16 consts r :: "i⇒i⇒bool" (infixr "r" 70)
17 -- "the betterness relation r, used in definition of 0"
18 abbreviation(input) mopt  :: "(i⇒bool)⇒(i⇒bool)" ("opt<_>")
19 where "opt<φ> ≡ (λv. ( φ(v) ∧ (∀x. ((φ(x) → v r x) )) )"
20 abbreviation(input) msubset :: "σ⇒σ⇒bool" (infix "⊆" 53)
21 where "φ ⊆ ψ ≡ ∀x. φ x → ψ x"
22 abbreviation(input) mcond  :: "σ⇒σ⇒σ" ("◊<_>")
23 where "◊<ψ|φ> ≡ λw. opt<φ> ⊆ ψ"
24
25 abbreviation(input) valid  :: "σ⇒bool" ("⊨"[81]109)
26 where "|p| ≡ ∀w. p w"
27 definition cjactual :: "σ ⇒ bool" ("⊨i"[7]105)
28 where "|p|i ≡ p(aw)"
29
30 lemma True nitpick [satisfy, user_axioms, show_all, expect=genuine] oops
31

```

Output | Query | Sledgehammer | Symbols

(isabelle,isabelle,UTF-8-Isabelle)tm r o UG 17/11/2016 3:26 PM

Fig. 1. Shallow semantical embedding of  $\mathbf{E}$  in Isabelle/HOL.

```

52 section {* Chisholm Scenario *}
53
54 consts go :: "o" tell :: "o"
55
56
57
58 context (*Chisholm Scenario*)
59 assumes
60 ax1: "[ O<go|T> ]" (*It ought to be that a certain man go to help his neighbours.*) and
61
62 ax2: "[ O<tell|go > ]"(*It ought to be that if he goes he tell them he is coming.*) and
63
64 ax3: "[ O<-tell|-go> ]" (*If he does not go, he ought not to tell them he is coming.*) and
65
66 ax4 : "[~go]i" (*He does not go.*)
67
68
69 begin
70
71 lemma True nitpick [satisfy, user_axioms, show_all, expect=genuine] oops
72
73 end
74

```

Nitpick found a model for card i = 1:

```

Constants:
aw = i1
go = (λx. _) (i1 := False)
op r = (λx. _) (i1 := (λx. _) (i1 := False))
tell = (λx. _) (i1 := False)

```

**Fig. 2.** The Chisholm paradox scenario encoded in  $\mathbf{E}$  (the shallow semantical embedding of  $\mathbf{E}$  in Isabelle/HOL as displayed in Fig. 1 is imported here). Nitpick confirms consistency the encoded statements.