# ELLIPTIC PROBLEMS <br> WITH NONLOCAL BOUNDARY CONDITIONS <br> AND FELLER SEMIGROUPS 

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#### Abstract

This monograph is devoted to the following interrelated problems: the solvability and smoothness of elliptic linear equations with nonlocal boundary conditions and the existence of Feller semigroups that appear in the theory of multidimensional diffusion processes.


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## INTRODUCTION

I. This monograph is based on the author's doctoral dissertation and is devoted to the following interrelated problems: the solvability and smoothness of elliptic linear equations with nonlocal boundary conditions and the existence of Feller semigroups that appear in the theory of multidimensional diffusion processes.

One-dimension nonlocal problems were studied by Sommerfeld [99], Il'in [45], Il'in and Moiseev [46], Krall [55], Picone [68], Skubachevskii [97], Tamarkin [104], Shkalikov [80], and others.

In 1932, Carleman [12] studied a problem of finding a holomorphic function in a bounded domain $G$ that satisfies the following condition: the value of an unknown function at a point $y$ of the boundary $\partial G$ is related to the value at each point $\Omega(y)$, where $\Omega: \partial G \rightarrow \partial G$ is a smooth nondegenerate transformation, $\Omega(\Omega(y))=y, y \in \partial G$. In [12], this problem was reduced to a singular equation with a shift. Further studies of singular integral equations with shift that maps a boundary of the domain to itself and generates a finite group (see detailed bibliography in [62]) and studies of elliptic equations with a shift of the domain to itself (see [4]) are related to such a statement of the problem. Beals [5], Browder [11], Vishik [106], and Schechter [77] studied elliptic equations with nonlocal boundary conditions. Conditions that guarantee the fulfillment of the coercivity inequality were imposed on abstract operators. In some cases, restrictions were imposed on an adjoint operator.

In 1969, Bitsadze and Samarskii (see [9]) considered a completely different nonlocal elliptic problem appearing in the theory of plasma: find a harmonic function in a bounded domain $G$ that satisfies nonlocal conditions, which are related to a value of the function on a manifold $\Gamma_{1} \subset \partial G$ with values on some manifold inside the domain $G$; the Dirichlet condition was imposed on the set $\partial G \backslash \Gamma_{1}$. If the domain is rectangular, this problem was solved in [9] by reduction to a Fredholm equation of the second type and applying the maximum principle. In the case of an auxiliary domain and general nonlocal conditions, this problem was formulated as unsolved [75] (we specify [55], where the importance of development of the theory of nonlocal boundary-value problems was mentioned).

Bitsadze [7, 8], Gushchin [42], Gushchin and Mikhailov [43, 44], Eidelman and Zhiratau [14], Il'in and Moiseev [47], Kishkis [50, 51], Paneah [67], Roitberg and Sheftel' [73, 74], Soldatov [78], and others studied different versions and generalizations of nonlocal problems containing transformations of variables that mapped a boundary to closure of the domain. Special attention was devoted to the solvability nonlocal problems. Moiseev [60,61] and Mustafin [63] considered spectral properties of nonlocal problems in the multidimensional case. Note that, in the cited publications, the twodimensional case, or second-order equations, or strict conditions are imposed on the geometry of the support of nonlocal terms (e.g., it is assumed that the support of nonlocal terms lies inside the domain or intersects with a part of the boundary where the "local" Dirichlet condition is given).

The foundations of the theory of linear elliptic $2 m$-order equations with general nonlocal boundary conditions were stated by Skubachevskii and his colleagues (see [52, 69, 70, 82, 84, 85, 87-90, 93-98]). A classification according to a type of nonlocal conditions was conducted, a priori estimations were proved, left and right regularizers were constructed in Sobolev and weight spaces (depending on the type of nonlocal conditions), and asymptotic expansions of solutions near singular points were obtained. Spectral properties and properties of indexes of the corresponding operators were studied for some problems. In particular, it was shown that properties of a problem significantly depend on
the geometry of the support of nonlocal terms. Let us illustrate some possible cases by the following example.

Let $G \subset \mathbb{R}^{n}(n \geq 2)$ be a bounded domain with the boundary $\partial G=\Gamma_{1} \cup \Gamma_{2} \cup \mathcal{K}$, where $\Gamma_{\sigma}$ are open, connected (in the topology of $\partial G$ ), $(n-1)$-dimensional manifolds of class $C^{\infty}$, and $\mathcal{K}=\overline{\Gamma_{1}} \cap \overline{\Gamma_{2}}$ is an $(n-2)$-dimensional, connected manifold of class $C^{\infty}$ (if $n=2$, then $\mathcal{K}=\left\{g_{1}, g_{2}\right\}$, where $g_{1}$ and $g_{2}$ are endpoints of the curves $\overline{\Gamma_{1}}$ and $\overline{\Gamma_{2}}$ ) without a boundary. Let the domain $G$ be diffeomorphic to an $n$-dimensional dihedral angle in a neighborhood of each point $g \in \mathcal{K}$. Let us consider the following nonlocal problem in the domain $G$ :

$$
\begin{array}{ll}
\Delta u=f_{0}(y), & y \in G, \\
\left.u\right|_{\Gamma_{\sigma}}-\left.b_{\sigma}(y) u\left(\Omega_{\sigma}(y)\right)\right|_{\Gamma_{\sigma}}=0, & \sigma=1,2 . \tag{2}
\end{array}
$$

Here $b_{\sigma} \in C^{\infty}\left(\mathbb{R}^{2}\right) ; \Omega_{\sigma}$ is an infinitely differentiable, nondegenerate transformation that maps some neighborhood $\mathcal{O}_{\sigma}$ of the manifold $\Gamma_{\sigma}$ to the set $\Omega_{\sigma}\left(\mathcal{O}_{\sigma}\right)$ in such a way that $\Omega_{\sigma}\left(\Gamma_{\sigma}\right) \subset G$. Points of set $\mathcal{K}$ are called conjugation points of nonlocal boundary conditions.
A. L. Skubachevskii proposed the following classification:
(1) $\Gamma_{2}=\varnothing$ and $\Omega_{1}\left(\Gamma_{1}\right)=\Omega_{1}(\partial G) \subset G$ (Fig. 1);
(2) $\Gamma_{2} \neq \varnothing$ and $\Omega_{\sigma}\left(\overline{\Gamma_{\sigma}}\right) \cap \mathcal{K}=\varnothing, \sigma=1,2$ (Fig. 2);
(3) $\Gamma_{2} \neq \varnothing$ and $\Omega_{\sigma}\left(\overline{\Gamma_{\sigma}}\right) \cap \mathcal{K} \neq \varnothing, \sigma=1$ or 2 (Fig. 3).


Fig. 1. Domain $G$ with boundary $\partial G=\Gamma_{1}$.


Fig. 2. $\Omega_{\sigma}\left(\overline{\Gamma_{\sigma}}\right) \cap \mathcal{K}=\varnothing$.


Fig. 3. $\Omega_{\sigma}\left(\overline{\Gamma_{\sigma}}\right) \cap \mathcal{K} \neq \varnothing$.

The first type of problems (and generalizations to the case of abstract nonlocal operators in boundary conditions) is the best studied because the properties of nonlocal problems are similar to the corresponding "local" problem (when $b_{\sigma}(y) \equiv 0$ ). In particular, a nonlocal problem is a Fredholm problem in Sobolev spaces and its index equals the index of a "localized" problem; the corresponding
problem with a spectral parameter is uniquely solvable for sufficiently large parameters (see [82, 84, $97]$ ). If the spectrum of a local problem is discrete, then the nonlocal problem has a discrete spectrum and the system of root functions forms an Abel basis in the corresponding Sobolev space (see [69, 70]).

The situation is substantially more difficult for the second and third types. For the second class, a curve $\Omega_{\sigma}\left(\Gamma_{\sigma}\right)$ can cross (it can also be tangent) the boundary of the domain. In the general case, it even can partially coincide with the boundary. For problems of the third type, we assume that nonlocal terms are not tangent to the boundary of the domain at conjugation points; this is important for the method used. In [85, 98], it is shown that if the support of nonlocal terms crosses the boundary of the domain, then the areas of solutions can have power singularities near a conjugation point of the boundary conditions. This can happen even if the boundary is infinitely smooth and the right-hand side is infinitely differentiable. Therefore, such problems were considered in special weight spaces (they consider singularities of solutions). It turned out that Kondrat'ev spaces are the most convenient. Kondrat'ev introduced such spaces in for the study of "local" boundary problems in domains with corner or conical points. In [52, 85, 88, 89], the Fredholm solvability of nonlocal problems in Kondrat'ev spaces was proved, and in [90], it was shown that if the support of nonlocal terms does not cross conjugation points of boundary conditions (Fig. 2), then the index of nonlocal problem is equal to the index of the corresponding local problem; otherwise (Fig. 3), this is not true.

Note that nonlocal elliptic problems with tangent approach of a curve $\Omega_{\sigma}\left(\Gamma_{\sigma}\right)$ to the boundary of a domain at conjugation points were studied in $[8,51]$ by methods of the theory of complex functions. Nevertheless, the general theory of nonlocal boundary-value problems is not developed enough.

Independently of the papers mentioned above, nonlocal problems arose in the theory of multidimensional diffusion processes that describe the behavior of a particle in a domain $G$ from the point of view of the probability theory. In [15, 16], it was shown that any one-dimensional $(n=1)$ diffusion process corresponds to some nonnegative, continuous contracting semigroup in the space $C(\bar{G})$ or in some of its subspaces. Further, such semigroups are called Feller semigroups. Moreover, Feller obtained necessary and sufficient conditions for a second-order ordinary differential operator to be a generator (infinitesimal generating operator) of this semigroup. He obtained nonlocal boundary conditions that give the domain of the operator.

In the multidimensional case ( $n \geq 2$ ), Ventsel [105] obtained a general form of a generator of a Feller semigroup. He proved that this generator is an elliptic, second-order differential operator (possibly, with generation), and its domain consists of continuous (once or twice continuously differentiable depending on the process) functions satisfying nonlocal boundary conditions. A nonlocal term is an integral of a function over a closure of a domain with respect to a nonnegative Borel measure $\mu(y, d \eta)$, $y \in \partial G$.

In the most difficult case, when the measure is atomic, the nonlocal case can have form (2). They have the following probabilistic sense: a particle hitting a point $y \in \Gamma_{\sigma}$ can either hit a point $\Omega_{\sigma}(y)$ with probability $b_{\sigma}\left(0 \leq b_{\sigma} \leq 1\right)$ (such behavior of a particle is called "jump"), or be assimilated by a boundary with probability $1-b_{\sigma}$. In the latter case, the process terminates.

In the general case, boundary conditions include the derivatives of an unknown function up to the second order. This corresponds to the assimilation and reflection of a particle from the boundary, diffusion along the boundary, and viscosity.

The following problem is unsolved for $n \geq 2$. Let an elliptic integro-differential operator be given; its domain is described by general nonlocal boundary conditions (see [105]). Is such an operator (or its closure) a generator of a Feller semigroup?

There are two types of nonlocal boundary conditions: transversal and nontransversal. In the transversal case, the order of nonlocal terms is less than the order of local terms. In the nontransversal case, these orders coincide. Sato and Ueno [76], Bony, Courrege, and Priouret [10], Watanabe [109], Taira [101-103], Ishikawa [48], and others studied the transversal case. Skubachevskii [86, 91, 92] proposed a method of studying the nontransversal case. This method is based on the idea of separation
of nonlocal terms from local terms and the Hille-Yosida theorem. This idea was used earlier (see [82, 85]). Further, this method was developed in [17-20].

Let us specify important applications arising in the theory of functional-differential equations (see [97] and the references therein), in the theory of parabolic problems with nonlocal boundary conditions (see [79]), in modelling of multilayered plates and shells in aerospace engineering [65, 66, 97], in thermic control problems in the description of processes in chemical reactors and climate-control systems (see [40]), in the theory of parabolic problems with nonlocal boundary conditions (see [13, 40]), and in control theory (see, e.g., [3]). In addition, let us mention a monograph of Bensoussan and Lions (see [6]), where elliptic integro-differential operators were studied in connection with stochastic control theory.

Lately, the theory of nonlocal nonlinear equations and inequalities and applications are being developed. Let us mention the paper [59] where differential inequalities with nonlocal terms are studied (see also bibliography there).
II. Until now (see [24, 85, 88, 89]), in the general theory of elliptic problems with nonlocal boundary conditions, it was assumed that transformations $\Omega_{\sigma}$ near conjugation points of boundary conditions are linear, that is, they are compositions of operators of shift, rotation, and homothety. In Chap. 1, we consider a problem with nonlinear transformations. It turns out (see [25]) that such a problem is not a small or compact perturbation of the corresponding problem with linear transformations. Nevertheless, it is shown that the operator of the problem remains a Fredholm operator in Kondrat'yev weight spaces and its index does not change after passage to nonlinear transformations. For simplicity, we assume that $n=2$; however, the corresponding results are valid in the case where $n \geq 3$ (see [25]).

The problem of whether a unbounded nonlocal operator in $L_{2}(G)$ is a Fredholm operator when the support of nonlocal terms approaches the boundary of a domain was studied when nonlocal conditions were given at shifts of the boundary (see [83, 87]) or when the Dirichlet problem for a second-order equation was nonlocally perturbed (see [42-44]). The solvability of nonlocal elliptic problems in the Sobolev spaces $W^{l+2 m}(G)=W_{2}^{l+2 m}(G)$ (where $2 m$ is the order of the elliptic equation, $l \geq 0$ ) has not yet been studied. The main difficulty is that solutions of a nonlocal problem can have singularities near some points and, in general, do not belong to a "necessary" Sobolev space. Such problems are studied in Chaps. 2-4. We show that the Fredholm solvability of a bounded operator in the Sobolev space $W^{l+2 m}(G)$ is defined by the eigenvalues of some auxiliary function $\tilde{\mathcal{L}}(\lambda)$ (they depend on a complex parameter $\lambda$ ), the structure of Jordan chains corresponding to these eigenvalues, and by some algebraic relations between the elliptic operator and operators in the nonlocal boundary conditions. We study nonlocal boundary conditions with arbitrary right-hand sides. An unbounded operator in $L_{2}(G)$ given at generalized solutions of a nonlocal problem (i.e., functions from $W^{\ell}(G), 0 \leq \ell \leq 2 m-1$ ), is a Fredholm operator for any position of eigenvalues of the operator-valued function $\tilde{\mathcal{L}}(\lambda)$. The stability of the index of nonlocal operators in $L_{2}(G)$ when an elliptic equation and boundary conditions are perturbed by minor terms has been studied via the notion of the spread between unbounded operators (see [49]).

In [53], a $2 m$-order elliptic equation with the Dirichlet boundary condition was solved and a question on the smoothness near an corner or conical point of generalized solutions from the Sobolev space $W^{m}(G)$ was considered. In particular, it is proved that solutions can be made arbitrary smooth due to decrease of the angle. In the case of nonlocal boundary conditions, the situation is substantially different. In [85, 98], it is shown that the smoothness of generalized solutions can be violated near a smooth boundary or a vertex of a small angle. On the other hand, generalized solutions near a vertex of an angle that is more than $\pi$ can be smooth if there are nonlocal terms with sufficiently large coefficients. In Chap. 5, we study the smoothness of generalized solution from $W^{\ell}(G), 0 \leq \ell \leq 2 m-1$, of $2 m$-order elliptic equations; we consider nonlocal boundary conditions with both zero and arbitrary right-hand sides.

In [86, 92], the question of whether Feller semigroups in the nontransversal case exist is considered under conditions where the coefficients of nonlocal operators decrease as the argument tends to the boundary of the domain. In $[19,20]$, boundary conditions in the case where the coefficients of nonlocal terms near conjugation points are less than 1 are considered. It is proved that a nonlocal problem (after reducing to the boundary) can be considered as a perturbation of a "local" Dirichlet problem. The extreme case where the coefficients of nonlocal terms equal 1 remains unstudied until now (the coefficients cannot be more than 1 ; see [105]). In Chap. 6 , we study nontransversal nonlocal conditions that allow the extreme case. We obtain sufficient conditions for Borel measure $\mu(y, d \eta)$ (its support is inside the closure of the domain) that guarantee that the corresponding operator is a generator of a Feller semigroup. Both bounded and unbounded perturbations of an elliptic operator were studied.
III. Let us describe the structure of the monograph and state the main results corresponding to the chapters.

This monograph consists of the Introduction, six chapters, and the bibliography list.
Introduce the notation

$$
K=\left\{y \in \mathbb{R}^{2}: r>0, \omega_{1}<\omega<\omega_{2}\right\}, \quad \gamma_{\sigma}=\left\{y \in \mathbb{R}^{2}: r>0, \omega=\omega_{\sigma}\right\}, \quad \sigma=1,2
$$

where $\omega$ and $r$ are the polar coordinates of a point $y, \omega_{1}<0<\omega_{2}$, and $\omega_{2}-\omega_{1}<2 \pi$.
Denote by $\mathcal{O}_{\varepsilon}(0)$ the $\varepsilon$-neighborhood of the origin. Let us introduce the following sets:

$$
K^{\varepsilon}=K \cap \mathcal{O}_{\varepsilon}(0), \quad \gamma_{\sigma}^{\varepsilon}=\gamma_{\sigma} \cap \mathcal{O}_{\varepsilon}(0), \quad \sigma=1,2
$$

Let $G \subset \mathbb{R}^{2}$ be a bounded domain with the boundary $\partial G$. The boundary contains the origin. We assume that $\overline{G \cap \mathcal{O}_{\varepsilon}(0)}=\overline{K^{\varepsilon}}$ for some $\varepsilon>0$, and the boundary of the domain $G$ is infinitely smooth in a neighborhood of every point $y \in \partial G \backslash\{0\}$.

In Chaps. $1-5$, we consider a nonlocal boundary-value problem

$$
\begin{align*}
\mathbf{P}(y, D) u & =f_{0}(y), \quad y \in G  \tag{3}\\
\mathbf{B}_{\mu} u \equiv \mathbf{B}_{\mu}^{0} u+\mathbf{B}_{\mu}^{1} u+\mathbf{B}_{\mu}^{2} u & =f_{\mu}(y), \quad y \in \partial G \backslash\{0\} ; \mu=1, \ldots, m \tag{4}
\end{align*}
$$

where $\mathbf{P}(y, D)$ is a $2 m$-order, properly elliptic operator in $\bar{G}$ with smooth complex-valued coefficients;

$$
\mathbf{B}_{\mu}^{0} u=\left.B_{\mu 0}(y, D) u(y)\right|_{\partial G \backslash\{0\}}, \quad \mathbf{B}_{\mu}^{1} u= \begin{cases}\left.\left(B_{\mu 1}(y, D) u\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}^{\varepsilon}}, & y \in \gamma_{\sigma}^{\varepsilon}, \sigma=1,2 \\ 0, & y \in \partial G \backslash \mathcal{O}_{\varepsilon}(0)\end{cases}
$$

$B_{\mu 0}(y, D)$ and $B_{\mu 1}(y, D)$ are $m_{\mu}$-order differential operators with smooth (on $\partial G \backslash\{0\}$ ) complex-valued coefficients (as $y \in \partial G$ tends to the origin, the coefficients and all their derivatives have, generally speaking, one-sided limits); the support of the coefficients of the operators $B_{\mu 1}(y, D)$ lies inside a sufficiently small neighborhood of the origin; $\Omega_{\sigma}$ are diffeomorphisms given in some neighborhood $\gamma_{\sigma}^{\varepsilon}$ such that $\Omega_{\sigma}\left(\gamma_{\sigma}^{\varepsilon}\right) \subset G, \Omega_{\sigma}(0)=0$ and the curves $\Omega_{\sigma}\left(\gamma_{\sigma}^{\varepsilon}\right)$ have a nontangent approach to the boundary $\partial G$ at the origin; $\mathbf{B}_{\mu}^{2} u=\tilde{\mathbf{B}}_{\mu}^{2}\left(\left.u\right|_{G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(0)}}\right)\left(\varkappa_{1}>0\right)$ are abstract nonlocal operators corresponding to the nonlocal terms with supports lying outside a $\varkappa_{1}$-neighborhood of the origin; if $y \in \partial G$, the system of operators $\left\{B_{\mu 0}(y, D)\right\}_{\mu=1}^{m}$ satisfies the cover condition (the Lopatinskii condition) relative to the operator $\mathbf{P}(y, D)$ (at the origin, the values of coefficients of operators $B_{\mu 0}(y, D)$ are understood in the sense of one-sided limits).

A vector

$$
\begin{equation*}
\mathbf{B}_{1}^{2} u(y)=\left.b(y) u(\Omega(y))\right|_{\partial G \backslash\{0\}} \tag{5}
\end{equation*}
$$

can be an example of the operator $\mathbf{B}_{\mu}^{2}$. Here $b \in C^{\infty}\left(\mathbb{R}^{2}\right)$ and the restrictions of the transformation $\Omega$ to $\partial G \backslash\{0\}$ and $\overline{\gamma_{\sigma}^{\varepsilon}}(\sigma=1,2)$ are smooth nongenerate transformations; moreover,

$$
\Omega(\partial G \backslash\{0\}) \subset G, \quad \overline{\Omega(\partial G \backslash\{0\})} \subset \bar{G} \backslash \mathcal{O}_{\varkappa_{1}}(0)
$$

For any $l \geq 0$ and any $a \in \mathbb{R}$, we denote by $H_{a}^{l}(G)$ the completion of the set of infinitely differentiable in $\bar{G}$ functions with supports in $\bar{G} \backslash\{0\}$ with respect to the norm

$$
\|u\|_{H_{a}^{l}(G)}=\left(\sum_{|\alpha| \leq l} \int_{G} \rho^{2(a-l+|\alpha|)}\left|D^{\alpha} u\right|^{2} d y\right)^{1 / 2},
$$

where $\rho=\rho(y)=\operatorname{dist}(y,\{0\})$. For $l \geq 1$, we denote by $H_{a}^{l-1 / 2}(\partial G \backslash\{0\})$ the trace space on $\partial G \backslash\{0\}$ with the norm

$$
\|\psi\|_{H_{a}^{l-1 / 2}(\partial G \backslash\{0\})}=\inf \|u\|_{H_{a}^{l}(G)}, \quad u \in H_{a}^{l}(G):\left.u\right|_{\partial G \backslash\{0\}}=\psi .
$$

Assume that

$$
\mathcal{H}_{a}^{l}(G, \partial G)=H_{a}^{l}(G) \times \prod_{\mu=1}^{m} H_{a}^{l+2 m-m_{\mu}-1 / 2}(\partial G \backslash\{0\}),
$$

where $a \in \mathbb{R}$ and $l \geq 0$ is an integer such that $l+2 m-m_{\mu} \geq 1$.
In Chap. 1, we study the problem of whether the operator

$$
\mathbf{L}=\left\{\mathbf{P}(y, D), \mathbf{B}_{\mu}\right\}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G)
$$

is a Fredholm operator and its index is stable when we change the transformation $\Omega_{\sigma}$ to $\hat{\Omega}_{\sigma}$. Both transformations have the same linear part near the origin.

In Sec. 1, auxiliary results from the theory of linear operators are proved and definitions of functional spaces are given.

In Sec. 2, the statement of nonlocal problem (3), (4) is studied in weight spaces in a bounded domain and the statement of a model problem in a plane angle is considered. We introduce a model operator

$$
\tilde{\mathcal{L}}(\lambda): W^{l+2 m}\left(\omega_{1}, \omega_{2}\right) \rightarrow W^{l}\left(\omega_{1}, \omega_{2}\right) \times \mathbb{C}^{2 m}, \quad \lambda \in \mathbb{C},
$$

which corresponds to a nonlocal problem near the origin. This problem should be written in the polar coordinates $\omega$ and $r$ (with the subsequent Mellin transformation $r \mapsto \lambda$ ). Spectral properties of the operator $\tilde{\mathcal{L}}(\lambda)$ play a key role in the study of the solvability and smoothness of solutions of problem (3), (4).

In Sec. 3, we study properties of nonlocal operators with nonlinear transformations near the origin.
In Sec. 4, we prove an a priori estimate of solutions of problem (3), (4) and construct the right regularizer in weight spaces under the condition that the straight $\operatorname{line} \operatorname{Im} \lambda=a+1-l-2 m$ does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Hence,

$$
\mathbf{L}=\left\{\mathbf{P}(y, D), \mathbf{B}_{\mu}\right\}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G)
$$

is a Fredholm operator. In this section, we prove that the index of the operator $\mathbf{L}$ is determined by the linear part of the transform $\Omega_{\sigma}$.

Chapters 2-6 are devoted to properties of solutions of problem (3), (4). In Chaps. 2 and 3 we study strong solutions from Sobolev spaces, in Chaps. 4 and 5 we study generalized solutions from Sobolev spaces, and in Chap. 6 we consider solutions in spaces of continuous functions. The corresponding results are based on the solvability of the same problem in weight spaces. Taking into account results of Chap. 1, we consider transformations $\Omega_{\sigma}$ that are linear near the origin. We prove theorems of Chaps. 2-6 for the general case where there is a finite number of corner points on the boundary $\partial G$. They divide $\partial G$ in a finite number of parts; each part corresponds to its boundary condition that contains, generally speaking, several nonlocal terms. In the Introduction, we restrict ourself to problem (3), (4) in the domain $G$.

In Chap. 2, we study model nonlocal problems in plane angles.
In Sec. 5, we introduce functional spaces.

In Sec. 6, we consider the statement of the nonlocal problem (3), (4) in a bounded domain and model problems in plane angles in Sobolev spaces. The operators of model problems in plane angles are defined by the formulas

$$
\mathfrak{L} u=\left\{\mathbf{P}(y, D) u, \quad \mathbf{B}_{\sigma \mu}(y, D) u\right\}, \quad \mathcal{L} v=\left\{\mathcal{P}(D) v, \mathcal{B}_{\sigma \mu}(D) v\right\},
$$

where

$$
\begin{aligned}
\mathbf{B}_{\sigma \mu}(y, D) u & =\left.B_{\mu 0}(y, D) u\right|_{\gamma \delta}+\left.\left(B_{\mu 1}(y, D) u\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma \delta} ^{\varepsilon}, \\
\mathcal{B}_{\sigma \mu}(D) v & =\left.B_{\mu 0}(D) v\right|_{\gamma_{\sigma}^{\varepsilon}}+\left.\left(B_{\mu 1}(D) v\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}^{\varepsilon}},
\end{aligned}
$$

$B_{\mu 0}(D)$ and $B_{\mu 1}(D)$ are the principal homogeneous parts of operators $B_{\mu 0}(0, D)$ and $B_{\mu 1}(0, D)$.
In Chap. 2, we study properties of the operators $\mathfrak{L}$ and $\mathcal{L}$ in Sobolev spaces. These properties play a key role in the study of the solvability and smoothness of generalized solutions of nonlocal problems in bounded domains.

In Sec. 7, we construct "solutions" of model problems in Sobolev spaces with accuracy to functions that have a zero of certain order at the origin. We consider the following two situations: when the straight line $\operatorname{Im} \lambda=a+1-l-2 m$ does not contain eigenvalues of operator $\tilde{\mathcal{L}}(\lambda)$ and when it contains only a regular eigenvalue (i.e., such an eigenvalue $\lambda_{0}$ for which there is no adjoint vectors and for any eigenvector $\varphi(\omega)$ the function $r^{i \lambda_{0}} \varphi(\omega)$ being written in rectangular coordinates $y$ is a polynomial). In the second case, concordance conditions (integral conditions) for coordinates are imposed on the right-hand sides of both equation and nonlocal boundary conditions. These conditions arise since the operators of the problem are related by certain algebraic relations. (More precisely, algebraic relations arise between operators $D_{y}^{|\alpha|} \mathcal{P}(D),|\alpha|=l-1$, and $D_{\tau_{\sigma}}^{l+2 m-m_{\mu}-1} \mathcal{B}_{\sigma \mu}(D)$, where $\tau_{\sigma}$ is a unit vector parallel to $\gamma_{\sigma}^{\varepsilon}$.)

Chapter 3 is devoted to the solvability of problem (3), (4) in a plane bounded domain in Sobolev spaces.

We say that a function $\psi$ belongs to the space $W^{l-1 / 2}(\partial G \backslash\{0\}), l \geq 1$, if $\psi \in W^{l-1 / 2}\left(\partial G \backslash \overline{\mathcal{O}_{\delta}(0)}\right)$ for all $\delta>0$ and its restriction to $\gamma_{\sigma}^{\varepsilon}$ belongs to $W^{l-1 / 2}\left(\gamma_{\sigma}^{\varepsilon}\right)$. We assume that the operators $\mathbf{B}_{\mu}^{2}$ acts boundedly from the space $W^{l+2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}(0)}}\right)$ to the space $W^{l+2 m-m_{\mu}-1 / 2}(\partial G \backslash\{0\})$ (cf. (5)). Recall that if $y \in \partial G$ tends to the origin, then the coefficients of the operators $B_{\mu 0}(y, D)$ and $B_{\mu 1}(y, D)$ and all their derivatives have, generally speaking, only one-sided limits; therefore, we must consider trace spaces given on the sets $\partial G \backslash\{0\}$.

Introduce the notation

$$
\mathcal{W}^{l}(G, \partial G)=W^{l}(G) \times \prod_{\mu=1}^{m} W^{l+2 m-m_{\mu}-1 / 2}(\partial G \backslash\{0\}),
$$

where $l \geq 0$ is an integer such that $l+2 m-m_{\mu} \geq 1$.
In Sec. 8, we prove that the bounded operator

$$
\mathbf{L}=\left\{\mathbf{P}(y, D), \mathbf{B}_{\mu}\right\}: W^{l+2 m}(G) \rightarrow \mathcal{W}^{l}(G, \partial G)
$$

is a Fredholm operator if and only if the straight line $\operatorname{Im} \lambda=1-l-2 m$ does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$.

In Sec. 9, we consider the operator

$$
\mathbf{L}_{a}=\left\{\mathbf{P}(y, D), \mathbf{B}_{\mu}\right\}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G) \dot{+} \mathcal{R}_{a}^{l}(G, \partial G),
$$

which acts in weight spaces with a small weight index $a>0$. Here $\mathcal{R}_{a}^{l}(G, \partial G)$ is some finite-dimensional space of functions that have a singularity at the origin. It arises owing to the fact that if $a \leq l+2 m-1$, then the inclusion $u \in H_{a}^{l+2 m}(G)$, generally speaking, does not imply the inclusion $\mathbf{L}_{a} u \in \mathcal{H}_{a}^{l}(G, \partial G)$. Indeed, one can find a function $u \in H_{a}^{l+2 m}(G)$ for which the concordance conditions are violated, i.e., $\mathbf{B}_{\mu}^{2} u$ belongs to $W^{l+2 m-m_{\mu}-1 / 2}(\partial G \backslash\{0\})$, but it does not belong to $H_{a}^{l+2 m-m_{\mu}-1 / 2}(\partial G)$.

We prove that if the straight line $\operatorname{Im} \lambda=a+1-l-2 m$ does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$, then $\mathbf{L}_{a}$ is a Fredholm operator. In particular, it follows that if the right-hand side of problem (3), (4) belongs to $\mathcal{H}_{a}^{l}(G, \partial G)$ and satisfies a finite number of orthogonality conditions, then there exists a solution $u \in H_{a}^{l+2 m}(G)$.

In Sec. 10, we consider the case where the straight line $\operatorname{Im} \lambda=1-l-2 m$ contains only regular eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$. In this case, by virtue of the results of Sec. 8, L: $W^{l+2 m}(G) \rightarrow$ $\mathcal{W}^{l}(G, \partial G)$ is not a Fredholm operator (its image is not closed). Hence the operator

$$
\hat{\mathbf{L}}=\left\{\mathbf{P}(y, D), \mathbf{B}_{\mu}\right\}: W^{l+2 m}(G) \rightarrow \hat{\mathcal{S}}^{l}(G, \partial G) \dot{+} \mathcal{R}^{l}(G, \partial G)
$$

is set to correspondence with problem (3), (4). Here $\hat{\mathcal{S}}^{l}(G, \partial G)$ is the set of right-hand sides of problem (3), (4) from the Sobolev space $\mathcal{W}^{l}(G, \partial G)$, which satisfy integral concordance conditions near the origin (cf. Sec. 7), and $\mathcal{R}^{l}(G, \partial G)$ is a finite-dimensional subset in $\mathcal{W}^{l}(G, \partial G)$. We prove that $\hat{\mathbf{L}}$ is a Fredholm operator.

In Secs. 8-10, we prove the Fredholm property of nonlocal operators by the common scheme. The fact that the kernel of problem (3), (4) has a finite dimension in "proper" weight spaces yields the fact that the kernel has a finite dimension. To prove the fact that the image has a finite dimension, we construct the right regularizer using results of Chap. 2.

In Sec. 11, we use results of Sec. 10 and show that if there is only regular eigenvalue on the straight line $\operatorname{Im} \lambda=1-l-2 m$, then the problem with homogeneous boundary conditions (unlike the problem with inhomogeneous boundary conditions) can be a Fredholm problem. Concordance conditions can be fulfilled for any vector of right-hand sides that has a zero component corresponding to right-hand sides of conditions. Therefore, properties of the problem can be improved.

In Sec. 12, we give examples that illustrate general theorems of Chap. 3. Here, we can observe the following effects.
(1) Even in the case of infinitely smooth boundary, a nonlocal problem cannot be a Fredholm problem in Sobolev spaces for arbitrary small coefficients of nonlocal terms. On the other hand, such a problem can became a Fredholm problem.
(2) The Fredholm solvability of a nonlocal problem in the Sobolev spaces $W^{l+2 m}(G)$ depends on the index $l$. For example, a problem can be a Fredholm problem for even $l$ and have a nonclosed image for odd $l$.
Such effects are due to the following. If the coefficients of nonlocal terms and the index $l$ are changed in the Sobolev space $W^{l+2 m}(G)$, then the mutual position of eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ and the straight line $\operatorname{Im} \lambda=1-l-2 m$, the structure of Jordan chains, and the structure of algebraic relationships between the operators $D_{y}^{|\alpha|} \mathcal{P}(D),|\alpha|=l-1$, and $D_{\tau_{\sigma}}^{l+2 m-m_{\mu}-1} \mathcal{B}_{\sigma \mu}(D)$ change.

In Chap. 4, we study generalized solutions of problem (3), (4).
Let us fix an integer $\ell$ such that $0 \leq \ell \leq 2 m-1$. In Sec. 13, we denote a generalized solution of problem (3), (4) as a function that belongs to $W^{\ell}(G) \cap W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\delta}(0)}\right)$ for all $\delta>0$ and satisfies Eq. (3) almost everywhere and the boundary conditions (4) in the sense of traces. Thus, a generalized solution can have a singularity near the origin that is a point of conjugation of nonlocal conditions.

In Sec. 14, we prove the Fredholm solvability of the unbounded operator

$$
\mathbf{P}: \mathrm{D}(\mathbf{P}) \subset L_{2}(G) \rightarrow L_{2}(G),
$$

which acts by the formula

$$
\begin{gathered}
\mathbf{P} u=\mathbf{P}(y, D) u, \\
u \in \mathrm{D}(\mathbf{P})=\left\{u \in W^{\ell}(G) \cap W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\delta}(0)}\right) \forall \delta>0: \mathbf{B}_{\mu} u=0, \mathbf{P}(y, D) u \in L_{2}(G)\right\} .
\end{gathered}
$$

In Sec. 15, we state that the index of the operator $\mathbf{P}: \mathrm{D}(\mathbf{P}) \subset L_{2}(G) \rightarrow L_{2}(G)$ does not change if we add minor terms to the elliptic equation, and in Sec. 16 we prove the stability of the index when we add to boundary conditions nonlocal terms with coefficients that have a zero of a certain order at the
origin. Both cases have the same difficulty. When the operator $\mathbf{P}$ is perturbed, its domain changes. To prove the stability of the index, we use the notion of the spread between unbounded operators (see [49]) and reduction to operators that act in weight spaces.

In Sec. 17, we show that if we add to boundary conditions nonlocal terms with arbitrary small coefficients that do not equal zero at the origin, then the index of the operator $\mathbf{P}$ can change. We give examples of the operator $\mathbf{P}$ whose spectrum occupies the whole complex plane.

Chapter 5 is devoted to the smoothness of generalized solutions $u \in W^{\ell}(G)$ of problem (3), (4) under the condition that $f_{0} \in L_{2}(G)$ and $f_{\mu} \in W^{2 m-m_{\mu}-1 / 2}(\partial G \backslash\{0\})$.

Let us denote the set of all eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ that lie in the strip $1-2 m<\operatorname{Im} \lambda<1-\ell$ by $\Lambda$ (in particular, the set $\Lambda$ can be empty).

In Sec. 18, we assume that the following condition holds.
Condition 1. The straight line $\operatorname{Im} \lambda=1-2 m$ does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ and all eigenvalues from the set $\Lambda$ are regular.

We obtain sufficient conditions for any generalized condition of problems (3), (4) to belong to $W^{2 m}(G)$. These conditions are formulated in terms of eigenvectors that correspond to eigenvalues from the set $\Lambda$. These conditions are called "conditions on the structure of eigenvectors."

In Sec. 19, we study the so-called borderline case. Namely, we assume that the following condition holds.

Condition 2. The straight line $\operatorname{Im} \lambda=1-2 m$ contains exactly one eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$; this eigenvalue is regular. All eigenvalues from the set $\Lambda$ are also regular.

In addition to the "conditions on the structure of eigenvectors," we obtain (integral) concordance conditions. If the right-hand sides of $f_{\mu}$ and the operators $\mathbf{B}_{\mu}^{0}, \mathbf{B}_{\mu}^{1}$, and $\mathbf{B}_{\mu}^{2}$ satisfy these conditions, then any generalized condition of problem (3), (4) belongs to $W^{2 m}(G)$. In Secs. 18 and 19, we consider boundary conditions with zero and arbitrary right-hand sides.

In Sec. 20, we study nonlocal boundary conditions of a special type. For such boundary conditions, "conditions on the structure of eigenvectors" do not affect the smoothness of generalized solutions of problem (3), (4).

In Sec. 21, we considered the cases where the smoothness can be violated. We show that both conditions 1 and 2 and the "conditions on the structure of eigenvectors" are substantial in the general case.

In Sec. 22, we give an example that illustrates the results of Secs. 18-21.
In Chap. 6, we study the problem on the existence of Feller semigroups arising in the theory of multidimensional diffusion processes, when we describe the motion of a particle in a domain in terms of the probability theory. Let us consider a second-order elliptic differential operator $\mathbf{P}(y, D)$ with smooth real coefficients such that $\mathbf{P}(y, D) 1 \leq 0$ and $y \in \bar{G}$. The domain of the operator $\mathbf{P}(y, D)$ is given by nonlocal boundary conditions of nontransversal type (cf. (4))

$$
\begin{align*}
u(y)-b_{\sigma}(y) u\left(\Omega_{\sigma}(y)\right)-\int_{\bar{G}} u(\eta) \beta(y, d \eta) & =0, \quad y \in \gamma_{\sigma}^{\varepsilon}, \quad \sigma=1,2, \\
u(y)-\int_{\bar{G}} u(\eta) \beta(y, d \eta) & =0, \quad y \in \partial G \backslash \mathcal{O}_{\varepsilon}(0),  \tag{6}\\
u(0) & =0 .
\end{align*}
$$

Here $b_{\sigma} \in C^{\infty}\left(\mathbb{R}^{2}\right)$ are real functions such that $\operatorname{supp} b_{\sigma} \subset \mathcal{O}_{\varepsilon}(0)$ and $0 \leq b_{\sigma}(y) \leq 1$ and $\beta(y, \cdot)$ is a nonnegative Borel measure.

Nonlocal conditions (6) can also be written in the following form:

$$
\begin{equation*}
u(y)-\int_{\bar{G}} u(\eta) \mu(y, d \eta)=0, \quad y \in \partial G \tag{7}
\end{equation*}
$$

Here $\mu(0, \cdot)=0$ and $\mu(y, \cdot)=\delta(y, \cdot)+\beta(y, \cdot)$ for $y \in \partial G \backslash\{0\} ;$ moreover, $\delta(y, \cdot)$ is a nonnegative atomic measure defined by the formula

$$
\delta(y, Q)= \begin{cases}b_{\sigma}(y) \chi_{Q}\left(\Omega_{\sigma}(y)\right), & y \in \gamma_{\sigma}^{\varepsilon}, \quad \sigma=1,2 \\ 0, & y \in \partial G \backslash \mathcal{O}_{\varepsilon}(0)\end{cases}
$$

Here $Q$ is an arbitrary Borel set and $\chi_{Q}$ is the characteristic function of the set $Q$. The nonlocal condition (7) is a particular case of a boundary condition obtained in [105].

In Sec. 23, we prove theorems on the unique solvability of the elliptic equation

$$
\mathbf{P}(y, D) u-q u=f_{0}(y), \quad y \in G, \quad q>0
$$

with nonlocal boundary conditions (6) in the case where $\beta(y, \cdot) \equiv 0$ in the space of continuous functions.
The study of the solvability of nonlocal problems in spaces of continuous functions is based on theorems on the unique solvability in weight spaces (see [41]) and results on the asymptotic behavior of solutions of nonlocal problems (see [26, 96], where results on the asymptotic behavior are generalized to the case of systems of differential equations with nonlocal boundary conditions; these systems are analytic in the Douglis-Nirenberg sense).

In Secs. 24 and 25, we use results of Sec. 23 and the Hille-Yosida theorem and obtained sufficient conditions for the measure $\mu(y, \cdot)$ in terms of the geometrical structure of the support of the measure. These conditions guarantee the existence of a Feller semigroup that corresponds to nonlocal boundary conditions (6). We assume that $\mu(y, \cdot)=\delta(y, \cdot)+\beta(y, \cdot)$ (see above); moreover, the measure $\beta(y, \cdot)$ can be represented as the sum of three nonnegative Borel measures: the first has a support separated from the origin, the second possesses some smallness property, and the third generates a compact operator in the corresponding spaces.

Note that in [19, 20], the conditions $0 \leq b_{\sigma}(0)<1$ or $0 \leq b_{\sigma}(0)<1 / 2, \sigma=1,2$, are assumed to be fulfilled (depending on the structure of the measure $\beta(y, \cdot)$ ). In this paper, we assume that $0 \leq b_{\sigma}(0) \leq 1, \sigma=1,2$, and

$$
\begin{equation*}
b_{1}(0)+b_{2}(0)<2 . \tag{8}
\end{equation*}
$$

In Sec. 24, we obtain conditions for the measure $\beta(y, \cdot)$ under which the unbounded operator

$$
\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})
$$

defined by the formula

$$
\begin{gather*}
\mathbf{P}_{B} u=\mathbf{P}(y, D) u+\mathbf{P}_{1} u \\
u \in \mathrm{D}\left(\mathbf{P}_{B}\right)=\left\{u \in C_{B}(\bar{G}): \mathbf{P}(y, D) u+\mathbf{P}_{1} u \in C_{B}(\bar{G})\right\}, \tag{9}
\end{gather*}
$$

is a generator of a Feller semigroup. Here $C_{B}(\bar{G})$ is the set of continuous in $\bar{G}$ functions that satisfy nonlocal conditions (6); $\mathbf{P}_{1}$ is a bounded in $C(\bar{G})$ operator such that if a function $u \in C(\bar{G})$ has a positive maximum at a point $y^{0} \in G$, then $\mathbf{P}_{1} u\left(y^{0}\right) \leq 0$. Then the operator $\mathbf{P}_{1}$ is a generalization of a bounded integral operator of the form

$$
\mathbf{P}_{1} u(y)=\int_{\bar{G}}[u(\eta)-u(y)] m(y, d \eta), \quad y \in \bar{G},
$$

where $m(y, \cdot)$ is a nonnegative Borel measure on $\bar{G}$.

In Sec. 25 , we study the case of an unbounded in $C(\bar{G})$ operator $\mathbf{P}_{1}$, which generalizes the operator of the form

$$
\mathbf{P}_{1} u(y)=\int_{F}[u(y+z(y, \eta))-u(y)-(\nabla u(y), z(y, \eta))] m(y, \eta) \pi(d \eta), \quad y \in G
$$

(cf. $[6,20,21,102]$ ), where $F$ is a space with a $\sigma$-algebra $\mathcal{F}$ and a Borel measure $\pi$, and the vector-valued function $z(y, \eta)$ and the scalar-valued function $m(y, \eta)$ are bounded and $\pi$-measured with respect to $\eta$, $m(y, \eta) \geq 0$, and $y+z(y, \eta) \in \bar{G}$ for all $y \in \bar{G}$ and $\eta \in F$. The operator $\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ of the form (9) is, generally speaking, nonclosed. We obtain conditions on the measure $\beta(y, \cdot)$ under which the closure of the operator $\mathbf{P}_{B}$ is a generator of a Feller semigroup.

In Sec. 26, we construct examples in which some of above-mentioned conditions on the coefficients of nonlocal conditions, the structure of transformations of variables, or the Borel measure $\beta(y, \cdot)$ are violated. We show that the closure of the corresponding operator $\mathbf{P}_{B}$ is not a generator of a Feller semigroup. To do this, we prove that the image of the operator $\overline{\mathbf{P}_{B}}-q \mathbf{I}$ does not coincide with $C_{B}(\bar{G})$ for some $q>0$ and we apply the Hille-Yosida theorem.

In particular, we construct an example in which the Dirichlet condition is given in the original and in some punctured neighborhood $b_{1}(y)=b_{2}(y)=1$, i.e., condition (8) is violated. From the point of view of the probability theory, this means the following: the origin is a point of process termination; nevertheless, jumps with probability 1 from an arbitrary small neighborhood of the origin occur.

In this paper, we consider the two-dimensional case. However, we note that the results of Chap. 1 on the solvability of nonlocal problems in Kondrat'ev weight spaces are also valid in $\mathbb{R}^{n}, n \geq 3$, when the boundary contains edge-type singularities. Similarly to [54], some results on the smoothness of generalized solutions can also be generalized to the $n$-dimensional case. To generalize results on the existence of Feller semigroups to the $n$-dimensional case, further development of the theory of nonlocal elliptic problems is needed: the study of the asymptotic behavior of solutions near edge-type singularities on the boundaries and the solvability in weight spaces and Sobolev spaces (based on $\left.L_{p}(G), p>2\right)$ and in Hölder spaces.

The author is deeply gratitude to Prof. A. L. Skubachevskii for his constant attention and support for many years.

## Chapter 1

## NONLOCAL ELLIPTIC PROBLEMS WITH NONLINEAR TRANSFORMATIONS OF VARIABLES

## 1. Some Definitions and Results from Linear Operators. Functional Spaces

1.1. Some definitions and results from the theory of linear operators. In this section, we recall some definitions from the theory of linear operators and we prove two lemmas that will be used below.

Let $H_{1}$ and $H_{2}$ be Hilbert spaces and $\mathbf{P}: \mathrm{D}(\mathbf{P}) \subset H_{1} \rightarrow H_{2}$ be a linear (generally speaking, unbounded) operator, where $\mathrm{D}(\mathbf{P})$ denotes the domain of the operator $\mathbf{P}$.

Definition 1.1. An operator $\mathbf{P}$ is called a Fredholm operator if it is closed, has a closed image, and the dimension of its kernel $\operatorname{ker} \mathbf{P}$ and the co-dimension of its image $\mathcal{R}(\mathbf{P})$ are finite. The number ind $\mathbf{P}=\operatorname{dim} \operatorname{ker} \mathbf{P}-\operatorname{codim} \mathcal{R}(\mathbf{P})$ is called the index of the Fredholm operator $\mathbf{P}$.

Let $\mathbf{A}: \mathrm{D}(\mathbf{A}) \subset H_{1} \rightarrow H_{2}$ be a linear operator.

Definition 1.2 (see $[49,56]$ ). An operator $\mathbf{A}$ is called compact with respect to $\mathbf{P}$ (or simply $\mathbf{P}$ compact) if $\mathrm{D}(\mathbf{P}) \subset \mathrm{D}(\mathbf{A})$ and for any sequence $u_{n} \in \mathrm{D}(\mathbf{P})$ such that $\left\{u_{n}\right\}$ and $\left\{\mathbf{P} u_{n}\right\}$ are bounded, the sequence $\left\{\mathbf{A} u_{n}\right\}$ contains a converging sequence.

Let us introduce the notion of a spread between linear operators. Let $\mathbf{S}: \mathrm{D}(\mathbf{S}) \subset H_{1} \rightarrow H_{2}$ be a linear operator. Introduce the following norm in the space $H_{1} \times H_{2}$ :

$$
\|(u, f)\|=\left(\|u\|_{H_{1}}^{2}+\|f\|_{H_{2}}^{2}\right)^{1 / 2} \quad \forall(u, f) \in H_{1} \times H_{2} .
$$

Introduce the notation

$$
\delta(\mathbf{P}, \mathbf{S})=\sup _{\substack{u \in \mathrm{D}(\mathbf{P}): \\\|(u, \mathbf{P} u)\|=1}} \operatorname{dist}((u, \mathbf{P} u), \operatorname{Gr} \mathbf{S}),
$$

where $\mathrm{Gr} \mathbf{S}$ is the graph of the operator $\mathbf{S}$.
Definition 1.3 (see [49]). The number $\hat{\delta}(\mathbf{P}, \mathbf{S})=\max \{\delta(\mathbf{P}, \mathbf{S}), \delta(\mathbf{S}, \mathbf{P})\}$ is called the spread between operators $\mathbf{P}$ and $\mathbf{S}$.

The spread between operators allows one to evaluate the "closeness" of two unbounded operators that have, generally speaking, different domains (see [49]).

We prove the following two auxiliary results.
Lemma 1.1. Let $H, H_{1}$, and $H_{2}$ be Hilbert spaces, A : $H \rightarrow H_{1}$ be a linear bounded operator, and $\mathbf{T}: H \rightarrow H_{2}$ be a compact operator. Assume that for some $\delta, c>0$, and $f \in H$, the following inequality holds:

$$
\begin{equation*}
\|\mathbf{A} f\|_{H_{1}} \leq \delta\|f\|_{H}+c\|\mathbf{T} f\|_{H_{2}} . \tag{1.1}
\end{equation*}
$$

Then there exist operators $\mathbf{M}, \mathbf{F}: H \rightarrow H_{1}$ such that

$$
\mathbf{A}=\mathbf{M}+\mathbf{F},
$$

$\|\mathbf{M}\| \leq 2 \delta$, and the operator $\mathbf{F}$ is finite-dimensional.
Proof. As is known (see [71, Chap. 5, Sec. 85]), any compact operator is a limit in the operator norm of a sequence of finite-dimensional operators. Thus, there exist operators $\mathbf{M}_{0}, \mathbf{F}_{0}: H \rightarrow H_{2}$ such that $\mathbf{T}=\mathbf{M}_{0}+\mathbf{F}_{0},\left\|\mathbf{M}_{0}\right\| \leq c^{-1} \delta$, and the operator $\mathbf{F}_{0}$ is finite-dimensional. This and (1.1) imply that

$$
\begin{equation*}
\|\mathbf{A} f\|_{H_{1}} \leq 2 \delta\|f\|_{H}+c\left\|\mathbf{F}_{0} f\right\|_{H_{2}} \quad \forall f \in H \tag{1.2}
\end{equation*}
$$

Denote the orthogonal complement of the kernel of the operator $\mathbf{F}_{0}$ in $H$ by $\operatorname{ker}\left(\mathbf{F}_{0}\right)^{\perp}$. Since the finitedimensional operator $\mathbf{F}_{0}$ maps $\operatorname{ker}\left(\mathbf{F}_{0}\right)^{\perp}$ to its image bijectively, we see that the subspace $\operatorname{ker}\left(\mathbf{F}_{0}\right)^{\perp}$ is finite-dimensional. Denote the identity operator in $H$ by $\mathbf{I}$ and the orthogonal projector to $\operatorname{ker}\left(\mathbf{F}_{0}\right)^{\perp}$ by $\mathbf{P}_{0}$. Obviously, $\mathbf{A P}_{0}: H \rightarrow H_{1}$ is a finite-dimensional operator. Moreover, since $\mathbf{I}-\mathbf{P}_{0}$ is an orthogonal projector to $\operatorname{ker}\left(\mathbf{F}_{0}\right)$, we obtain $\mathbf{F}_{0}\left(\mathbf{I}-\mathbf{P}_{0}\right)=0$. Hence, substituting the function $\left(\mathbf{I}-\mathbf{P}_{0}\right) f$ instead of $f$ to (1.2), we obtain

$$
\left\|\mathbf{A}\left(\mathbf{I}-\mathbf{P}_{0}\right) f\right\|_{H_{1}} \leq 2 \delta\left\|\left(\mathbf{I}-\mathbf{P}_{0}\right) f\right\|_{H} \leq 2 \delta\|f\|_{H} \quad \forall f \in H
$$

Setting $\mathbf{M}=\mathbf{A}\left(\mathbf{I}-\mathbf{P}_{0}\right)$ and $\mathbf{F}=\mathbf{A} \mathbf{P}_{0}$, we complete the proof.
Lemma 1.2. Let $H$ be a Hilbert space and $\mathbf{I}$ be a identity operator in the space $H$. Let $\mathbf{M}_{\delta}$ and $\mathbf{S}_{\delta}=\mathbf{U}_{\delta}+\mathbf{F}_{\delta}$ and $\delta>0$ be families of bounded in $H$ operators such that

$$
\begin{equation*}
\left\|\mathbf{M}_{\delta}\right\| \leq c_{1} \delta, \quad\left\|\mathbf{U}_{\delta}\right\| \leq c_{2} \tag{1.3}
\end{equation*}
$$

where $c_{1}$ and $c_{2}>0$ are independent of $\delta$ and the operators $\mathbf{F}_{\delta}$ and $\mathbf{S}_{\delta}^{2}$ are compact. Then, for any sufficiently small $\delta>0$, the operators

$$
\mathbf{L}_{\delta}=\mathbf{I}+\mathbf{M}_{\delta}+\mathbf{S}_{\delta}
$$

are Fredholm operators.

Proof. To prove the lemma, we construct right and left regularizers for the operator $\mathbf{L}_{\delta}$. We have

$$
\begin{aligned}
& \mathbf{L}_{\delta}\left(\mathbf{I}-\left(\mathbf{M}_{\delta}+\mathbf{S}_{\delta}\right)\right)=\mathbf{I}-\mathbf{M}_{\delta}^{2}-\mathbf{M}_{\delta} \mathbf{S}_{\delta}-\mathbf{S}_{\delta} \mathbf{M}_{\delta}-\mathbf{S}_{\delta}^{2} \\
&=\left(\mathbf{I}-\mathbf{M}_{\delta}^{2}-\mathbf{M}_{\delta} \mathbf{U}_{\delta}-\mathbf{U}_{\delta} \mathbf{M}_{\delta}\right)-\left(\mathbf{M}_{\delta} \mathbf{F}_{\delta}+\mathbf{F}_{\delta} \mathbf{M}_{\delta}+\mathbf{S}_{\delta}^{2}\right)
\end{aligned}
$$

It follows from (1.3) that

$$
\left\|\mathbf{M}_{\delta}^{2}+\mathbf{M}_{\delta} \mathbf{U}_{\delta}+\mathbf{U}_{\delta} \mathbf{M}_{\delta}\right\| \leq c_{3} \delta
$$

where $c_{3}>0$ is independent of $\delta$. This and [56, Theorem 16.2] imply that for all sufficiently small $\delta>0$, the operators $\mathbf{I}-\mathbf{M}_{\delta}^{2}-\mathbf{M}_{\delta} \mathbf{U}_{\delta}-\mathbf{U}_{\delta} \mathbf{M}_{\delta}$ are Fredholm operators. Then we use the compactness of the operators $\mathbf{F}_{\delta}$ and $\mathbf{S}_{\delta}^{2}$ and apply [56, Theorem 16.4]. We see that the operators $\mathbf{L}_{\delta}\left(\mathbf{I}-\left(\mathbf{M}_{\delta}+\mathbf{S}_{\delta}\right)\right)$ are also Fredholm operators. It follows from [56, Theorem 15.2] that there exist bounded operators $\mathbf{R}_{1 \delta}$ and compact operators $\mathbf{T}_{1 \delta}$ such that

$$
\begin{equation*}
\mathbf{L}_{\delta}\left(\mathbf{I}-\left(\mathbf{M}_{\delta}+\mathbf{S}_{\delta}\right)\right) \mathbf{R}_{1 \delta}=\mathbf{I}+\mathbf{T}_{1 \delta} . \tag{1.4}
\end{equation*}
$$

Similarly, we prove the existence of bounded operators $\mathbf{R}_{2 \delta}$ and compact operators $\mathbf{T}_{2 \delta}$ such that

$$
\begin{equation*}
\mathbf{R}_{2 \delta}\left(\mathbf{I}-\left(\mathbf{M}_{\delta}+\mathbf{S}_{\delta}\right)\right) \mathbf{L}_{\delta}=\mathbf{I}+\mathbf{T}_{2 \delta} . \tag{1.5}
\end{equation*}
$$

Now the lemma follows from (1.4), (1.5), and [56, Theorems 14.3 and 15.2].

### 1.2. Functional spaces.

1.2.1. Spaces of continuous and infinitely differentiable functions. Let $X$ be a nonempty set in $\mathbb{R}^{n}$, $n \geq 1$. We denote the set of continuous in $X$ functions by $C(X)$. If $X$ is a compact, then $C(X)$ is a Banach space with the norm

$$
\|u\|_{C(X)}=\max _{y \in X}|u(y)|, \quad u \in C(X) .
$$

Let $X$ and $M$ be closed sets and the set $X$ be nonempty. We denote by $C^{\infty}(X)$ the set of restrictions to $X$ of infinitely differentiable in $\mathbb{R}^{n}$ functions. Denote the set of infinitely differentiable in $\mathbb{R}^{n}$ functions with compact supports in $X \backslash M$ by $C_{0}^{\infty}(X \backslash M)$.

For any domain $Q \subset \mathbb{R}^{n}$ and any $l \geq 0$ (in what follows, the number $l$ is assumed to be integer whenever the contrary is not stated), we denote the set of $l$ times infinitely differentiable in $Q$ (in $\bar{Q}$ ) functions by $C^{l}(Q)$ (respectively, by $C^{l}(\bar{Q})$ ). In particular, $C^{0}(Q)=C(Q)$ and $C^{0}(\bar{Q})=C(\bar{Q})$.

### 1.2.2. The domain $G$ and the angle $K$. Introduce the notation

$$
K=\left\{y \in \mathbb{R}^{2}: r>0, \omega_{1}<\omega<\omega_{2}\right\}, \quad \gamma_{\sigma}=\left\{y \in \mathbb{R}^{2}: r>0, \omega=\omega_{\sigma}\right\}, \quad \sigma=1,2,
$$

where $\omega$ and $r$ are the polar coordinates of a point $y, \omega_{1}<0<\omega_{2}$, and $\omega_{2}-\omega_{1}<2 \pi$.
Denote the $\varepsilon$-neighborhood of the origin by $\mathcal{O}_{\varepsilon}(0)$.
Introduce the sets

$$
K^{\varepsilon}=K \cap \mathcal{O}_{\varepsilon}(0), \quad \gamma_{\sigma}^{\varepsilon}=\gamma_{\sigma} \cap \mathcal{O}_{\varepsilon}(0), \quad \sigma=1,2 .
$$

In this chapter, we denote by $G \subset \mathbb{R}^{2}$ a bounded domain with boundary $\partial G$ containing the origin. We assume that $\overline{G \cap \mathcal{O}_{\varepsilon}(0)}=\overline{K^{\varepsilon}}$ for some $\varepsilon>0$. Assume that in a neighborhood of any point $y \in \partial G \backslash\{0\}$, the boundary of the domain $G$ is infinitely smooth.
1.2.3. Sobolev spaces. For any domain $Q \subset \mathbb{R}^{n}$ and any $l \geq 0$, we denote the Sobolev space with the norm

$$
\|u\|_{W^{l}(Q)}=\left(\sum_{|\alpha| \leq l} \int_{G}\left|D^{\alpha} u\right|^{2} d y\right)^{1 / 2}
$$

by $W^{l}(Q)=W_{2}^{l}(Q)$ (for $l=0$, we assume that $\left.W^{0}(Q)=L_{2}(Q)\right)$. In the sequel, $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$, $|\alpha|=\alpha_{1}+\cdots+\alpha_{n}, D^{\alpha}=D_{y_{1}}^{\alpha_{1}} \ldots D_{y_{n}}^{\alpha_{n}}, D_{y_{j}}=-i \partial / \partial y_{j}$. If it is necessary to specify the variables with respect to which we differentiate the function $u$, then we write $D_{y}^{\alpha} u, D_{y^{\prime}}^{\alpha} u$, etc. For $l \geq 1$, let us introduce the space $W^{l-1 / 2}\left(\Gamma_{1}\right)$ of traces on a smooth curve $\Gamma_{1} \subset \bar{Q}$ with the norm

$$
\|\psi\|_{W^{l-1 / 2}\left(\Gamma_{1}\right)}=\inf \|u\|_{W^{l}(Q)}, \quad u \in W^{l}(Q):\left.u\right|_{\Gamma_{1}}=\psi .
$$

We denote by $W_{\text {loc }}^{l}(Q)$ the set consisting of all distributions $u$ on $Q$ such that $\varphi u \in W^{l}(Q)$ for any $\varphi \in C_{0}^{\infty}(Q)$.
1.2.4. Kondrat'ev weight spaces. Let us consider the following cases: $Q=K, Q=K^{d}(d>0)$, and $Q=G$. For any $l \geq 0$ and any $a \in \mathbb{R}$, we denote the completion of the set $C_{0}^{\infty}(\bar{Q} \backslash\{0\})$ with respect to the norm

$$
\|u\|_{H_{a}^{l}(Q)}=\left(\sum_{|\alpha| \leq l} \int_{Q} \rho^{2(a-l+|\alpha|)}\left|D^{\alpha} u\right|^{2} d y\right)^{1 / 2}
$$

by $H_{a}^{l}(Q)=H_{a}^{l}(Q,\{0\})$, where $\rho=\rho(y)=\operatorname{dist}(y,\{0\})$. For $l \geq 1$, we denote the space of traces on a smooth curve $\Gamma_{1} \subset \bar{Q}$ with the norm

$$
\|\psi\|_{H_{a}^{l-1 / 2}\left(\Gamma_{1}\right)}=\inf \|u\|_{H_{a}^{l}(Q)}, \quad u \in H_{a}^{l}(Q):\left.u\right|_{\Gamma_{1}}=\psi
$$

by $H_{a}^{l-1 / 2}\left(\Gamma_{1}\right)$.

## 2. The statement of the problem in a bounded domain

2.1. The statement of the problem. Let us consider linear differential operators $\mathbf{P}(y, D)$ and $B_{\mu s}(y, D)$ of orders $2 m$ and $m_{\mu}$, respectively, with complex coefficients ( $\mu=1, \ldots, m ; s=0,1$ ). Assume that the coefficients of the operators $\mathbf{P}(y, D)$ are infinitely smooth in $\bar{G}$ and the coefficients of the operators $B_{\mu s}(y, D)$ are infinitely smooth in $(\partial G \backslash\{0\}) \cup \overline{\gamma_{\sigma}^{\varepsilon}}, \sigma=1,2$.

Let us formulate conditions on the operators $\mathbf{P}(y, D)$ and $B_{\mu 0}(y, D)$ that correspond to a "local" elliptic problem (see, e.g., [57, Chap. 2, Sec. 1].
Condition 2.1. The operator $\mathbf{P}(y, D)$ is properly elliptic on $\bar{G}$.
Condition 2.2. For any $y \in \overline{\gamma_{\sigma}^{\varepsilon}}$ and any $y \in \partial G \backslash \mathcal{O}_{\varepsilon}(0)$, the system of operators $\left\{B_{\mu 0}(y, D)\right\}_{\mu=1}^{m}$ satisfies the covering condition (Lopatinskii condition) with respect to the operator $\mathbf{P}(y, D)$.

We emphasize that the normality of the operators $B_{\mu 0}(y, D)$ is not provided.
Let us introduce the operators $\mathbf{B}_{\mu}^{0}: H_{a}^{l+2 m}(G) \rightarrow H_{a}^{l+2 m-m_{\mu}-1 / 2}(\partial G)$ by the formula

$$
\mathbf{B}_{\mu}^{0} u=\left.B_{\mu 0}(y, D) u(y)\right|_{\partial G} .
$$

Everywhere in this chapter, $a \in \mathbb{R}$ and $l \geq 0$ is an integer such that $l+2 m-m_{\mu} \geq 1$.
Define the operators that correspond to nonlocal conditions near the origin. Let $\Omega_{\sigma}, \sigma=1,2$, be infinitely differentiable, nondegenerate transformations that map some neighborhood $\mathcal{O}_{\sigma}$ of a curve $\overline{\gamma_{\sigma}^{\varepsilon}}$ to the set $\Omega_{\sigma}\left(\mathcal{O}_{\sigma}\right)$ such that $\Omega_{\sigma}\left(\gamma_{\sigma}^{\varepsilon}\right) \subset G$ and

$$
\begin{equation*}
\Omega_{\sigma}(0)=0 . \tag{2.1}
\end{equation*}
$$

Choose small $\varepsilon$ (see Remark 2.2 below) such that there exists a neighborhood $\mathcal{O}_{\varepsilon_{1}}(0)$ such that $\mathcal{O}_{\varepsilon_{1}}(0) \supset \mathcal{O}_{\varepsilon}(0)$ and
(1) $\overline{G \cap \mathcal{O}_{\varepsilon_{1}}(0)}=K^{\varepsilon_{1}}$;
(2) $\Omega_{\sigma}\left(\mathcal{O}_{\varepsilon}(0)\right) \subset \mathcal{O}_{\varepsilon_{1}}(0)$.

Condition 2.3. For $y \in \mathcal{O}_{\varepsilon}(0)$, the transformation $\Omega_{\sigma}$ has the form

$$
\Omega_{\sigma}: y \mapsto \mathcal{G}_{\sigma} y+o(|y|),
$$

where $\mathcal{G}_{\sigma}$ is the composition of the rotation by the angle $-\omega_{\sigma}$ about the origin and the dilation with the center at the origin and the coefficient $\chi_{\sigma}>0, \sigma=1,2$.

Thus, the operator $\mathcal{G}_{\sigma}$ maps the ray $\gamma_{\sigma}$ of the angle $K$ to the half-line $\left\{y \in \mathbb{R}^{2}: r>0, \omega=0\right\}$ lying inside the angle $K$.
Remark 2.1. In particular, condition 2.3 means that the curve $\Omega_{\sigma}\left(\overline{\gamma_{\sigma}^{\bar{\varepsilon}}}\right)$ approaches the boundary $\partial G$ at the point 0 but does not touch it.

In this chapter, we use the notation

$$
\begin{equation*}
d_{1}=\min _{\sigma=1,2}\left\{1, \chi_{\sigma}\right\} / 2, \quad d_{2}=2 \max _{\sigma=1,2}\left\{1, \chi_{\sigma}\right\} . \tag{2.2}
\end{equation*}
$$

Let us choose a number $\varepsilon_{0}, 0<\varepsilon_{0} \leq \varepsilon$, satisfying the condition

$$
\mathcal{O}_{\varepsilon_{0}}(0) \subset \Omega_{\sigma}\left(\mathcal{O}_{\varepsilon}(0)\right) \subset \mathcal{O}_{\varepsilon_{1}}(0)
$$

Let us consider a function $\zeta \in C^{\infty}\left(\mathbb{R}^{2}\right)$ such that

$$
\begin{equation*}
\zeta(y)=1\left(y \in \mathcal{O}_{\varepsilon_{0} / 2}(0)\right), \quad \zeta(y)=0\left(y \notin \mathcal{O}_{\varepsilon_{0}}(0)\right) . \tag{2.3}
\end{equation*}
$$

Introduce the bounded operators $\mathbf{B}_{\mu}^{1}: H_{a}^{l+2 m}(G) \rightarrow H_{a}^{l+2 m-m_{\mu}-1 / 2}(\partial G)$ by the formula

$$
\mathbf{B}_{\mu}^{1} u= \begin{cases}\left.\left(B_{\mu 1}(y, D)(\zeta u)\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}^{\varepsilon}}, & y \in \gamma_{\sigma}^{\varepsilon}, \quad \sigma=1,2, \\ 0, & y \in \partial G \backslash \mathcal{O}_{\varepsilon}(0),\end{cases}
$$

where $\left(B_{\mu 1}(y, D) v\right)\left(\Omega_{i s}(y)\right)=\left.B_{\mu 1}\left(y^{\prime}, D_{y^{\prime}}\right) v\left(y^{\prime}\right)\right|_{y^{\prime}=\Omega_{\sigma}(y)}$.
Since $\mathbf{B}_{\mu}^{1} u=0$ when $\operatorname{supp} u \subset \bar{G} \backslash \overline{\mathcal{O}_{\varepsilon_{0}}(0)}$, we say that the operators $\mathbf{B}_{\mu}^{1}$ correspond to nonlocal terms with supports near the origin.

Introduce the notation

$$
G_{\rho}=\{y \in G: \operatorname{dist}(y, \partial G)>\rho\} .
$$

Let us consider a bounded operator

$$
\mathbf{B}_{\mu}^{2}: H_{a}^{l+2 m}(G) \rightarrow H_{a}^{l+2 m-m_{\mu}-1 / 2}(\partial G)
$$

satisfying the following condition.
Condition 2.4. There exist numbers $\varkappa_{1}>\varkappa_{2}>0$ and $\rho>0$ such that the following inequalities hold:

$$
\begin{array}{ll}
\left\|\mathbf{B}_{\mu}^{2} u\right\|_{H_{a}^{l+2 m-m_{\mu}-1 / 2}(\partial G)} \leq c_{1}\|u\|_{W^{l+2 m}\left(G \backslash \overline{\mathcal{O}_{1}(0)}\right)} & \forall u \in W^{l+2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(0)}\right) \\
\left\|\mathbf{B}_{\mu}^{2} u\right\|_{W^{l+2 m-m_{\mu}-1 / 2}\left(\partial G \backslash \overline{\mathcal{O}_{2}(0)}\right)} \leq c_{2}\|u\|_{W^{l+2 m}\left(G_{\rho}\right)} & \forall u \in W^{l+2 m}\left(G_{\rho}\right), \tag{2.5}
\end{array}
$$

where $\mu=1, \ldots, m, c_{1}$, and $c_{2}>0$ are independent of $u$.
It follows from (2.4) that $\mathbf{B}_{\mu}^{2} u=0$ if $\operatorname{supp} u \subset \mathcal{O}_{\varkappa_{1}}(0)$. This is why we say that the operators $\mathbf{B}_{\mu}^{2}$ correspond to nonlocal terms with supports outside the origin.

Note that we do not assume a priori the presence of any relation between the numbers $\varkappa_{1}, \varkappa_{2}$, and $\rho$ in Condition 2.4 and the number $\varepsilon_{0}$ in Condition 2.3.

We study the nonlocal elliptic problem

$$
\begin{align*}
\mathbf{P}(y, D) u & =f_{0}(y), & y \in G,  \tag{2.6}\\
\mathbf{B}_{\mu} u \equiv \mathbf{B}_{\mu}^{0} u+\mathbf{B}_{\mu}^{1} u+\mathbf{B}_{\mu}^{2} u & =f_{\mu}(y), & y \in \partial G ; \quad \mu=1, \ldots, m . \tag{2.7}
\end{align*}
$$

Problem (6.9), (6.10) from Sec. 6.2 (see Chap. 2) for $N=1$ can serve as an example of problem (2.6), (2.7) (see [25] for details).

Introduce the following operator corresponding to problem (2.6), (2.7) in weight spaces:

$$
\begin{equation*}
\mathbf{L}=\left\{\mathbf{P}(y, D), \mathbf{B}_{\mu}\right\}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G), \tag{2.8}
\end{equation*}
$$

where

$$
\mathcal{H}_{a}^{l}(G, \partial G)=H_{a}^{l}(G) \times \prod_{\mu=1}^{m} H_{a}^{l+2 m-m_{\mu}-1 / 2}(\partial G) .
$$

Remark 2.2. Let us show that the number $\varepsilon_{0}$ in the definition of nonlocal operators $\mathbf{B}_{\mu}^{1}$ can be chosen arbitrarily small.

Let the number $\hat{\varepsilon}_{0}$ be such that $0<\hat{\varepsilon}_{0}<\varepsilon_{0}$. Let us consider a function $\hat{\zeta} \in C^{\infty}\left(\mathbb{R}^{2}\right)$ such that

$$
\hat{\zeta}(y)= \begin{cases}1, & y \in \mathcal{O}_{\hat{\varepsilon}_{0} / 2}(\mathcal{K}), \\ 0, & y \notin \mathcal{O}_{\hat{\varepsilon}_{0}}(\mathcal{K}),\end{cases}
$$

and introduce the operator $\hat{\mathbf{B}}_{\mu}^{1}: H_{a}^{l+2 m}(G) \rightarrow H_{a}^{l+2 m-m_{\mu}-1 / 2}(\partial G)$ by the formula

$$
\hat{\mathbf{B}}_{\mu}^{1} u= \begin{cases}\left.\left(B_{\mu 1}(y, D)(\hat{\zeta} u)\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}^{\varepsilon}}, & y \in \gamma_{\sigma}^{\varepsilon}, \quad \sigma=1,2, \\ \hat{\mathbf{B}}_{\mu}^{1} u=0, & y \in \partial G \backslash \mathcal{O}_{\varepsilon}(0) .\end{cases}
$$

Obviously,

$$
\mathbf{B}_{\mu}^{0}+\mathbf{B}_{\mu}^{1}+\mathbf{B}_{\mu}^{2}=\mathbf{B}_{\mu}^{0}+\hat{\mathbf{B}}_{\mu}^{1}+\hat{\mathbf{B}}_{\mu}^{2},
$$

where

$$
\hat{\mathbf{B}}_{\mu}^{2}=\mathbf{B}_{\mu}^{1}-\hat{\mathbf{B}}_{\mu}^{1}+\mathbf{B}_{\mu}^{2} .
$$

It is easy to see that the operator $\mathbf{B}_{\mu}^{1}-\hat{\mathbf{B}}_{\mu}^{1}$ for some $\varkappa_{1}, \varkappa_{2}$, and $\rho$ satisfies Condition 2.4. Thus, we can always choose $\varepsilon_{0}$ arbitrary small (perhaps, due to a modification of the operator $\mathbf{B}_{\mu}^{2}$ and numbers $\varkappa_{1}, \varkappa_{2}$, and $\rho$ ).
2.2. Reduction to model problems in plane angles. In problem (2.6), (2.7), the behavior of solutions in a neighborhood of the origin requires special attention. Let us consider the following model problem in a plane angle. Assume that

$$
\begin{equation*}
\mathbf{B}_{\mu}^{2}=0, \quad \mu=1, \ldots, m . \tag{2.9}
\end{equation*}
$$

Then, by Condition 2.3 , problem (2.6), (2.7) has the following form in a $\varepsilon$-neighborhood of the origin:

$$
\begin{gather*}
\mathbf{P}(y, D) U=f_{0}(y), \quad y \in K^{\varepsilon},  \tag{2.10}\\
\left.B_{\sigma \mu 0}(y, D) U\right|_{\gamma_{\sigma}}+\left.\left(B_{\sigma \mu 1}(y, D) U\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}}=f_{\sigma \mu}(y), \quad y \in \gamma_{\sigma}^{\varepsilon}, \tag{2.11}
\end{gather*}
$$

where $\sigma=1,2 ; \mu=1, \ldots, m ; B_{\sigma \mu s}(y, D), s=0,1$, are linear differential operators of order $m_{\mu}$ with variable coefficients of the class $C^{\infty}$.

We denote by $\mathcal{P}(D)$ and $B_{\sigma \mu S}(D)$ the principal homogeneous parts of the operators $\mathbf{P}(0, D)$ and $B_{\sigma \mu s}(0, D)$ respectively. Assume that

$$
\begin{aligned}
& \mathcal{B}_{\sigma \mu}^{\Omega}(D) U=\left.B_{\sigma \mu 0}(D) U\right|_{\gamma_{\sigma}}+\left.\left(B_{\sigma \mu 1}(D) U\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}}, \\
& \mathcal{B}_{\sigma \mu}(D) U=\left.B_{\sigma \mu 0}(D) U\right|_{\gamma_{\sigma}}+\left.\left(B_{\sigma \mu 1}(D) U\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}} .
\end{aligned}
$$

Introduce the operators

$$
\begin{aligned}
\mathcal{L}^{\Omega} & =\left\{\mathcal{P}(D), \mathcal{B}_{\sigma \mu}^{\Omega}(D)\right\}: H_{a}^{l+2 m}(K) \rightarrow \mathcal{H}_{a}^{l}(K, \gamma), \\
\mathcal{L} & =\left\{\mathcal{P}(D), \mathcal{B}_{\sigma \mu}(D)\right\}: H_{a}^{l+2 m}(K) \rightarrow \mathcal{H}_{a}^{l}(K, \gamma)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}_{a}^{l}(K, \gamma) & =H_{a}^{l}(K) \times \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma) \\
\mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma) & =\prod_{\sigma=1,2} \prod_{\mu=1}^{m} H_{a}^{l+2 m-m_{\mu}-1 / 2}\left(\gamma_{\sigma}\right) .
\end{aligned}
$$

We assume that the operator $\mathcal{L}^{\Omega}$ is defined on the set of functions with supports that are concentrated in a neighborhood of the origin (in particular, with supports such that $\Omega_{\sigma}(y) \in K$ for $y \in \operatorname{supp} U$ ).

It is obvious that the operators $\mathcal{L}^{\Omega}$ and $\mathcal{L}$ agree with the model problems with nonlinear and linearized transformations, respectively.

We rewrite the operators $\mathcal{P}(D)$ and $B_{\sigma \mu s}(D)$ in the polar coordinates in the following form:

$$
\mathcal{P}(D)=r^{-2 m} \tilde{\mathcal{P}}\left(\omega, D_{\omega}, r D_{r}\right), \quad B_{\sigma \mu}(D)=r^{-m_{\mu}} \tilde{B}_{\sigma \mu}\left(\omega, D_{\omega}, r D_{r}\right),
$$

where $D_{\omega}=-i \frac{\partial}{\partial \omega}, D_{r}=-i \frac{\partial}{\partial r}$. Consider the following operator (it is an analytic operator-function depending on a parameter $\lambda \in \mathbb{C})$ :

$$
\tilde{\mathcal{L}}(\lambda): W^{l+2 m}\left(\omega_{1}, \omega_{2}\right) \rightarrow W^{l}\left(\omega_{1}, \omega_{2}\right) \times \mathbb{C}^{2 m}
$$

defined by the formula

$$
\begin{equation*}
\tilde{\mathcal{L}}(\lambda) \varphi=\left\{\tilde{\mathcal{P}}\left(\omega, D_{\omega}, \lambda\right) \varphi, \tilde{\mathcal{B}}_{\sigma \mu}\left(\omega, D_{\omega}, \lambda\right) \varphi\right\} \tag{2.12}
\end{equation*}
$$

where

$$
\tilde{\mathcal{B}}_{\sigma \mu}\left(\omega, D_{\omega}, \lambda\right) \varphi=\left.\tilde{B}_{\sigma \mu 0}\left(\omega, D_{\omega}, \lambda\right) \varphi\right|_{\omega=\omega_{\sigma}}+\left.\left(\chi_{\sigma}\right)^{i \lambda-m_{\mu}} \tilde{B}_{\sigma \mu 1}\left(\omega, D_{\omega}, \lambda\right) \varphi\left(\omega-\omega_{\sigma}\right)\right|_{\omega=\omega_{\sigma}} .
$$

Note that the set of eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ is discrete (see [88]).

## 3. Nonlinear Transformations Near the Origin

3.1. Nonlinear transformations in polar coordinates. It was shown in [25] that the operator corresponding to a problem with nonlinear transformations of variables is not a small or compact perturbation of an operator corresponding to the problem with linearized transformations. Therefore, to prove the Fredholm solvability of a problem with nonlinear transformations, we first obtain a priori estimates and construct a right regularizer. To do this, we study some properties of the transformations $\Omega_{\sigma}$ near the origin.

Applying the Taylor formula, we easily obtain the following statement, which will be used in the proof of lemma on the representation of transformations $\Omega$ in the polar coordinates (see Lemma 3.2).
Lemma 3.1. Let $h=h(r)$ be a function such that $\left|D_{r}^{k} h\right| \leq c_{k}$ as $0 \leq r \leq \varrho, k=0, \ldots, k_{0}$, where $c_{k}>0$ is independent of $r$. Let $f(r)=r^{-l} h(r)$ for some $l \in \mathbb{N}$. If $|f| \leq c$, then $\left|D_{r}^{k} f\right| \leq C_{k}$ as $0 \leq r \leq \varrho, k=1, \ldots, k_{0}$, where $C_{k}>0$ is independent of $r$.

The following lemma describes the structure of nonlinear transformation $\Omega_{\sigma}$ in the polar coordinates. Such a representation is the most convenient when we deal with weight spaces.
Lemma 3.2. If $\varrho$ is sufficiently small, then the representation $\left.\Omega_{\sigma}(y)\right|_{\gamma_{\sigma}^{o}}$ can be written in the polar coordinates as follows:

$$
\begin{equation*}
\left(\omega_{\sigma}, r\right) \mapsto\left(\Phi_{\sigma}(r), \chi_{\sigma} r+R_{\sigma}(r)\right), \quad r \leq \varrho, \tag{3.1}
\end{equation*}
$$

where $\Phi_{\sigma}(r)$ and $R_{\sigma}(r)$ are infinitely smooth functions such that

$$
\begin{gather*}
\left|\Phi_{\sigma}\right| \leq c \varrho, \quad\left|R_{\sigma}\right| \leq c \varrho r  \tag{3.2}\\
\left|D_{r}^{k} \Phi_{\sigma}\right| \leq c_{k}, \quad\left|D_{r}^{k}\left(R_{\sigma} / r\right)\right| \leq c_{k} \tag{3.3}
\end{gather*}
$$

Here $k \geq 1$; $c$ and $c_{k}>0$ are independent of $\varrho$.
Proof. Let $\Omega_{\sigma}(y)=\left(\Omega_{\sigma}^{1}(y), \Omega_{\sigma}^{2}(y)\right)$. Using Eq. (2.1) and the Taylor formula in the neighborhood of $r=0$, we obtain

$$
\begin{equation*}
\Omega_{\sigma}^{i}\left(r \cos \omega_{\sigma}, r \sin \omega_{\sigma}\right)=\left(\frac{\partial \Omega_{\sigma}^{i}}{\partial y_{1}}(0) \cos \omega_{\sigma}+\frac{\partial \Omega_{\sigma}^{i}}{\partial y_{2}}(0) \sin \omega_{\sigma}\right) r+O\left(r^{2}\right) . \tag{3.4}
\end{equation*}
$$

Note that

$$
\frac{\partial \Omega_{\sigma}^{1}}{\partial y_{1}}(0) \cos \omega_{\sigma}+\frac{\partial \Omega_{\sigma}^{1}}{\partial y_{2}}(0) \sin \omega_{\sigma}, \quad \frac{\partial \Omega_{\sigma}^{2}}{\partial y_{1}}(0) \cos \omega_{\sigma}+\frac{\partial \Omega_{\sigma}^{2}}{\partial y_{2}}(0) \sin \omega_{\sigma}
$$

does not vanish simultaneously; this follows from the nondegeneracy of the Jacobian of the transformation $y \mapsto \Omega_{\sigma}(y)$ at the origin. For example, let

$$
\begin{equation*}
\frac{\partial \Omega_{\sigma}^{1}}{\partial y_{1}}(0) \cos \omega_{\sigma}+\frac{\partial \Omega_{\sigma}^{1}}{\partial y_{2}}(0) \sin \omega_{\sigma} \neq 0 \tag{3.5}
\end{equation*}
$$

Then, by Eq. (3.4), if $\varrho$ is sufficiently small, then we have

$$
\begin{equation*}
\Omega_{\sigma}^{1} \neq 0 \quad \text { as } r \leq \varrho \tag{3.6}
\end{equation*}
$$

and the transformation $\left.\Omega_{\sigma}\right|_{\gamma_{\sigma}^{o}}$ has the following form in the polar coordinates:

$$
\begin{equation*}
\left(\omega_{\sigma}, r\right) \mapsto\left(\arctan \frac{\Omega_{\sigma}^{2}}{\Omega_{\sigma}^{1}}+\pi l, \sqrt{\sum_{i=1,2}\left(\Omega_{\sigma}^{i}\right)^{2}}\right) . \tag{3.7}
\end{equation*}
$$

Here $l=0$ if $\Omega_{\sigma}^{1}>0$ and $\Omega_{\sigma}^{2} \geq 0 ; l=1$ if $\Omega_{\sigma}^{1}<0 ; l=2$ if $\Omega_{\sigma}^{1}>0$ and $\Omega_{\sigma}^{2}<0$.
Equation (3.4) and the Taylor formula yield

$$
\begin{aligned}
& \arctan \frac{\Omega_{\sigma}^{2}}{\Omega_{\sigma}^{1}}=\arctan \frac{\frac{\partial \Omega_{\sigma}^{2}}{\partial y_{1}}(0) \cos \omega_{\sigma}+\frac{\partial \Omega_{\sigma}^{2}}{\partial y_{2}}(0) \sin \omega_{\sigma}}{\frac{\partial \Omega_{\sigma}^{1}}{\partial y_{1}}(0) \cos \omega_{\sigma}+\frac{\partial \Omega_{\sigma}^{1}}{\partial y_{2}}(0) \sin \omega_{\sigma}}+O(r), \\
& \sqrt{\sum_{i=1,2}\left(\Omega_{\sigma}^{i}\right)^{2}}=r \sqrt{\sum_{i=1,2}\left(\frac{\partial \Omega_{\sigma}^{i}}{\partial y_{1}}(0) \cos \omega_{\sigma}+\frac{\partial \Omega_{\sigma}^{i}}{\partial y_{2}}(0) \sin \omega_{\sigma}\right)^{2}}+O\left(r^{2}\right) .
\end{aligned}
$$

Assuming that

$$
\begin{aligned}
\omega_{k q} & =\arctan \frac{\frac{\partial \Omega_{\sigma}^{2}}{\partial y_{1}}(0) \cos \omega_{\sigma}+\frac{\partial \Omega_{\sigma}^{2}}{\partial y_{2}}(0) \sin \omega_{\sigma}}{\frac{\partial \Omega_{\sigma}^{1}}{\partial y_{1}}(0) \cos \omega_{\sigma}+\frac{\partial \Omega_{\sigma}^{1}}{\partial y_{2}}(0) \sin \omega_{\sigma}}+\pi l \\
\chi_{\sigma} & =\sqrt{\sum_{i=1,2}\left(\frac{\partial \Omega_{\sigma}^{i}}{\partial y_{1}}(0) \cos \omega_{\sigma}+\frac{\partial \Omega_{\sigma}^{i}}{\partial y_{2}}(0) \sin \omega_{\sigma}\right)^{2}}
\end{aligned}
$$

we obtain formula (3.1) and inequalities (3.2).
Prove the first inequality in (3.3). By (3.6), $\left|\Omega_{\sigma}^{2} / \Omega_{\sigma}^{1}\right| \leq c$ for $r \leq \varrho$. Therefore, by Eqs. (3.1) and (3.7), it suffices to prove the boundedness of the partial derivatives $D_{r}^{k} \frac{\Omega_{\sigma}^{2}}{\Omega_{\sigma}^{1}}$. Let us write

$$
\frac{\Omega_{\sigma}^{2}}{\Omega_{\sigma}^{1}}=\frac{r^{-1} \Omega_{\sigma}^{2}}{r^{-1} \Omega_{\sigma}^{1}} .
$$

It follows from Eqs. (3.4) and (3.5) that $r^{-1} \Omega_{\sigma}^{1} \neq 0$ for $r \leq \varrho$; therefore, it suffices to prove that

$$
\left|D_{r}^{k}\left(r^{-1} \Omega_{\sigma}\right)\right| \leq c_{k} .
$$

However, $\Omega_{\sigma}$ is an infinitely smooth transformation for $r \leq \varrho$; since $\Omega_{\sigma}(0)=0$, we have $\Omega_{\sigma}=O(r)$. Hence, $\left|r^{-1} \Omega_{\sigma}\right| \leq c$. Now the desired statement follows from Lemma 3.1.

Similarly, we prove the second inequality in (3.3). It follows from Eqs. (3.1) and (3.7) that

$$
\frac{R_{\sigma}(r)}{r}=\sqrt{\sum_{i=1,2} \frac{\left(\Omega_{\sigma}^{i}\right)^{2}}{r^{2}}}-\chi_{\sigma} .
$$

By Eqs. (3.4) and (3.5), we have $\sum_{i=1,2}\left(\Omega_{\sigma}^{i}\right)^{2} / r^{2} \neq 0$ for $r \leq \varrho$; hence, it suffices to show that

$$
\left|D_{r}^{k} \sum_{i=1,2}\left(\Omega_{\sigma}^{i}\right)^{2} / r^{2}\right| \leq c_{k}
$$

However, $\sum_{i=1,2}\left(\Omega_{\sigma}^{i}\right)^{2}$ is an infinitely smooth function if $r \leq \varrho$. Since $\Omega_{\sigma}(0)=0$, we have that $\sum_{i=1,2}\left(\Omega_{\sigma}^{i}\right)^{2}=O\left(r^{2}\right)$. Hence, $\left|\sum_{i=1,2}\left(\Omega_{\sigma}^{i}\right)^{2} / r^{2}\right| \leq c$, and the desired statement follows from Lemma 3.1.

Introduce the notation $\delta=\min \left(-\omega_{1}, \omega_{2}\right) / 2$. Let $\varrho$ be so small that

$$
\begin{equation*}
\left|\Phi_{\sigma}\right| \leq \delta / 2, \quad\left|R_{\sigma}\right| \leq \chi_{\sigma} r / 2, \quad r \leq \varrho / d_{1} . \tag{3.8}
\end{equation*}
$$

The existence of such a $\varrho$ follows from Lemma 3.2.
Let us introduce infinitely differentiable functions $\zeta_{i}(\omega)$ and $\zeta_{\sigma i}(\omega), i=0, \ldots, 4$, such that

$$
\begin{gather*}
\zeta_{i}(\omega)=1 \quad \text { for } \quad|\omega| \leq \delta / 2^{i+1}, \quad \zeta_{i}(\omega)=0 \quad \text { for } \quad|\omega| \geq \delta / 2^{i}, \\
\zeta_{\sigma i}(\omega)=\zeta_{i}\left(\omega-\omega_{\sigma}\right) . \tag{3.9}
\end{gather*}
$$

Obviously, $\zeta_{\sigma i}(\omega)=1$ for $\left|\omega-\omega_{\sigma}\right| \leq \delta / 2^{i+1}$ and $\zeta_{\sigma i}(\omega)=0$ for $\left|\omega-\omega_{\sigma}\right| \geq \delta / 2^{i}$.
Consider the transformation $\tilde{\Omega}_{\sigma}(y)$, acting in the polar coordinates by the formula

$$
\begin{equation*}
(\omega, r) \mapsto\left(\omega-\omega_{\sigma}+\Phi_{\sigma}(r), \chi_{\sigma} r+R_{\sigma}(r)\right) . \tag{3.10}
\end{equation*}
$$

According to Lemma 3.2, we have $\left.\tilde{\Omega}_{\sigma}(y)\right|_{\gamma_{\sigma}^{o}}=\left.\Omega_{\sigma}(y)\right|_{\gamma_{\sigma}^{\rho}}$; therefore, in the sequel, we can assume that the transformation $\Omega_{\sigma}(y)$ is defined by formula (3.10). Generally speaking, the transformation $\Omega_{\sigma}(y)$ can have a singularity at the origin since the new transformation $\Omega_{\sigma}(y)$ coincides with the old $\Omega_{\sigma}(y)$ only on $\gamma_{\sigma}^{\varrho}$.

Definition 3.1. For any function $W(y)$, we set $\widehat{W}(y)=W\left(\Omega_{\sigma}\left(\mathcal{G}_{\sigma}^{-1} y\right)\right)$.
By Lemma 3.2, the transformation $\Omega_{\sigma}\left(\mathcal{G}_{\sigma}^{-1} y\right)$ in the polar coordinates has the form

$$
\begin{equation*}
(\omega, r) \mapsto\left(\omega+\Phi_{\sigma}^{\prime}(r), r+R_{\sigma}^{\prime}(r)\right) \tag{3.11}
\end{equation*}
$$

where $\Phi_{\sigma}^{\prime}(r)=\Phi_{\sigma}\left(\chi_{\sigma}^{-1} r\right)$ and $R_{\sigma}^{\prime}(r)=R_{\sigma}\left(\chi_{\sigma}^{-1} r\right)$. It is easy to see that $\Phi_{\sigma}^{\prime}$ and $R_{\sigma}^{\prime}$ also satisfy inequalities (3.2) and (3.3).

### 3.2. Properties of operators containing nonlinear transformations in weight spaces.

Lemma 3.3. If $\varrho>0$ is sufficiently small, then for any function $W \in H_{a}^{l}(K)$ such that $\operatorname{supp} W \subset \overline{K^{\varrho}}$, we have $\zeta_{1} \widehat{W} \in H_{a}^{l}(K)$ and

$$
\left\|\zeta_{1} \widehat{W}\right\|_{H_{a}^{l}(K)} \leq c\|W\|_{H_{a}^{l}(K)}
$$

where $c>0$ is independent of $W$ and $\varrho$.
Proof. To prove the lemma, we use the following obvious statement:

$$
\begin{equation*}
W \in H_{a}^{l}(K) \quad \Longleftrightarrow \quad D^{\alpha} W \in H_{a+|\alpha|-l}^{0}(K), \quad|\alpha| \leq l . \tag{3.12}
\end{equation*}
$$

Formula (3.11) and inequalities (3.8) imply that the transformation (3.11) maps the set $K^{\varrho} \cap\{y:|\omega|<\delta\}$ to $K$. Moreover, it follows from inequalities (3.2) and (3.3) that the modulus of the Jacobian of transformation (3.11) is bounded and nonzero in $K^{\varrho} \cap\{y:|\omega|<\delta\}$ if $\varrho$ are small. Hence, the lemma is valid for $l=0$ with the function $\zeta_{0}$ instead of $\zeta_{1}$.

Let us consider functions $\zeta_{0}^{p} \in C_{0}^{\infty}(\mathbb{R}), p=0, \ldots, l$, such that $\zeta_{0}^{0}=\zeta_{0}, \zeta_{0}^{l}=\zeta_{1}$, and $\zeta_{0}^{p-1}(\omega)=1$ for $\omega \in \operatorname{supp} \zeta_{0}^{p}, p=1, \ldots, l$. Assume that the lemma is valid for $l=p-1$ with the function $\zeta_{0}^{p-1}$ instead of $\zeta_{1}$; prove that it is valid for $l=p$ with the function $\zeta_{0}^{p}$ instead of $\zeta_{1}(p \geq 1)$. Let $W \in H_{a}^{p}(K)$; then $\frac{1}{r} \frac{\partial W}{\partial \omega}, \frac{\partial W}{\partial r} \in H_{a}^{p-1}(K)$. Hence, by the induction hypothesis, $\zeta_{0}^{p-1} \frac{\widehat{1} \frac{\partial W}{\partial \omega}}{\frac{\partial \omega}{p-1}} \zeta_{0}^{p-1} \frac{\widehat{\partial W}}{\partial r} \in H_{a}^{p-1}(K)$. These inclusions, the relations

$$
\begin{gather*}
\frac{1}{r} \frac{\partial \widehat{W}_{k}}{\partial \omega}=\widehat{\frac{1}{r}} \frac{\widehat{\partial W}}{\partial \omega}\left(1+\frac{R_{\sigma}^{\prime}}{r}\right)  \tag{3.13}\\
\frac{\partial \widehat{W}_{k}}{\partial r}=\frac{1}{r} \frac{\partial W}{\partial \omega}\left(1+\frac{R_{\sigma}^{\prime}}{r}\right) r \frac{\partial \Phi_{\sigma}^{\prime}}{\partial r}+\frac{\widehat{\partial W}}{\partial r}\left(1+\frac{\partial R_{\sigma}^{\prime}}{\partial r}\right),
\end{gather*}
$$

inequalities (3.2) and (3.3), and [53, Lemma 2.1] imply that

$$
\begin{equation*}
\zeta_{0}^{p-1} \frac{1}{r} \frac{\partial \widehat{W}}{\partial \omega}, \zeta_{0}^{p-1} \frac{\partial \widehat{W}}{\partial r} \in H_{a}^{p-1}(K) \tag{3.14}
\end{equation*}
$$

Moreover, the inclusion $W \in H_{a}^{p}(K)$, the embedding $H_{a}^{p}(K) \subset H_{a-p}^{0}(K)$, and the statement of the lemma for $l=0$ yield $\zeta_{0}^{p} \widehat{W} \in H_{a-p}^{0}(K)$. This and Eqs. (3.12) and (3.14) imply that $D^{\alpha}\left(\zeta_{0}^{p} \widehat{W}\right) \in$ $H_{a+|\alpha|-p}^{0}(K),|\alpha| \leq p$. Applying Eq. (3.12) again, we obtain the desired statement.

Thus, the operator $W \mapsto \zeta_{1} \widehat{W}$ is bounded in $H_{a}^{l}(K)$.
Lemma 3.4. For any function $W \in H_{a}^{l}(K)$ such that $\operatorname{supp} W \subset \overline{K^{\varrho}}$ and for any multi-index $\gamma$, $1 \leq|\gamma| \leq l$, the following inequality holds:

$$
\begin{equation*}
\left\|\zeta_{2} D^{\gamma} \widehat{W}-\zeta_{2} \widehat{D^{\gamma} W}\right\|_{H_{a}^{l-|\gamma|}(K)} \leq c \varrho\|W\|_{H_{a}^{l}(K)} \tag{3.15}
\end{equation*}
$$

Here $c>0$ is independent of $W$ and $\varrho$.
Proof. Consider functions $\zeta_{1}^{p} \in C_{0}^{\infty}(\mathbb{R}), p=1, \ldots, l$, such that $\zeta_{1}^{1}=\zeta_{1}, \zeta_{1}^{l}=\zeta_{2}$, and $\zeta_{1}^{p-1}(\omega)=1$ for $\omega \in \operatorname{supp} \zeta_{1}^{p}, p=2, \ldots, l$.

Let $|\gamma|=1$; then it suffices to prove inequality (3.15) for the case where there are operators $\frac{1}{r} \frac{\partial}{\partial \omega}$ and $\frac{\partial}{\partial r}$ instead of the operator $D^{\gamma}$. For example, consider the operator $\frac{1}{r} \frac{\partial}{\partial \omega}$ (the remaining operators
are considered similarly). The first relation in (3.13) and the Leibnitz formula yield the relation

$$
\begin{aligned}
\left\|\zeta_{1}^{1} \frac{1}{r} \frac{\partial \widehat{W}}{\partial \omega}-\zeta_{1}^{1} \frac{\widehat{1} \frac{\partial W}{\partial \omega}}{\partial \omega}\right\|_{H_{a}^{l-1}(K)}^{2} & =\left\|\zeta_{1}^{1} \frac{1}{r} \frac{\partial W}{\partial \omega} \frac{R_{\sigma}^{\prime}}{r}\right\|_{H_{a}^{l-1}(K)}^{2} \\
& \leq k_{1} \sum_{|\alpha| \leq l-1} \sum_{|\beta| \leq|\alpha|} \int_{K} r^{2(a+|\alpha|-(l-1))}\left|D^{\alpha-\beta} \frac{R_{\sigma}^{\prime}}{r}\right|^{2}\left|D^{\beta}\left(\zeta_{1}^{1} \frac{\widehat{1} \frac{\partial W}{\partial \omega}}{\partial \omega}\right)\right|^{2} d y .
\end{aligned}
$$

This, the last inequality in Eq. (3.2), and the last inequality in (3.3) imply that

$$
\begin{equation*}
\left\|\zeta_{1}^{1} \frac{1}{r} \frac{\partial \widehat{W}}{\partial \omega}-\zeta_{1}^{1} \frac{\widehat{1} \frac{\partial W}{\partial \omega}}{\|_{H_{a}^{l-1}(K)}^{2}}\right\|_{2} \varrho^{2} \| \zeta_{1}^{1} \frac{\widehat{1} \frac{\partial W}{\partial \omega}}{\|_{H_{a}^{l-1}(K)}^{2}} \tag{3.16}
\end{equation*}
$$

Estimate (3.16) and Lemma 3.3 prove the lemma for $|\gamma|=1$ with the function $\zeta_{1}^{1}$ instead of $\zeta_{2}$.
Assume that the lemma is valid for $1 \leq|\gamma| \leq p-1$ with the function $\zeta_{1}^{p-1}$ instead of $\zeta_{2}$. Prove that it is valid for $|\gamma|=p$ with the function $\zeta_{1}^{\bar{p}}$ instead of $\zeta_{2}(p \geq 2)$. We have

$$
\begin{align*}
& \left\|\zeta_{1}^{p} D^{\gamma} \widehat{W}-\zeta_{1}^{p} \widehat{D^{\gamma} W}\right\|_{H_{a}^{l-|\gamma|}(K)} \\
& \quad \leq\left\|\zeta_{1}^{p} D^{|\gamma|-1}\left(D^{1} \widehat{W}\right)-\zeta_{1}^{p} D^{|\gamma|-1} \widehat{D^{1} W}\right\|_{H_{a}^{l-|\gamma|}(K)}+\left\|\zeta_{1}^{p} D^{|\gamma|-1} \widehat{D^{1} W}-\zeta_{1}^{p} D^{|\gamma|-1}\left(D^{1} W\right)\right\|_{H_{a}^{l-|\gamma|}(K)} \\
& \quad \leq k_{3}\left(\left\|\zeta_{1}^{p-1} D^{1} \widehat{W}-\zeta_{1}^{p-1} \widehat{D^{1} W}\right\|_{H_{a}^{l-1}(K)}+\left\|\zeta_{1}^{p} D^{|\gamma|-1} \widehat{D^{1} W}-\zeta_{1}^{p} D^{|\gamma|-1}\left(D^{1} W\right)\right\|_{H_{a}^{l-|\gamma|}(K)}\right) \tag{3.17}
\end{align*}
$$

Here $D^{|\gamma|-1}$ and $D^{1}$ are some generalized derivatives of orders $|\gamma|-1$ and 1 , respectively. By the induction hypothesis, the following estimates are valid for each of the two norms on the right-hand side of Eq. (3.17):

$$
\begin{gathered}
\left\|\zeta_{1}^{p-1} D^{1} \widehat{W}-\zeta_{1}^{p-1} \widehat{D^{1} W}\right\|_{H_{a}^{l-1}(K)} \leq k_{4} \varrho\|W\|_{H_{a}^{l}(K)} \\
\left\|\zeta_{1}^{p} D^{|\gamma|-1} \widehat{D^{1} W}-\zeta_{1}^{p} D|\gamma| \widehat{-1}\left(D^{1} W\right)\right\|_{H_{a}^{l-|\gamma|}(K)} \leq k_{5} \varrho\left\|D^{1} W\right\|_{H_{a}^{l-1}(K)} \leq k_{6 \varrho}\|W\|_{H_{a}^{l}(K)}
\end{gathered}
$$

The desired statement follows from here and Eq. (3.17).
The multiplier $\varrho$ appears in estimate (3.15) since both terms on the left-hand side of the inequality contain the same transformation of variables $\Omega_{\sigma}\left(\mathcal{G}_{\sigma}^{-1} y\right)$, but the minuend is a derivative from the transformed function and the subtrahend is a transformation of a derivative.
Lemma 3.5. The following inequality holds for any function $U \in H_{a}^{l+2 m}(K)$ such that $\operatorname{supp} U \subset \overline{K^{\varrho}}$ :

$$
\begin{align*}
&\left\|\left.\left(B_{\sigma \mu 1} U\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}-\left.\left(B_{\sigma \mu 1} U\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}}\right\|_{H_{a}^{l+2 m-m_{\mu}-1 / 2}\left(\gamma_{\sigma}\right)} \\
& \leq c\left(\varrho\|U\|_{H_{a}^{l+2 m}(K)}+\left\|\zeta_{3} U-\zeta_{3} \widehat{U}\right\|_{H_{a}^{l+2 m}(K)}\right), \tag{3.18}
\end{align*}
$$

where $c>0$ is independent of $U$ and $\varrho$.
Proof. Using the continuity of the trace operator in weight spaces, we obtain

$$
\begin{gather*}
\left\|\left.\left(B_{\sigma \mu 1} U\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}-\left.\left(B_{\sigma \mu 1} U\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}}\right\|_{H_{a}^{l+2 m-m_{\mu}-1 / 2}\left(\gamma_{\sigma}\right)} \leq k_{1}\left\|\zeta_{4} B_{\sigma \mu 1} U-\zeta_{4} \widehat{B_{\sigma \mu 1} U}\right\|_{H_{a}^{l+2 m-m_{\mu}}(K)} \\
\leq k_{1}\left(\left\|\zeta_{4} B_{\sigma \mu 1} U-\zeta_{4} B_{\sigma \mu 1} \widehat{U}\right\|_{H_{a}^{l+2 m-m_{\mu}(K)}}+\left\|\zeta_{4} B_{\sigma \mu 1} \widehat{U}-\zeta_{4} \widehat{B_{\sigma \mu 1} U}\right\|_{H_{a}^{l+2 m-m_{\mu}}(K)}\right) . \tag{3.19}
\end{gather*}
$$

Let us estimate the first norm on the right-hand side of inequality (3.19):

$$
\begin{equation*}
\left\|\zeta_{4} B_{\sigma \mu 1} U-\zeta_{4} B_{\sigma \mu 1} \widehat{U}\right\|_{H_{a}^{l+2 m-m_{\mu}}(K)} \leq k_{2}\left\|\zeta_{3} U-\zeta \zeta_{3} \widehat{U}\right\|_{H_{a}^{l+2 m}(K)} . \tag{3.20}
\end{equation*}
$$

The second norm on the right-hand side of inequality (3.19) is estimated by Lemma 3.4:

$$
\begin{equation*}
\left\|\zeta_{4} B_{\sigma \mu 1} \widehat{U}-\zeta_{4} \widehat{B_{\sigma \mu 1} U}\right\|_{H_{a}^{l+2 m-m_{\mu}}(K)} \leq k_{3} \varrho\|U\|_{H_{a}^{l+2 m}(K)} \tag{3.21}
\end{equation*}
$$

The statement of the lemma follows from Eqs. (3.19)-(3.21).
Note that the right-hand side of inequality (3.18) contains the norm of the difference of the nontransformed and transformed functions. We need the following lemma to estimate such differences.

Lemma 3.6. The following inequality holds for any function $W \in H_{a+1}^{1}(K)$ such that $\operatorname{supp} W \subset \overline{K^{\varrho}}$ :

$$
\begin{equation*}
\left\|\zeta_{1} W-\zeta_{1} \widehat{W}\right\|_{H_{a}^{0}(K)} \leq c \varrho\|W\|_{H_{a+1}^{1}(K)} \tag{3.22}
\end{equation*}
$$

where $c>0$ is independent of $W$ and $\varrho$.
Proof. Writing the arguments of the functions $W$ and $\widehat{W}$ in the polar coordinates, we obtain the inequality

$$
\begin{align*}
\left\|\zeta_{1} W-\zeta_{1} \widehat{W}\right\|_{H_{a}^{0}(K)} \leq \| \zeta_{1} W(\omega, r) & -\zeta_{1} W\left(\omega+\Phi_{\sigma}^{\prime}(r), r\right) \|_{H_{a}^{0}(K)} \\
& +\left\|\zeta_{1} W\left(\omega+\Phi_{\sigma}^{\prime}(r), r\right)-\zeta_{1} W\left(\omega+\Phi_{\sigma}^{\prime}(r), r+R_{\sigma}^{\prime}(r)\right)\right\|_{H_{a}^{0}(K)} \tag{3.23}
\end{align*}
$$

Let us estimate the square of the first norm on the right-hand side of Eq. (3.23), using the Schwartz inequality:

$$
\begin{aligned}
\left\|\zeta_{1} W(\omega, r)-\zeta_{1} W\left(\omega+\Phi_{\sigma}^{\prime}(r), r\right)\right\|_{H_{a}^{0}(K)}^{2}= & \int_{0}^{\infty} r^{2 a} r d r \int_{\omega_{1}}^{\omega_{2}}\left|\zeta_{1} \int_{\omega}^{\omega+\Phi_{\sigma}^{\prime}(r)} \frac{\partial W}{\partial \omega^{\prime}} d \omega^{\prime}\right|^{2} d \omega \\
& \left.\leq\left.\int_{0}^{\infty} r^{2 a} r d r \int_{\omega_{1}}^{\omega_{2}}\left|\zeta_{1}\right|^{2}\left|\Phi_{\sigma}^{\prime}(r)\right| \cdot\left|\int_{\omega}^{\omega+\Phi_{\sigma}^{\prime}(r)}\right| \frac{\partial W}{\partial \omega^{\prime}}\right|^{2} d \omega^{\prime} \right\rvert\, d \omega
\end{aligned}
$$

Taking into consideration the restrictions on the supports of functions $W$ and $\zeta_{1}$ and inequalities (3.8), we change the order of integrating with respect to $\omega$ and $\omega^{\prime}$; as a result, using Eq. (3.2), we obtain

$$
\begin{aligned}
& \left\|\zeta_{1} W(\omega, r)-\zeta_{1} W\left(\omega+\Phi_{\sigma}^{\prime}(r), r\right)\right\|_{H_{a}^{0}(K)}^{2} \\
& \leq k_{1} \int_{0}^{\infty} r^{2 a} r\left|\Phi_{\sigma}^{\prime}(r)\right|^{2} d r \int_{\omega_{1}}^{\omega_{2}}\left|\frac{\partial W}{\partial \omega}\right|^{2} d \omega \\
& \quad \leq k_{2} \varrho^{2} \int_{0}^{\infty} r^{2(a+1)} r d r \int_{\omega_{1}}^{\omega_{2}}\left|\frac{1}{r} \frac{\partial W}{\partial \omega}\right|^{2} d \omega \leq k_{3} \varrho^{2}\|W\|_{H_{a+1}^{1}(K)}^{2} .
\end{aligned}
$$

Similarly, we estimate the square of the second norm on the right-hand side of Eq. (3.23).
Thus, the multiplier $\varrho$ appears in Eq. (3.22) if we increase the differentiability index by 1 (the left-hand side of (3.22) contains the norm in $H_{a}^{0}(K)$ and the right-hand side contains the norm in $H_{a+1}^{1}(K)$ ). This happens because in this case (unlike Eq. (3.15)), we estimate the difference of two functions, one of which does not contain a transformation of variables and the other contains it.

## 4. Fredholm Solvability of Nonlocal Problems and the Stability of the Index

4.1. A priori estimates of solutions. In this section, we prove an a priori estimate for the operator $\mathbf{L}$ that guarantees the finite dimension if its kernel and closeness of its image.

Condition 4.1. The straight line $\operatorname{Im} \lambda=a+1-l-2 m$ does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$.

Lemma 4.1. Let conditions 2.1-2.4 and 4.1 hold. Then

$$
\|u\|_{H_{a}^{l+2 m}(G)} \leq c\left(\|\mathbf{L} u\|_{\mathcal{H}_{a}^{l}(G, \partial G)}+\|u\|_{H_{a+1-l-2 m}^{0}(G)}\right) \quad \forall u \in H_{a}^{l+2 m}(G) .
$$

Here $c>0$ is independent of $u$.
Proof. Similarly to the proof of [41, Theorem 4.1], we use the principle of partition of unity and the Leibnitz formula and reduce the proof of Lemma 4.1 to the proof of the following a priori estimate for sufficiently small $\varrho>0$ :

$$
\begin{equation*}
\|U\|_{H_{a}^{l+2 m}(K)} \leq c\left\|\mathcal{L}^{\Omega} U\right\|_{\mathcal{H}_{a}^{l}(K, \gamma)} \quad \forall U \in H_{a}^{l+2 m}(K), \operatorname{supp} U \subset \overline{K^{\varrho}}, \tag{4.1}
\end{equation*}
$$

where $c>0$ is independent of $U$.
Let us prove Eq. (4.1). By condition 4.1 and [88, Theorem 2.1], the operator $\mathcal{L}: H_{a}^{l+2 m}(K) \rightarrow$ $\mathcal{H}_{a}^{l}(K, \gamma)$ has a bounded inverse operator. Hence applying Lemma 3.5, for $U \in H_{a}^{l+2 m}(K)$, $\operatorname{supp} U \subset \overline{K^{\varrho}}$, we obtain the following inequality:

$$
\begin{align*}
& \|U\|_{H_{a}^{l+2 m}(K)} \leq k_{1}\|\mathcal{L} U\|_{\mathcal{H}_{a}^{l}(K, \gamma)} \\
& \quad \leq k_{2}\left(\left\|\mathcal{L}^{\Omega} U\right\|_{\mathcal{H}_{a}^{l}(K, \gamma)}+\varrho\|U\|_{H_{a}^{l+2 m}(K)}+\left\|\zeta_{3} U-\zeta_{3} \widehat{U}\right\|_{H_{a}^{l+2 m}(K)}\right) \tag{4.2}
\end{align*}
$$

where $k_{1}, k_{2}, \cdots>0$ are independent of $U$ and $\varrho$.
Let us estimate the last norm in Eq. (4.2). According to [58, Theorem 4.1], we have

$$
\begin{equation*}
\left\|\zeta_{3} U-\zeta_{3} \widehat{U}\right\|_{H_{a}^{l+2 m}(K)} \leq k_{3}\left(\left\|\mathcal{P}(D)\left(\zeta_{3} U-\zeta_{3} \widehat{U}\right)\right\|_{H_{a}^{l}(K)}+\left\|\zeta_{3} U-\zeta_{3} \widehat{U}\right\|_{H_{a-l-2 m}^{0}(K)}\right) . \tag{4.3}
\end{equation*}
$$

By Lemma 3.6 and the continuity of the embedding $H_{a}^{l+2 m}(K) \subset H_{a-l-2 m+1}^{1}(K)$, we have

$$
\begin{equation*}
\left\|\zeta_{3} U-\zeta_{3} \widehat{U}\right\|_{H_{a-l-2 m}^{0}(K)} \leq k_{4} \varrho\|U\|_{H_{a}^{l+2 m}(K)} \tag{4.4}
\end{equation*}
$$

To estimate the first norm on the right-hand side of Eq. (4.3), we use the Leibnitz formula and Lemmas 3.3 and 3.4:

$$
\begin{align*}
& \left\|\mathcal{P}(D)\left(\zeta_{3} U-\zeta_{3} \widehat{U}\right)\right\|_{H_{a}^{l}(K)} \leq k_{5}\left(\left\|\zeta_{3} \mathcal{P}(D) U\right\|_{H_{a}^{l}(K)}+\left\|\zeta_{3} \mathcal{P}(D) \widehat{U}\right\|_{H_{a}^{l}(K)}\right. \\
& \left.+\sum_{|\beta| \leq 2 m-1} \sum_{|\gamma|=2 m-|\beta|}\left\|D^{\gamma} \zeta_{3} D^{\beta} U-D^{\gamma} \zeta_{3} D^{\beta} \widehat{U}\right\|_{H_{a}^{l}(K)}\right) \leq k_{6}\left(\|\mathcal{P}(D) U\|_{H_{a}^{l}(K)}\right. \\
& \left.\quad+\varrho\|U\|_{H_{a}^{l+2 m}(K)}+\sum_{|\beta| \leq 2 m-1} \sum_{|\gamma|=2 m-|\beta|}\left\|D^{\gamma} \zeta_{3} D^{\beta} U-D^{\gamma} \zeta_{3} D^{\beta} \widehat{U}\right\|_{H_{a}^{l}(K)}\right) . \tag{4.5}
\end{align*}
$$

Since $\left|D^{\gamma} \zeta_{3}\right| \leq k_{7} r^{-|\gamma|}\left|\zeta_{2}\right|$, we have

$$
\begin{align*}
& \sum_{|\beta| \leq 2 m-1} \sum_{|\gamma|=2 m-|\beta|}\left\|D^{\gamma} \zeta_{3} D^{\beta} U-D^{\gamma} \zeta_{3} D^{\beta} \widehat{U}\right\|_{H_{a}^{l}(K)} \\
& \leq \\
& \leq k_{8} \sum_{|\alpha| \leq l+2 m-1}\left\|\zeta_{2} D^{\alpha} U-\zeta_{2} D^{\alpha} \widehat{U}\right\|_{H_{a+|\alpha|-l-2 m}^{0}(K)} \\
& \leq k_{9} \sum_{|\alpha| \leq l+2 m-1}\left(\left\|\zeta_{2} D^{\alpha} U-\zeta_{2} \widehat{D^{\alpha} U}\right\|_{H_{a+|\alpha|-l-2 m}^{0}(K)}\right.  \tag{4.6}\\
&\left.\quad+\left\|\zeta_{2} \widehat{D^{\alpha} U}-\zeta_{2} D^{\alpha} \widehat{U}\right\|_{H_{a+|\alpha|-l-2 m}^{0}(K)}\right) .
\end{align*}
$$

Using Lemma 3.6 and the continuity of the embedding of the spaces

$$
H_{a}^{l+2 m}(K) \subset H_{a+1+|\alpha|-l-2 m}^{1+|\alpha|}(K)
$$

for $|\alpha| \leq l+2 m-1$, we obtain the inequality

$$
\begin{equation*}
\left\|\zeta_{2} D^{\alpha} U-\zeta_{2} \widehat{D^{\alpha} U}\right\|_{H_{a+|\alpha|-l-2 m}^{0}(K)} \leq k_{10} \varrho\left\|D^{\alpha} U\right\|_{H_{a+1+|\alpha|-l-2 m}^{1}(K)} \leq k_{11} \varrho\|U\|_{H_{a}^{l+2 m}(K)} \tag{4.7}
\end{equation*}
$$

Similarly, Lemma 3.4 implies that

$$
\begin{equation*}
\left\|\zeta_{2} \widehat{D^{\alpha} U}-\zeta_{2} D^{\alpha} \widehat{U}\right\|_{H_{a+|\alpha|-l-2 m}^{0}(K)} \leq k_{12} \varrho\|U\|_{H_{a}^{l+2 m}(K)} . \tag{4.8}
\end{equation*}
$$

Now estimate (4.1) follows from Eqs. (4.2)-(4.8) for sufficiently small $\varrho$.
By virtue of the compactness of the embedding $H_{a}^{l+2 m}(G) \subset H_{a+1-l-2 m}^{0}(G)$ (see [53, Lemma 3.5]), Lemma 4.1 implies that the operator $\mathbf{L}$ has a finite-dimensional kernel and a closed image.
4.2. Construction of a right regularizer. Fredholm solvability of nonlocal problems. In this case, we construct the right regularizer for the operator $\mathbf{L}$. This and Lemma 4.1 allow us to prove the Fredholm property of nonlocal boundary-value problem (2.6), (2.7).

Lemma 4.2. Let conditions 2.1-2.4 and 4.1 hold. Then there exist a bounded operator $\mathbf{R}$ : $\mathcal{H}_{a}^{l}(G, \partial G) \rightarrow H_{a}^{l+2 m}(G)$ and a compact operator $\mathbf{T}: \mathcal{H}_{a}^{l}(G, \partial G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G)$ such that

$$
\mathbf{L R}=\mathbf{I}+\mathbf{T},
$$

where $\mathbf{I}$ is the identity operator in $\mathcal{H}_{a}^{l}(G, \partial G)$.
Proof. 1. Similarly to the proof of [41, Theorem 5.2], we use the principle of partition of unity and the Leibnitz formula and reduce the proof of Lemma 4.2 to the proof of the following statement: for all sufficiently small $\varrho>0$, there exist bounded operators $\mathcal{R}$ and $\mathcal{M}$ and a compact operator $\mathcal{T}$, acting from $\left\{f \in \mathcal{H}_{a}^{l}(K, \gamma): \operatorname{supp} f \subset \mathcal{O}_{d_{1} \varrho / 4}(0)\right\}$ to $H_{a}^{l+2 m}(K), \mathcal{H}_{a}^{l}(K, \gamma)$, and $\mathcal{H}_{a}^{l}(K, \gamma)$, respectively, and satisfying the conditions

$$
\begin{equation*}
\mathcal{L}^{\Omega} \mathcal{R} f=f+\mathcal{M} f+\mathcal{T} f \tag{4.9}
\end{equation*}
$$

$\|\mathcal{M} f\|_{\mathcal{H}_{a}^{l}(K, \gamma)} \leq c \varrho\|f\|_{\mathcal{H}_{a}^{l}(K, \gamma)}$. Here $d_{1}$ is the number defined by formula (2.2), and $c>0$ is independent of $\varrho$ and $f$.

Let us construct the operators $\mathcal{R}, \mathcal{M}$ and $\mathcal{T}$ that satisfy relation (4.9).
Introduce a function $\psi_{\varrho}(y)=\psi(y / \varrho)$, where $\psi \in C^{\infty}\left(\mathbb{R}^{n}\right), \psi(y)=1$ for $|y| \leq 1$, and $\psi(y)=0$ for $|y| \geq 2$. Obviously, $\psi_{\varrho} \in C^{\infty}\left(\mathbb{R}^{n}\right)$ and $\psi_{\varrho}(y)=1$ for $|y| \leq \varrho$ and $\psi_{\varrho}(y)=0$ for $|y| \geq 2 \varrho$. Since we have $\left|D^{\alpha} \psi_{\varrho}\right| \leq c_{\alpha} r^{-|\alpha|}$, it follows from [53, Lemma 2.1] that

$$
\begin{equation*}
\left\|\psi_{\varrho} v\right\|_{H_{a}^{l+2 m}(K)} \leq c\|v\|_{H_{a}^{l+2 m}(K)} \quad \text { for all } \quad v \in H_{a}^{l+2 m}(K), \tag{4.10}
\end{equation*}
$$

where $c>0$ is independent of $\varrho$. Moreover, let $\psi_{\varrho}$, written in the polar coordinates, be independent of $\omega$.

Let $f^{\prime}=\left\{f_{\sigma \mu}\right\}$. By condition 4.1 and [88, Theorem 2.1], the operator $\mathcal{L}: H_{a}^{l+2 m}(K) \rightarrow \mathcal{H}_{a}^{l}(K, \gamma)$ has a bounded inverse operator. Hence we can define the operators

$$
\mathcal{R}_{0}: H_{a}^{l}(K) \rightarrow H_{a}^{l+2 m}(K), \quad \mathcal{R}^{\prime}: \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma) \rightarrow H_{a}^{l+2 m}(K)
$$

by the formulas

$$
\mathcal{R}_{0} f_{0}=\psi_{\varrho} \mathcal{L}^{-1}\left(f_{0}, 0\right), \quad \mathcal{R}^{\prime} f^{\prime}=\psi_{\varrho} \mathcal{L}^{-1}\left(0, f^{\prime}\right)
$$

Thus, the supports of the functions $\mathcal{R}_{0} f_{0}$ and $\mathcal{R}^{\prime} f^{\prime}$ are located in the ball of radius $2 \varrho$ centered at the origin.

Let the operators

$$
\mathcal{P}: H_{a}^{l+2 m}(K) \rightarrow H_{a}^{l}(K), \quad \mathcal{B}, \mathcal{B}^{\Omega}: H_{a}^{l+2 m}(K) \rightarrow \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)
$$

act by the formulas

$$
\mathcal{P} U=\mathcal{P}(D) U, \quad \mathcal{B} U=\left\{\mathcal{B}_{\sigma \mu}(D) U\right\}, \quad \mathcal{B}^{\Omega} U=\left\{\mathcal{B}_{\sigma \mu}^{\Omega}(D) U\right\} .
$$

Let us establish the connection between the operators $\mathcal{P}, \mathcal{B}, \mathcal{B}^{\Omega}$, and $\mathcal{R}_{0}, \mathcal{R}^{\prime}$. We use the following property of weight spaces (see [53, Lemma 3.5]): the embedding operator

$$
\begin{equation*}
\left\{v \in H_{a}^{l+1}(K): \operatorname{supp} v \subset \mathcal{O}_{d}(0), d>0\right\} \subset H_{a}^{l}(K) \tag{4.11}
\end{equation*}
$$

is compact for any $d>0$.
It follows from the Leibnitz formula, the boundedness of the support $\operatorname{supp} \psi_{\varrho}$, and the compactness of embedding (4.11) that

$$
\begin{equation*}
\mathcal{P} \mathcal{R}_{0} f_{0}=\psi_{\varrho} f_{0}+\mathcal{T}_{0} f_{0}, \quad \mathcal{P} \mathcal{R}^{\prime} f^{\prime}=\mathcal{T}^{\prime} f^{\prime} \tag{4.12}
\end{equation*}
$$

where $\mathcal{T}_{0}: H_{a}^{l}(K) \rightarrow H_{a}^{l}(K)$ and $\mathcal{T}^{\prime}: \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma) \rightarrow H_{a}^{l}(K)$ are compact operators. Similarly,

$$
\begin{equation*}
\mathcal{B} \mathcal{R}^{\prime} f^{\prime}=\psi_{\varrho} f^{\prime}+\left\{\left.\left(\psi_{\varrho}\left(\chi_{\sigma} y\right)-\psi_{\varrho}(y)\right)\left(B_{\sigma \mu 1} \mathcal{L}^{-1}\left(0, f^{\prime}\right)\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}\right\}+\mathcal{T}_{1} f^{\prime} \tag{4.13}
\end{equation*}
$$

where $\mathcal{T}_{1}$ is a compact operator in $\mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)$, and the braces show that the expression $\{\ldots\}$ is a vector whose components are defined by indices $\sigma$ and $\mu$.

Let us show that each term of the sum in Eq. (4.13) is a compact operator. Let $\zeta_{i}$ be functions introduced by formulas (3.9). Let us introduce the functions $\hat{\psi}_{0}, \hat{\psi}_{1} \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$,

$$
\begin{array}{cccc}
\hat{\psi}_{1}(y)=1 \quad \text { for } \quad 2 d_{1} \varrho \leq|y| \leq d_{2} \varrho, & \hat{\psi}_{1}(y)=0 \quad \text { outside } \quad d_{1} \varrho \leq|y| \leq 2 d_{2} \varrho \\
\hat{\psi}_{0}(y)=1 \quad \text { for } \quad d_{1} \varrho \leq|y| \leq 2 d_{2} \varrho, & \hat{\psi}_{0}(y)=0 \quad \text { outside } \quad d_{1} \varrho / 2 \leq|y| \leq 4 d_{2} \varrho
\end{array}
$$

where $d_{1}$ and $d_{2}$ are the numbers defined in Eq. (2.2). Then by the continuity of the trace operator in weight spaces, we have

$$
\begin{align*}
& \left\|\left.\left(\psi_{\varrho}\left(\chi_{\sigma} y\right)-\psi_{\varrho}(y)\right)\left(B_{\sigma \mu 1} \mathcal{L}^{-1}\left(0, f^{\prime}\right)\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}\right\|_{H_{a}^{l+2 m-m_{\mu}-1 / 2}\left(\gamma_{\sigma}\right)} \\
& \leq k_{2}\left\|\zeta_{2}\left(\psi_{\varrho}(y)-\psi_{\varrho}\left(\chi_{\sigma}^{-1} y\right)\right) B_{\sigma \mu 1} \mathcal{L}^{-1}\left(0, f^{\prime}\right)\right\|_{H_{a}^{l+2 m-m_{\mu}}(K)} \\
& \leq k_{3}\left\|\zeta_{1} \hat{\psi}_{1} \mathcal{L}^{-1}\left(0, f^{\prime}\right)\right\|_{H_{a}^{l+2 m}(K)} . \tag{4.14}
\end{align*}
$$

Since the support of the function $\hat{\psi}_{1}$ is bounded and separated from zero and the function $\zeta_{1}$ is zero near rays of the angle $K$, we can use [57, Chap. 2, Theorem 5.1]: applying the relation $\mathcal{P} \mathcal{L}^{-1}\left(0, f^{\prime}\right)=0$, we obtain from Eq. (4.14) the following inequality:

$$
\left\|\left.\left(\psi_{\varrho}\left(\chi_{\sigma} y\right)-\psi_{\varrho}(y)\right)\left(B_{\sigma \mu 1} \mathcal{L}^{-1}\left(0, f^{\prime}\right)\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}\right\|_{H_{a}^{l+2 m-m_{\mu}-1 / 2}\left(\gamma_{\sigma}\right)} \leq k_{4}\left\|\hat{\psi}_{0} \mathcal{L}^{-1}\left(0, f^{\prime}\right)\right\|_{H_{a}^{l+2 m-1}(K)}
$$

Since the support of the function $\hat{\psi}_{0}$ is bounded, it follows from the last inequality and the compactness of embedding (4.11) that

$$
\left\{\left.\left(\psi_{\varrho}\left(\chi_{\sigma} y\right)-\psi_{\varrho}(y)\right)\left(B_{\sigma \mu 1} \mathcal{L}^{-1}\left(0, f^{\prime}\right)\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}\right\}
$$

is a compact operator in $\mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)$. Together with Eq. (4.13), this yields

$$
\begin{equation*}
\mathcal{B} \mathcal{R}^{\prime} f^{\prime}=\psi_{\varrho} f^{\prime}+\mathcal{T}_{2} f^{\prime} \tag{4.15}
\end{equation*}
$$

where $\mathcal{T}_{2}$ is a compact operator $\mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)$.
Let us obtain the following formula for the composition $\mathcal{B}^{\Omega} \mathcal{R}^{\prime}$ from Eq. (4.15):

$$
\begin{equation*}
\mathcal{B}^{\Omega} \mathcal{R}^{\prime} f^{\prime}=\psi_{\varrho} f^{\prime}+\mathcal{T}_{2} f^{\prime}+\left\{\left.\left(B_{\sigma \mu 1} \mathcal{R}^{\prime} f^{\prime}\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}}-\left.\left(B_{\sigma \mu 1} \mathcal{R}^{\prime} f^{\prime}\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}\right\} . \tag{4.16}
\end{equation*}
$$

2. Let us introduce the operator $\mathcal{R}: \mathcal{H}_{a}^{l}(K, \gamma) \rightarrow H_{a}^{l+2 m}(K)$, acting by the formula

$$
\mathcal{R}\left(f_{0}, f^{\prime}\right)=\mathcal{R}_{0} f_{0}-\tilde{\mathcal{R}}^{\prime} \mathcal{B}^{\Omega} \mathcal{R}_{0} f_{0}+\mathcal{R}^{\prime} f^{\prime}
$$

Here $\tilde{\mathcal{R}}^{\prime}: \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma) \rightarrow H_{a}^{l+2 m}(K)$ is a bounded operator acting by the formula

$$
\tilde{\mathcal{R}}^{\prime} f^{\prime}=\psi_{\varrho}\left(d_{1} y / 2\right) \mathcal{L}^{-1}\left(0, f^{\prime}\right)
$$

Similarly to Eqs. (4.12) and (4.16), we prove that

$$
\begin{gather*}
\mathcal{P} \tilde{\mathcal{R}}^{\prime} f^{\prime}=\tilde{\mathcal{T}}^{\prime} f^{\prime}  \tag{4.17}\\
\mathcal{B}^{\Omega} \tilde{\mathcal{R}}^{\prime} f^{\prime}=\psi_{\varrho}\left(d_{1} y / 2\right) f^{\prime}+\mathcal{T}_{2}^{\prime} f^{\prime}+\left\{\left.\left(B_{\sigma \mu} \tilde{\mathcal{R}}^{\prime} f^{\prime}\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}}-\left.\left(B_{\sigma \mu 1} \tilde{\mathcal{R}}^{\prime} f^{\prime}\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}\right\}, \tag{4.18}
\end{gather*}
$$

where $\tilde{\mathcal{T}}^{\prime}$ and $\mathcal{T}_{2}^{\prime}$ are compact operators acting in the same spaces as operators $\mathcal{T}^{\prime}, \mathcal{T}_{2}$.
Let us show that $\mathcal{R}$ satisfies relation (4.9). It follows from formulas (4.12) and (4.17) that

$$
\begin{equation*}
\mathcal{P R}\left(f_{0}, f^{\prime}\right)=\psi_{\varrho} f_{0}+\mathcal{T}_{3}\left(f_{0}, f^{\prime}\right) \tag{4.19}
\end{equation*}
$$

where $\mathcal{T}_{3}: \mathcal{H}_{a}^{l}(K, \gamma) \rightarrow H_{a}^{l}(K)$ is a compact operator.
Using the fact that

$$
\psi_{\varrho}\left(d_{1} y / 2\right) \mathcal{B}^{\Omega} \mathcal{R}_{0} f_{0} \equiv \mathcal{B}^{\Omega} \mathcal{R}_{0} f_{0}
$$

and Eq. (4.18), we obtain the following relation:

$$
\begin{align*}
& \mathcal{B}^{\Omega} \mathcal{R}\left(f_{0}, f^{\prime}\right)=\mathcal{B}^{\Omega} \mathcal{R}_{0} f_{0}-\mathcal{B}^{\Omega} \tilde{\mathcal{R}}^{\prime} \mathcal{B}^{\Omega} \mathcal{R}_{0} f_{0}+ \mathcal{B}^{\Omega} \mathcal{R}^{\prime} f^{\prime} \\
&=-\mathcal{T}_{2}^{\prime} \mathcal{B}^{\Omega} \mathcal{R}_{0} f_{0}-\left\{\sum_{k, q, s}\right.( \\
&\left.\left(B_{\sigma \mu 1}\left[\tilde{\mathcal{R}}^{\prime} \mathcal{B}^{\Omega} \mathcal{R}_{0} f_{0}\right]_{k}\right) \Omega_{\sigma}(y)\right|_{\gamma_{\sigma}}  \tag{4.20}\\
&\left.\left.-\left.\left(B_{\sigma \mu 1}\left[\tilde{\mathcal{R}}^{\prime} \mathcal{B}^{\Omega} \mathcal{R}_{0} f_{0}\right]_{k}\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}\right)\right\}+\mathcal{B}^{\Omega} \mathcal{R}^{\prime} f^{\prime} .
\end{align*}
$$

Using Eq. (4.16), we obtain

$$
\begin{aligned}
& \mathcal{B}^{\Omega} \mathcal{R} g=\psi_{\varrho} f^{\prime}+\mathcal{T}_{6}\left(f_{0}, f^{\prime}\right)+\left\{\left.\left(B_{\sigma \mu 1} \mathcal{R}^{\prime} f^{\prime}\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}}-\left.\left(B_{\sigma \mu 1} \mathcal{R}^{\prime} f^{\prime}\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}\right\} \\
&-\left\{\left.\left(B_{\sigma \mu} \tilde{\mathcal{R}}^{\prime} \mathcal{B}^{\Omega} \mathcal{R}_{0} f_{0}\right) \Omega_{\sigma}(y)\right|_{\gamma_{\sigma}}-\left.\left(B_{\sigma \mu 1} \tilde{\mathcal{R}}^{\prime} \mathcal{B}^{\Omega} \mathcal{R}_{0} f_{0}\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}\right\},
\end{aligned}
$$

where $\mathcal{T}_{6}: \mathcal{H}_{a}^{l}(K, \gamma) \rightarrow \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)$ is a compact operator.
Let us consider the relation under the sum sign on the right-hand side of Eq. (4.20). By Lemma 3.5, we have

$$
\begin{align*}
\|\left.\left(B_{\sigma \mu 1} \mathcal{R}^{\prime} f^{\prime}\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}}-\left(B_{\sigma \mu 1} \mathcal{R}^{\prime} f^{\prime}\right) & \left.\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}} \|_{H_{a}^{l+2 m-m_{\mu}-1 / 2}\left(\gamma_{\sigma}\right)} \\
& \leq k_{5}\left(\varrho\left\|\mathcal{R}^{\prime} f^{\prime}\right\|_{H_{a}^{l+2 m}(K)}+\left\|\zeta_{3} \mathcal{R}^{\prime} f^{\prime}-\zeta_{3} \widehat{\mathcal{R}^{\prime} f^{\prime}}\right\|_{H_{a}^{l+2 m}(K)}\right) . \tag{4.21}
\end{align*}
$$

Repeating the reasoning of Eqs. (4.3)-(4.8) for $U=\mathcal{R}^{\prime} f^{\prime}$, from Eqs. (4.21) and (4.12) we obtain the following:

$$
\begin{aligned}
& \left\|\left.\left(B_{\sigma \mu 1} \mathcal{R}^{\prime} f^{\prime}\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}}-\left.\left(B_{\sigma \mu 1} \mathcal{R}^{\prime} f^{\prime}\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}\right\|_{H_{a}^{l+2 m-m_{\mu}-1 / 2}\left(\gamma_{\sigma}\right)} \\
& \leq k_{6}\left(\varrho\left\|\mathcal{R}^{\prime} f^{\prime}\right\|_{H_{a}^{l+2 m}(K)}+\left\|\mathcal{P} \mathcal{R}^{\prime} f^{\prime}\right\|_{H_{a}^{l}(K)}\right) \\
& \quad=k_{6}\left(\varrho\left\|\psi_{\varrho} \mathcal{L}^{-1}\left(0, f^{\prime}\right)\right\|_{H_{a}^{l+2 m}(K)}+\left\|\mathcal{T}^{\prime} f^{\prime}\right\|_{H_{a}^{l}(K)}\right) .
\end{aligned}
$$

This relation, inequality (4.10), and the boundedness of the operator $\mathcal{L}^{-1}: \mathcal{H}_{a}^{l}(K, \gamma) \rightarrow H_{a}^{l+2 m}(K)$ yield

$$
\begin{align*}
&\left\|\left.\left(B_{\sigma \mu 1} \mathcal{R}^{\prime} f^{\prime}\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}}-\left.\left(B_{\sigma \mu 1} \mathcal{R}^{\prime} f^{\prime}\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}\right\|_{H_{a}^{l+2 m-m \mu-1 / 2}\left(\gamma_{\sigma}\right)} \\
& \leq k_{7}\left(\varrho\left\|f^{\prime}\right\|_{\mathcal{H}_{a}^{l+2 m-\mathrm{m}-1 / 2}(\gamma)}+\left\|\mathcal{T}^{\prime} f^{\prime}\right\|_{H_{a}^{l}(K)}\right) . \tag{4.22}
\end{align*}
$$

Hence, by Lemma 1.1,

$$
\left.\left(B_{\sigma \mu 1} \mathcal{R}^{\prime} f^{\prime}\right)\left(\Omega_{\sigma}(y)\right)\right|_{\gamma_{\sigma}}-\left.\left(B_{\sigma \mu 1} \mathcal{R}^{\prime} f^{\prime}\right)\left(\mathcal{G}_{\sigma} y\right)\right|_{\gamma_{\sigma}}=\mathcal{M}_{\sigma \mu} f^{\prime}+\mathcal{F}_{\sigma \mu} f^{\prime}
$$

where

$$
\mathcal{M}_{\sigma \mu}, \mathcal{F}_{\sigma \mu}: \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma) \rightarrow H_{a}^{l+2 m-m_{\mu}-1 / 2}\left(\gamma_{\sigma}\right)
$$

are bounded operators; moreover, $\left\|\mathcal{M}_{\sigma \mu}\right\| \leq 2 k_{7} \varrho$, and the operator $\mathcal{F}_{\sigma \mu}$ is finite-dimensional.
Similarly, we prove that the summands in the second sum on the right-hand side of (4.20) can be represented in the form "an operator with a small norm + a finite-dimensional operator." This statement, Eqs. (4.20) and (4.19), and the assumption that $\operatorname{supp}\left(f_{0}, f^{\prime}\right) \subset \mathcal{O}_{\varrho}(0)$ yield Eq. (4.9).

Now we can prove the Fredholm property of the operator $\mathbf{L}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G)$.
Theorem 4.1. Let conditions 2.1-2.4 and 4.1 hold. Then $\mathbf{L}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G)$ is a Fredholm operator.
Proof. The Fredholm property of the operator $\mathbf{L}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G)$ follows form Lemmas 4.1 and 4.2 and [56, Theorems 7.1 and 15.2].

### 4.3. Stability of the index with respect to nonlinear perturbations of transformations.

 Show that the index of the problem is defined only by a linear part of transformations $\Omega_{\sigma}$ in a neighborhood of the origin.Denote transformations with the same properties as $\Omega_{\sigma}$ by $\hat{\Omega}_{\sigma}, \sigma=1,2$ (see Sec. 2). Let us consider the operators

$$
\hat{\mathbf{B}}_{\mu}^{1} u= \begin{cases}\left.\left(B_{\mu 1}(y, D)(\zeta u)\right)\left(\hat{\Omega}_{\sigma}(y)\right)\right|_{\gamma_{\sigma}^{\varepsilon}}, & y \in \gamma_{\sigma}^{\varepsilon}, \\ 0, & y \in \partial G \backslash \mathcal{O}_{\varepsilon}(0) .\end{cases}
$$

Introduce the operators $\hat{\mathbf{B}}_{\mu}=\mathbf{B}_{\mu}^{0}+\hat{\mathbf{B}}_{\mu}^{1}+\mathbf{B}_{\mu}^{2}$ and

$$
\hat{\mathbf{L}}=\left\{\mathbf{P}(y, D), \hat{\mathbf{B}}_{\mu}\right\}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G)
$$

(cf. (2.8)).
Theorem 4.2. Let conditions 2.1-2.4 and 4.1 hold; moreover, let condition 2.3 be valid for transformations $\Omega_{\sigma}$ and $\hat{\Omega}_{\sigma}$ with the same linear operator $\mathcal{G}_{\sigma}$. Then the operators $\mathbf{L}, \hat{\mathbf{L}}: H_{a}^{l+2 m}(G) \rightarrow$ $\mathcal{H}_{a}^{l}(G, \partial G)$ are Fredholm operators and ind $\mathbf{L}=\operatorname{ind} \hat{\mathbf{L}}$.
Proof. Let us introduce operators $\mathbf{L}_{t}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G)$ by the formula

$$
\mathbf{L}_{t} u=\left\{\mathbf{P}(y, D) u, \mathbf{B}_{\mu}+t\left(\hat{\mathbf{B}}_{\mu}-\mathbf{B}_{\mu}\right\} .\right.
$$

Obviously, $\mathbf{L}_{0}=\mathbf{L}, \mathbf{L}_{1}=\hat{\mathbf{L}}$.

The transformations $\Omega_{\sigma}$ and $\hat{\Omega}_{\sigma}$ coincide with accuracy up to infinitesimals in a neighborhood of the origin; therefore, by Theorem 4.1, the operators $\mathbf{L}_{t}$ are Fredholm operators for all $t$. For any $t_{0}$ and $t$,

$$
\left\|\mathbf{L}_{t} u-\mathbf{L}_{t_{0}} u\right\|_{\mathcal{H}_{a}^{l}(G, \partial G)} \leq k_{t_{0}}\left|t-t_{0}\right| \cdot\|u\|_{H_{a}^{l+2 m}(G)},
$$

where $k_{t_{0}}>0$ is independent of $t \in[0,1]$. Hence, by [56, Theorem 16.2], we have ind $\mathbf{L}_{t}=\operatorname{ind} \mathbf{L}_{t_{0}}$ for all $t$ from some sufficiently small neighborhood of the point $t_{0}$. This neighborhoods cover the interval $[0,1]$. Extracting a finite sub-covering, we obtain ind $\mathbf{L}=\operatorname{ind} \mathbf{L}_{0}=\operatorname{ind} \mathbf{L}_{1}=\operatorname{ind} \hat{\mathbf{L}}$.
Remark 4.1. The results of this chapter can be generalized to the case where the domain $G$ contains $\mathbb{R}^{n}, n \geq 2$, the boundary of the domain consists of $N$ open and connected (in the topology of $\partial G$ ) ( $n-1$ )-dimensional manifolds $\Gamma_{i}$ of the class $C^{\infty}$, and the domain $G$ is diffeomorphic to a $n$-dimensional dihedral angle (plane angle if $n=2$ ) in a neighborhood of every point $g \in \partial G \backslash \bigcup_{i=1}^{N} \Gamma_{i}$. The point $g \in \partial G \backslash \bigcup_{i=1}^{N} \Gamma_{i}$ is not necessarily a fixed point of transformations from nonlocal conditions, but it has a finite orbit (see [25] of details).

## Chapter 2

## STRONG SOLUTIONS OF NONLOCAL ELLIPTIC PROBLEMS IN PLANE ANGLES IN SOBOLEV SPACES

## 5. Functional Spaces

In this section, we introduce the notation and define functional spaces that will be used in Chaps. 26.

### 5.1. Spaces of continued functions.

5.1.1. Domain $G$ and angles $K_{j}$. Denote by $G \subset \mathbb{R}^{2}$ a bounded domain with boundary $\partial G$. Introduce the set $\mathcal{K} \subset \partial G$, consisting of $N$ points. Assume that $\partial G \backslash \mathcal{K}=\bigcup_{i=1}^{N} \Gamma_{i}$, where $\Gamma_{i}$ are open (in the topology of $\partial G$ ) curves of class $C^{\infty}$. For simplicity, we assume that the number of points of the set $\mathcal{K}$ is equal to the number of curves $\Gamma_{i}$ (all results can be easily generalized to the general case). Assume that in a neighborhood of each point $g_{j} \in \mathcal{K}$, the domain $G$ coincides with a plane angle

$$
K_{j}=\left\{y \in \mathbb{R}^{2}: r>0,|\omega|<\omega_{j}\right\}, \quad j=1, \ldots, N
$$

we denote the rays of this angle by

$$
\gamma_{j \sigma}=\left\{y \in \mathbb{R}^{2}: r>0, \omega=(-1)^{\sigma} \omega_{j}\right\}, \quad j=1, \ldots, N, \quad \sigma=1,2,
$$

where $\omega$ and $r$ are the polar coordinates with pole at the point $g_{j}, 0<\omega_{j}<\pi$.
5.1.2. Spaces of continuous functions. Let $X$ and $M$ be closed sets, where the set $X$ is nonempty. In addition to the spaces introduced in Sec. 1.2, we consider the spaces

$$
\begin{equation*}
C_{M}(X)=\{u \in C(X): u(y)=0, y \in X \cap M\} \tag{5.1}
\end{equation*}
$$

(if $X \cap M=\varnothing$, then we assume that $C_{M}(X)=C(X)$ ).
Let us also introduce the space of vector-valued functions

$$
\mathcal{C}_{M}(\partial G)=\prod_{i=1}^{N} C_{M}\left(\overline{\overline{\Gamma_{i}}}\right)
$$

with the norm

$$
\|\psi\|_{\mathcal{C}_{M}(\partial G)}=\max _{i=1, \ldots, N} \max _{y \in \overline{\Gamma_{i}}}\left\|\psi_{i}\right\|_{C\left(\overline{\Gamma_{i}}\right)}, \quad \psi=\left\{\psi_{i}\right\}, \quad \psi_{i} \in C_{M}\left(\overline{\Gamma_{i}}\right) .
$$

### 5.2. Sobolev spaces.

5.2.1. Spaces $W$ of scalar functions. Let us prove the following result for functions of the space introduced in Sec. 1.2.

Lemma 5.1. Let $f \in W^{l}\left(\mathbb{R}^{2}\right)$ and $D^{\alpha} f(0)=0,|\alpha| \leq l-2$, if $l \geq 2$. Then there exists a sequence $\left\{f^{s}\right\} \subset C_{0}^{\infty}\left(\mathbb{R}^{2}\right), s=1,2, \ldots$, such that $f^{s}(y)=0$ in some neighborhood of the origin (the neighborhood depends on s) and $f^{s} \rightarrow f$ in $W^{l}\left(\mathbb{R}^{2}\right)$.

Proof. As is known, the set $C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ is dense in $W^{l}\left(\mathbb{R}^{2}\right)$. On the other hand, the set

$$
\left\{u \in W^{l}\left(\mathbb{R}^{2}\right): D^{\alpha} u(0)=0,|\alpha| \leq l-2\right\}
$$

is a closed, finite-dimensional subspace in $W^{l}\left(\mathbb{R}^{2}\right)$ by the Sobolev embedding theorem and the Riesz theorem on the general form of linear continuous functionals in Hilbert spaces. Hence, by [56, Lemma 8.1], the set

$$
C_{0}^{\infty}\left(\mathbb{R}^{2}\right) \cap\left\{u \in W^{l}\left(\mathbb{R}^{2}\right): D^{\alpha} u(0)=0,|\alpha| \leq l-2\right\}
$$

is dense in $\left\{u \in W^{l}\left(\mathbb{R}^{2}\right): D^{\alpha} u(0)=0,|\alpha| \leq l-2\right\}$. Thus, it suffices to prove the lemma for a function $f \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ such that $D^{\alpha} f(0)=0,|\alpha| \leq l-2$.

Consider a function $\xi \in C_{0}^{\infty}[0, \infty)$ such that $0 \leq \xi(t) \leq 1$ and $\xi(t)=1$ for $t<1$ and $\xi(t)=0$ for $t>2$. Assume that

$$
\xi^{s}(y)=\xi\left(-\frac{\ln r}{s}\right)
$$

where $r=|y|$ and $s=1,2, \ldots$. Obviously, $0 \leq \xi^{s}(y) \leq 1, \xi^{s}(y)=0$ for $|y|<e^{-2 s}, \xi^{s}(y)=1$ for $|y|>e^{-s}$, and $\left|D^{\alpha} \xi^{s}(y)\right| \leq c_{\alpha} /\left(r^{|\alpha|} s\right)$ for any $|\alpha| \geq 1$, where $c_{\alpha}>0$ is independent of $s$ and $y$. It can be directly verified that the sequence $\xi^{s} f$ converges to $f$ in $W^{l}\left(\mathbb{R}^{2}\right)$ as $s \rightarrow \infty$.

For the domain $G$ and the part of the domain $\Gamma_{i}$ described in Sec. 5.1, we introduce Sobolev spaces of negative orders. For $l \geq 0$, let us denote the space dual ${ }^{1}$ to $W^{l}(G)$ with respect to the extension of the inner product in $L_{2}(G)$ by $W^{-l}(G) \stackrel{\text { def }}{=}\left(W^{l}(G)\right)^{*}$. The norm in $W^{-l}(G)$ is defined as follows:

$$
\|u\|_{W^{-l}(G)}=\sup _{0 \neq v \in W^{l}(G)} \frac{|\langle v, u\rangle|}{\|v\|_{W^{l}(G)}} .
$$

For $l \geq 1$, we denote the space dual to $W^{l-1 / 2}\left(\Gamma_{i}\right)$ with respect to the extension of the inner product in $L_{2}\left(\Gamma_{i}\right)$ by $W^{-(l-1 / 2)}\left(\Gamma_{i}\right) \stackrel{\text { def }}{=}\left(W^{l-1 / 2}\left(\Gamma_{i}\right)\right)^{*}$. The norm in $W^{-(l-1 / 2)}\left(\Gamma_{i}\right)$ is defined as follows:

$$
\|g\|_{W^{-(l-1 / 2)}\left(\Gamma_{i}\right)}=\sup _{0 \neq \psi \in W^{l-1 / 2}\left(\Gamma_{i}\right)} \frac{|\langle\psi, g\rangle|}{\|\psi\|_{W^{l-1 / 2}\left(\Gamma_{i}\right)}} .
$$

[^0]5.2.2. Spaces $W$ of vector-valued functions. For any $l \geq 0$ and integer $m_{i \mu}\left(0 \leq m_{i \mu} \leq l+2 m-1\right)$, we introduce the following spaces of vector-valued functions:
\[

$$
\begin{gather*}
\mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)=\prod_{i=1}^{N} \prod_{\mu=1}^{m} W^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)  \tag{5.2}\\
\mathcal{W}^{l}(G, \partial G)=W^{l}(G) \times \mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)
\end{gather*}
$$
\]

(in what follows, the number $m \geq 1$ is an integer).
In Sec. 22 (see Chap. 5), nonlocal perturbations of the Dirichlet problem for the Laplace operator are considered. In this case, $m=1, m_{i 1}=0$, and we denote spaces (5.2) as follows:

$$
\begin{equation*}
\mathcal{W}^{l+3 / 2}(\partial G)=\prod_{i=1}^{N} W^{l+3 / 2}\left(\Gamma_{i}\right), \quad \mathcal{W}^{l}(G, \partial G)=W^{l}(G) \times \mathcal{W}^{l+3 / 2}(\partial G) \tag{5.3}
\end{equation*}
$$

For any $l \geq 0$ and integer $m_{j \sigma \mu}\left(0 \leq m_{j \sigma \mu} \leq l+2 m-1\right)$, we introduce the spaces of vector-valued functions in plane angles

$$
\begin{gathered}
\mathcal{W}^{l}(K)=\prod_{j=1}^{N} W^{l}\left(K_{j}\right), \quad \mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)=\prod_{j=1}^{N} \prod_{\sigma=1,2} \prod_{\mu=1}^{m} W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right) \\
\mathcal{W}^{l}(K, \gamma)=\mathcal{W}^{l}(K) \times \mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)
\end{gathered}
$$

and the spaces of vector-valued functions on arcs

$$
\mathcal{W}^{l}(-\bar{\omega}, \bar{\omega})=\prod_{j=1}^{N} W^{l}\left(-\omega_{j}, \omega_{j}\right), \quad \mathcal{W}^{l}[-\bar{\omega}, \bar{\omega}]=\prod_{j=1}^{N}\left(W^{l}\left(-\omega_{j}, \omega_{j}\right) \times \mathbb{C}^{2 m}\right)
$$

where numbers $\omega_{j}$ determine spreads of angles $K_{j}$.
For any set $M$ and any $d>0$, we denote by $\mathcal{O}_{d}(M)$ a $d$-neighborhood of the set $M$ :

$$
\mathcal{O}_{d}(M)=\left\{y \in \mathbb{R}^{2}: \operatorname{dist}(y, M)<d\right\},
$$

where $\operatorname{dist}(y, M)=\inf _{\eta \in M}|y-\eta|$.
For any $d>0$, we set

$$
K_{j}^{d}=K_{j} \cap \mathcal{O}_{d}(\mathcal{K}), \quad \gamma_{j \sigma}^{d}=\gamma_{j \sigma} \cap \mathcal{O}_{d}(\mathcal{K}) .
$$

For any $d>0, l \geq 0$, and integers $m_{j \sigma \mu}\left(0 \leq m_{j \sigma \mu} \leq l+2 m-1\right)$, we introduce the spaces of vector-valued functions

$$
\begin{gathered}
\mathcal{W}^{l}\left(K^{d}\right)=\prod_{j=1}^{N} W^{l}\left(K_{j}^{d}\right), \quad \mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}\left(\gamma^{d}\right)=\prod_{j=1}^{N} \prod_{\sigma=1,2} \prod_{\mu=1}^{m} W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}^{d}\right), \\
\mathcal{W}^{l}\left(K^{d}, \gamma^{d}\right)=\mathcal{W}^{l}\left(K^{d}\right) \times \mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}\left(\gamma^{d}\right) .
\end{gathered}
$$

5.2.3. Spaces $S$ of scalar functions. For any $l \geq 2$, we denote by $S^{l}(G)$ the subspace of the space $W^{l}(G)$ consisting of functions $f_{0}$ that satisfy the relations

$$
\begin{equation*}
D^{\alpha} f_{0}(y)=0, \quad y \in \mathcal{K},|\alpha| \leq l-2 . \tag{5.4}
\end{equation*}
$$

For $l=0,1$, we assume that $S^{l}(G)=W^{l}(G)$.
For each curve $\Gamma_{i}, i=1, \ldots, N$, we denote its endpoints by $g_{i 1}$ and $g_{i 2}$. Recall that in some neighborhood of a point $g_{i 1}$ (respectively, $g_{i 2}$ ), the domain $G$ coincides with a plane angle and the curve $\Gamma_{i}$ coincides with the interval $I_{i 1}$ (respectively, $I_{i 2}$ ). Let $\tau_{i 1}$ (respectively, $\tau_{i 2}$ ) be a unit vector, which is parallel to the interval $I_{i 1}$ (respectively, $I_{i 2}$ ). Introduce the notation

$$
D_{\tau_{i 1}}=-i \frac{\partial}{\partial \tau_{i 1}}, \quad D_{\tau_{i 2}}=-i \frac{\partial}{\partial \tau_{i 2}}
$$

For any $l \geq 0$, we denote by $\mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)$ the subspace of the space $\mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)$ that consists of functions $\left\{f_{i \mu}\right\}$ such that

$$
\begin{equation*}
\left.D_{\tau_{i 1}}^{\beta} f_{i \mu}\right|_{y=g_{i 1}}=0,\left.\quad D_{\tau_{i 2}}^{\beta} f_{i \mu}\right|_{y=g_{i 2}}=0, \quad \beta \leq l+2 m-m_{i \mu}-2 . \tag{5.5}
\end{equation*}
$$

Assume that $\mathcal{S}^{l}(G, \partial G)=S^{l}(G) \times \mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)$.
5.2.4. Spaces $S$ of vector-valued functions. For any $l \geq 2$, we denote by $\mathcal{S}^{l}(K)$ the subspace of the space $\mathcal{W}^{l}(K)$ consisting of functions $\left\{f_{j}\right\}$ such that

$$
\begin{equation*}
\left.D^{\alpha} f_{j}\right|_{y=0}=0, \quad|\alpha| \leq l-2 \tag{5.6}
\end{equation*}
$$

Assume that $\mathcal{S}^{l}(K)=\mathcal{W}^{l}(K)$ for $l=0,1$.
For any $l \geq 0$, we denote by $\mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)$ the subspace of the space $\mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)$ consisting of functions $\left\{f_{j \sigma \mu}\right\}$ such that

$$
\begin{equation*}
\left.D_{\tau_{j \sigma}}^{\beta} f_{j \sigma \mu}\right|_{y=0}=0, \quad \beta \leq l+2 m-m_{j \sigma \mu}-2 . \tag{5.7}
\end{equation*}
$$

Here $\tau_{j \sigma}$ is the unit vector that has the same direction as the ray $\gamma_{j \sigma}, D_{\tau_{j \sigma}}^{\beta}=\frac{\partial^{\beta}}{\partial \tau_{j \sigma}^{\beta}}$. If $l+2 m-m_{j \sigma \mu}-2<$ 0 , then there are no corresponding conditions.

Let $\mathcal{S}^{l}(K, \gamma)=\mathcal{S}^{l}(K) \times \mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)$.
The spaces

$$
\mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}\left(\gamma^{d}\right), \quad \mathcal{S}^{l}\left(K^{d}\right), \quad \mathcal{S}^{l}\left(K^{d}, \gamma^{d}\right), \quad d>0
$$

are introduced similarly.

### 5.3. Kondrat'ev weight spaces.

5.3.1. Spaces of scalar functions. Let us consider the following cases:
(1) $Q=K_{j}, Q=K_{j}^{d}(d>0)$, or $Q=\mathbb{R}^{2}$; we set $\mathcal{M}=\{0\}$;
(2) $Q=G$; we set $\mathcal{M}=\mathcal{K}$.

For any $l \geq 0$ and any $a \in \mathbb{R}$, we denote by $H_{a}^{l}(Q)=H_{a}^{l}(Q, \mathcal{M})$ the completion of the set $C_{0}^{\infty}(\bar{Q} \backslash \mathcal{M})$ with respect to the norm

$$
\|u\|_{H_{a}^{l}(Q)}=\left(\sum_{|\alpha| \leq l} \int_{Q} \rho^{2(a-l+|\alpha|)}\left|D^{\alpha} u\right|^{2} d y\right)^{1 / 2}
$$

where $\rho=\rho(y)=\operatorname{dist}(y, \mathcal{M})$. For $l \geq 1$, we denote the trace space on the smooth curve $\Gamma \subset \bar{Q}$ with the norm

$$
\|\psi\|_{H_{a}^{l-1 / 2}(\Gamma)}=\inf \|u\|_{H_{a}^{l}(Q)}, \quad u \in H_{a}^{l}(Q):\left.u\right|_{\Gamma}=\psi
$$

by $H_{a}^{l-1 / 2}(\Gamma)$. In Chap. 6, we use, in addition to the standard norms, norms in weight spaces depending on a parameter $q>0$. Assume that

$$
\begin{align*}
\|u\|_{H_{a}^{l}(G)} & =\left(\|u\|_{H_{a}^{l}(G)}^{2}+q^{l}\|u\|_{H_{a}^{0}(G)}^{2}\right)^{1 / 2}, & & l \geq 0, \\
\|v\|_{H_{a}^{l-1 / 2}\left(\Gamma_{i}\right)} & =\left(\|v\|_{H_{a}^{l-1 / 2}\left(\Gamma_{i}\right)}^{2}+q^{l-1 / 2}\|v\|_{H_{a}^{0}\left(\Gamma_{i}\right)}^{2}\right)^{1 / 2}, & & l \geq 1, \tag{5.8}
\end{align*}
$$

where

$$
\|v\|_{H_{a}^{0}\left(\Gamma_{i}\right)}=\left(\int_{\Gamma_{i}} \rho^{2 a}|v(y)|^{1 / 2} d \Gamma\right)^{1 / 2}
$$

It follows form [41, Lemmas 7.1 and 7.2] that

$$
\begin{equation*}
\left\|\left.u\right|_{\Gamma_{i}}\right\|_{H_{a}^{l-1 / 2}\left(\Gamma_{i}\right)} \leq c\|u\|_{H_{a}^{l}(G)} \quad \forall u \in H_{a}^{l}(G) \tag{5.9}
\end{equation*}
$$

where $c>0$ is independent of $u$ and $q>0$.
We prove some auxiliary results on the connection between Sobolev spaces and weight Kondrat'ev spaces. Let us fix an arbitrary index $i$. Let $\Gamma=\Gamma_{i}$ and $g \in \bar{\Gamma} \backslash \Gamma$. Without loss of generality, we assume that $g=0$ and $\Gamma$ coincides with the axis $O y_{1}$ in a sufficiently small neighborhood $\mathcal{O}_{\varepsilon}(0)$ of the origin. Introduce the notation $G^{\varepsilon}=G \cap \mathcal{O}_{\varepsilon}(0)$ and $\Gamma^{\varepsilon}=\Gamma \cap \mathcal{O}_{\varepsilon}(0)$ and set $H_{a}^{l}\left(G^{\varepsilon}\right)=H_{a}^{l}\left(G^{\varepsilon},\{0\}\right)$.
Lemma 5.2. If $u \in W^{l}\left(G^{\varepsilon}\right)$ and $l \geq 1$, then the following statements are valid:
(1) $u(y)=P(y)+v(y)$ for $y \in G^{\varepsilon}$, where $P(y)=\sum_{|\alpha| \leq l-2} p_{\alpha} y^{\alpha}$ and $v \in W^{l}\left(G^{\varepsilon}\right) \cap H_{\delta}^{l}\left(G^{\varepsilon}\right)$ for all $\delta>0($ if $l=1$, we assume that $P(y) \equiv 0)$; in particular, $u \in H_{l-1+\delta}^{l}\left(G^{\varepsilon}\right) ;$
(2) $\left.D^{\alpha} u\right|_{y=0}=\left.D^{\alpha} P\right|_{y=0}$ for $|\alpha| \leq l-2$ (if $\left.l \geq 2\right)$;
(3) $\sum_{|\alpha| \leq l-2}\left|p_{\alpha}\right|+\|v\|_{H_{\delta}^{l}\left(G^{\varepsilon}\right)} \leq c_{\delta}\|u\|_{W^{l}\left(G^{\varepsilon}\right)}$, where $c_{\delta}>0$ is independent of $u$.

The proof follows from [53, Lemma 4.9] for $l=1$ and [53, Lemma 4.11] for $l \geq 2$.
Lemma 5.3. If $\psi \in W^{l-1 / 2}\left(\Gamma^{\varepsilon}\right)$ and $l \geq 1$, then the following statements hold:
(1) $\psi(r)=P_{1}(r)+\varphi(r)$ for $0<r<\varepsilon$, where $P_{1}(r)=\sum_{\beta=0}^{l-2} p_{\beta} r^{\beta}$ and $\varphi \in W^{l-1 / 2}\left(\Gamma^{\varepsilon}\right) \cap H_{\delta}^{l-1 / 2}\left(\Gamma^{\varepsilon}\right)$ for all $\delta>0\left(\right.$ if $l=1$, we assume $\left.P_{1}(r) \equiv 0\right)$; in particular, $\psi \in H_{l-1+\delta}^{l-1 / 2}\left(\Gamma^{\varepsilon}\right)$;
(2) $\left.\left(d^{\beta} \psi / d r^{\beta}\right)\right|_{r=0}=\left.\left(d^{\beta} P_{1} / d r^{\beta}\right)\right|_{r=0}$ for $\beta=0, \ldots, l-2$;
(3) $\sum_{\beta=0}^{l-2}\left|p_{\beta}\right|+\|\varphi\|_{H_{\delta}^{l-1 / 2}\left(\Gamma^{\varepsilon}\right)} \leq c_{\delta}\|\psi\|_{W^{l-1 / 2}\left(\Gamma^{\varepsilon}\right)}$, where $c_{\delta}>0$ is independent of $\psi$.

Proof. Consider a function $u \in W^{l}\left(G^{\varepsilon}\right)$ such that

$$
\left.u\right|_{\Gamma^{\varepsilon}}=\psi, \quad\|u\|_{W^{l}\left(G^{\varepsilon}\right)} \leq 2\|\psi\|_{W^{l-1 / 2}\left(\Gamma^{\varepsilon}\right)}
$$

and apply Lemma 5.2.
Lemma 5.4. Let $\psi \in W^{l-1 / 2}\left(\Gamma^{\varepsilon}\right), l \geq 2$, and let

$$
\begin{equation*}
\left.\frac{d^{s} \psi}{d r^{s}}\right|_{y=0}=0, \quad s=0, \ldots, k \tag{5.10}
\end{equation*}
$$

for a fixed $k \leq l-2$. Then $\psi \in H_{l-2-k+\delta}^{l-1 / 2}\left(\Gamma^{\varepsilon}\right)$ for all $\delta>0$ and

$$
\begin{equation*}
\|\psi\|_{H_{l-2-k+\delta}^{l-1 / 2}\left(\Gamma^{\varepsilon}\right)} \leq c \delta\|\psi\|_{W^{l-1 / 2}\left(\Gamma^{\varepsilon}\right)} \tag{5.11}
\end{equation*}
$$

where $c_{\delta}>0$ is independent of $\psi$.
Proof. It follows from Eq. (5.10) and Lemma 5.3 (items 1 and 2) that

$$
\begin{equation*}
\psi(r)=\sum_{\beta=k+1}^{l-2} p_{\beta} r^{\beta}+\varphi(r), \quad 0<r<\varepsilon, \tag{5.12}
\end{equation*}
$$

where

$$
\begin{equation*}
\varphi \in H_{\delta}^{l-1 / 2}\left(\Gamma^{\varepsilon}\right) \subset H_{l-2-k+\delta}^{l-1 / 2}\left(\Gamma^{\varepsilon}\right), \quad \delta>0 . \tag{5.13}
\end{equation*}
$$

If $k=l-2$, then there is no sum in (5.12) and the statement of the lemma follows from Eq. (5.13) and Lemma 5.3 (item 3).

If $k \leq l-3$, then the sum consists of summands of the form $r^{\beta}$, where $\beta \geq k+1$. It is directly verified that $r^{\beta} \in H_{l-2-k+\delta}^{l-1 / 2}\left(\Gamma^{\varepsilon}\right)$ for the $\beta$ mentioned and any $\delta>0$. Hence, the statement of the lemma follows from Eqs. (5.12) and (5.13) and Lemma 5.3 (item 3).

Lemma 5.5. Let $u \in H_{a+k}^{l-1 / 2}(\Gamma), k, l \in \mathbb{N}, a \in \mathbb{R}, B \in C^{\infty}\left(\overline{G^{\varepsilon}}\right),\left.D^{\alpha} B\right|_{y=0}=0$, and $|\alpha| \leq k-1$. Then

$$
\|B u\|_{H_{a}^{l}\left(G^{\varepsilon}\right)} \leq c\|u\|_{H_{a+k}^{l-1 / 2}\left(G^{\varepsilon}\right)}
$$

Proof. It follows from the Taylor formula that $\left|D^{\sigma} B\right|=O\left(r^{k-|\sigma|}\right)$ for $\sigma$; hence,

$$
\begin{aligned}
& \|B u\|_{H_{a}^{l}\left(G^{\varepsilon}\right)}^{2}=\sum_{|\alpha| \leq l} \int_{G^{\varepsilon}} r^{2(a+|\alpha|-l)}\left|D^{\alpha}(B u)\right|^{2} d y \\
& \quad \leq c_{2} \sum_{|\sigma|+|\zeta| \leq l} \int_{G^{\varepsilon}} r^{2(a+|\sigma|+|\zeta|-l)}\left|D^{\sigma} B\right|^{2}\left|D^{\zeta} u\right|^{2} d y \\
& \quad \leq c_{3} \sum_{|\zeta| \leq l} \int_{G^{\varepsilon}} r^{2(a+k+|\zeta|-l)}\left|D^{\zeta} u\right|^{2} d y=c_{3}\|u\|_{H_{a+k}^{l}(K)}^{2}
\end{aligned}
$$

The lemma is proved.
Lemma 5.6. Let $\psi \in H_{a+k}^{l-1 / 2}\left(\Gamma^{\varepsilon}\right), k, l \in \mathbb{N}, a \in \mathbb{R}, b \in C^{\infty}\left(\overline{\Gamma^{\varepsilon}}\right),\left.\frac{\partial^{s} b}{\partial r^{s}}\right|_{r=0}=0$, and $s=0, \ldots k-1$. Then

$$
\begin{equation*}
\|b \psi\|_{H_{a}^{l-1 / 2}(\Gamma)} \leq c\|\psi\|_{H_{a+k}^{l-1 / 2}(\Gamma)} \tag{5.14}
\end{equation*}
$$

Proof. Denote the extension of the function $b\left(y_{1}\right)$ to $\mathbb{R}$ by $\hat{b} \in C^{\infty}(\mathbb{R})$ and introduce a function $B\left(y_{1}, y_{2}\right)=\hat{b}\left(y_{1}\right),\left(y_{1}, y_{2}\right) \in \mathbb{R}^{2}$. Obviously,

$$
\begin{equation*}
B \in C^{\infty}(\bar{K}),\left.\quad D^{\sigma} B\right|_{y=0}=0, \quad|\sigma| \leq k-1 \tag{5.15}
\end{equation*}
$$

Let $u \in H_{a+k}^{l}\left(G^{\varepsilon}\right)$ be an extension of the function $\psi$ such that

$$
\begin{equation*}
\|u\|_{H_{a+k}^{l}(K)} \leq c_{1}\|\psi\|_{H_{a+k}^{l-1 / 2}(\gamma)}, \tag{5.16}
\end{equation*}
$$

where $c_{1}>0$ is independent of $\psi$. Now Eqs. (5.15) and (5.16) and Lemma 5.5 yield the statement of the lemma.
Lemma 5.7. Let $u \in W^{1}\left(\mathbb{R}^{2}\right)$ and $u(y)=0$ as $|y| \geq 1$. Then

$$
\left\|u(y)-u\left(\mathcal{G}_{0} y\right)\right\|_{H_{0}^{1}\left(\mathbb{R}^{2}\right)} \leq c\|u\|_{W^{1}\left(\mathbb{R}^{2}\right)}
$$

where $\mathcal{G}_{0}$ is a composition of the operator of rotation by the angle $\omega_{0}\left(-\pi<\omega_{0} \leq \pi\right)$ and dilation centered at the origin with coefficient $\chi_{0}>0$.
Proof. Let us write the function $u$ in the polar coordinates $(\omega, r)$; then we have

$$
u(y)-u\left(\mathcal{G}_{0} y\right)=u(\omega, r)-u\left(\omega+\omega_{0}, \chi_{0} r\right)=v_{1}+v_{2},
$$

where $v_{1}(\omega, r)=u(\omega, r)-u\left(\omega+\omega_{0}, r\right)$ and $v_{2}(\omega, r)=u\left(\omega+\omega_{0}, r\right)-u\left(\omega+\omega_{0}, \chi_{0} r\right)$.
Consider the function $v_{1}$. By [53, Lemma 4.15],

$$
\int_{0}^{\infty} r^{-1}\left|v_{1}(0, r)\right|^{2} d r \leq k_{1}\|u\|_{W^{1}\left(\mathbb{R}^{2}\right)}
$$

It follows from here and [53, Lemma 4.8] that $v_{1} \in H_{0}^{1}\left(\mathbb{R}^{2}\right)$ and

$$
\begin{equation*}
\left\|v_{1}\right\|_{H_{0}^{1}\left(\mathbb{R}^{2}\right)} \leq k_{2}\|u\|_{W^{1}\left(\mathbb{R}^{2}\right)} \tag{5.17}
\end{equation*}
$$

It remains to show that

$$
\begin{equation*}
\int_{\mathbb{R}^{2}} r^{-2}\left|v_{2}\right|^{2} d y \leq k_{3}\|u\|_{W^{1}\left(\mathbb{R}^{2}\right)} \tag{5.18}
\end{equation*}
$$

For $\chi_{0}>1$ (the case where $0<\chi_{0}<1$ is considered similarly), we have

$$
\int_{\mathbb{R}^{2}} r^{-2}\left|v_{2}\right|^{2} d y=\int_{-\pi}^{\pi} d \omega \int_{0}^{\infty} r^{-1}\left|v_{2}(\omega, r)\right|^{2} d r=\int_{-\pi+\omega_{0}}^{\pi+\omega_{0}} d \omega \int_{0}^{\infty} r^{-1} d r\left|\int_{r}^{\chi_{0} r} \frac{\partial u(\omega, t)}{\partial t} d t\right|^{2} .
$$

Using the Schwartz inequality and changing the limits of integration, we obtain estimate (5.18):

$$
\begin{aligned}
\int_{\mathbb{R}^{2}} r^{-2}\left|v_{2}\right|^{2} d y \leq\left(\chi_{0}-1\right) \int_{-\pi+\omega_{0}}^{\pi+\omega_{0}} d \omega & \int_{0}^{\infty} d r \int_{r}^{\chi_{0} r}\left|\frac{\partial u(\omega, t)}{\partial t}\right|^{2} d t \\
& =\frac{\left(\chi_{0}-1\right)^{2}}{\chi_{0}} \int_{-\pi+\omega_{0}}^{\pi+\omega_{0}} d \omega \int_{0}^{\infty}\left|\frac{\partial u(\omega, t)}{\partial t}\right|^{2} t d t \leq \frac{\left(\chi_{0}-1\right)^{2}}{\chi_{0}}\|u\|_{W^{1}\left(\mathbb{R}^{2}\right)}^{2}
\end{aligned}
$$

The lemma is proved.
We also will need weight spaces of negative and fractional orders. For $l \geq 1$, we denote the space dual to $H_{-a}^{l-1 / 2}\left(\Gamma_{i}\right)$ with respect to the extension of the inner product in $L_{2}\left(\Gamma_{i}\right)$ by

$$
H_{a}^{-(l-1 / 2)}\left(\Gamma_{i}\right) \stackrel{\text { def }}{=}\left(H_{-a}^{l-1 / 2}\left(\Gamma_{i}\right)\right)^{*} .
$$

The norm in $H_{a}^{-(l-1 / 2)}\left(\Gamma_{i}\right)$ is defined as follows:

$$
\|g\|_{H_{a}^{-(l-1 / 2)}\left(\Gamma_{i}\right)}=\sup _{0 \neq \psi \in H_{-a}^{l-1 / 2}\left(\Gamma_{i}\right)} \frac{|\langle\psi, g\rangle|}{\|\psi\|_{H_{-a}^{l-1 / 2}\left(\Gamma_{i}\right)}} .
$$

Lemma 5.8. For any $l \in \mathbb{Z}$ and $a \in \mathbb{R}$, the space $H_{a+1}^{l+1 / 2}\left(\Gamma_{i}\right)$ is a dense subset in $H_{a}^{l-1 / 2}\left(\Gamma_{i}\right)$ and

$$
\|g\|_{H_{a}^{l-1 / 2}\left(\Gamma_{i}\right)} \leq c\|g\|_{H_{a+1}^{l+1 / 2}\left(\Gamma_{i}\right)} \quad \forall g \in H_{a+1}^{l+1 / 2}\left(\Gamma_{i}\right)
$$

where $c>0$ is independent of $g$.
Proof. If $l=1,2, \ldots$, then the conclusion of the lemma follows from the definition of weight spaces. The case where $l=-1,-2, \ldots$ follows from duality considerations.

Let us consider the case where $l=0$. Let $g \in H_{a+1}^{1 / 2}\left(\Gamma_{i}\right)$. Using equivalent norms in weight spaces (see [58, Lemma 1.3]), it is easy to verify that

$$
\begin{equation*}
\left|\int_{\Gamma_{i}} \psi \bar{g} d \Gamma\right| \leq c\|\psi\|_{H_{-a}^{1 / 2}\left(\Gamma_{i}\right)}\|g\|_{H_{a+1}^{1 / 2}\left(\Gamma_{i}\right)} \quad \forall \psi \in H_{-a}^{1 / 2}\left(\Gamma_{i}\right), \quad g \in H_{a+1}^{1 / 2}\left(\Gamma_{i}\right) . \tag{5.19}
\end{equation*}
$$

By virtue of (5.19) and the relation $g \in H_{a+1}^{1 / 2}\left(\Gamma_{i}\right)$, we obtain

$$
\|g\|_{H_{a}^{-1 / 2}\left(\Gamma_{i}\right)}=\sup _{0 \neq \psi \in H_{-a}^{1 / 2}\left(\Gamma_{i}\right)} \frac{|\langle\psi, g\rangle|}{\|\psi\|_{H_{-a}^{1 / 2}\left(\Gamma_{i}\right)}} \leq c\|g\|_{H_{a+1}^{1 / 2}\left(\Gamma_{i}\right)} .
$$

Now we prove that $H_{a+1}^{1 / 2}\left(\Gamma_{i}\right)$ is a dense subset in $H_{a}^{-1 / 2}\left(\Gamma_{i}\right)$. Assume the contrary. Then there exists ${ }^{2}$ a nonzero element

$$
g \in\left(H_{a}^{-1 / 2}\left(\Gamma_{i}\right)\right)^{*}=H_{-a}^{1 / 2}\left(\Gamma_{i}\right)
$$

[^1]such that $\langle\psi, g\rangle=0$ for all $\psi \in H_{a+1}^{1 / 2}\left(\Gamma_{i}\right)$. Hence,
$$
\int_{\Gamma_{i}} \psi \bar{g} d \Gamma=\langle\psi, g\rangle=0 \quad \forall \psi \in C_{0}^{\infty}\left(\Gamma_{i}\right) .
$$

Taking into account the fact that $C_{0}^{\infty}\left(\Gamma_{i}\right)$ is dense in $H_{-a}^{1 / 2}\left(\Gamma_{i}\right)$, we conclude that $g=0$.
5.3.2. Spaces of vector-valued functions. For any $l \geq 0, a \in \mathbb{R}$, and integer $m_{i \mu} \geq 0$ ( $l+2 m-m_{i \mu} \geq 1$ ), we introduce spaces of vector-valued functions

$$
\begin{gather*}
\mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)=\prod_{i=1}^{N} \prod_{\mu=1}^{m} H_{a}^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right),  \tag{5.20}\\
\mathcal{H}_{a}^{l}(G, \partial G)=H_{a}^{l}(G) \times \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) .
\end{gather*}
$$

In Sec. 22 (see Chap. 5) and in Chap. 6, we consider nonlocal perturbations of the Dirichlet problem for a second-order differential operator. In this case, $m=1$ and $m_{i 1}=0$. Denote the spaces (5.20) as follows:

$$
\begin{equation*}
\mathcal{H}_{a}^{l+3 / 2}(\partial G)=\prod_{i=1}^{N} H_{a}^{l+3 / 2}\left(\Gamma_{i}\right), \quad \mathcal{H}_{a}^{l}(G, \partial G)=H_{a}^{l}(G) \times \mathcal{H}_{a}^{l+3 / 2}(\partial G) \tag{5.21}
\end{equation*}
$$

In Chap. 6, we will use the following norms that depend on a parameter $q>0$ (along with standard norms):

$$
\begin{gather*}
\|\psi\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)}=\left(\sum_{i=1}^{N}\left\|\psi_{i}\right\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i}\right)}^{2}\right)^{1 / 2}, \quad \psi=\left\{\psi_{i}\right\},  \tag{5.22}\\
\|(f, \psi)\|_{\mathcal{H}_{a}^{l}(G, \partial G)}=\left(\|f\|_{H_{a}^{l}(G)}^{2}+\|\psi\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)}^{2}\right)^{1 / 2}
\end{gather*}
$$

where $\|\cdot\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i}\right)}$ and $\|\cdot\|_{H_{a}^{l}(G)}^{2}$ are norms defined in (5.8), $l \geq 0$.
In Sec. 25 (see Chap. 6), we will use the following Banach spaces with norms depending on a parameter $q>0$ :

- $H_{\mathcal{N}, a}^{l+2}(G)=C_{\mathcal{N}}(\bar{G}) \cap H_{a}^{l+2}(G)$ with the norm

$$
\|u\|_{H_{\mathcal{N}, a}^{l+2}(G)}=\|u\|_{C_{\mathcal{N}}(\bar{G})}+\|u\|_{H_{a}^{l+2}(G)} ;
$$

- $H_{\mathcal{N}, a}^{l+3 / 2}\left(\Gamma_{i}\right)=C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right) \cap H_{a}^{l+3 / 2}\left(\Gamma_{i}\right)$ with the norm

$$
\|v\|_{H_{\mathcal{N}, a}^{l+3 / 2}\left(\Gamma_{i}\right)}=\|v\|_{C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)}+\|v\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i}\right)} ;
$$

- $\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)=\prod_{i=1}^{N} H_{\mathcal{N}, a}^{l+3 / 2}\left(\Gamma_{i}\right)=\mathcal{C}_{\mathcal{N}}(\partial G) \cap \mathcal{H}_{a}^{l+3 / 2}(\partial G)$ with the norm

$$
\|\psi\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)}=\sum_{i=1}^{N}\left\|\psi_{i}\right\|_{H_{\mathcal{N}, a}^{l+3 / 2}\left(\Gamma_{i}\right)}, \quad \psi=\left\{\psi_{i}\right\} .
$$

Here $\mathcal{N}$ is a closed subset of $\partial G$ and $l \geq 0$.

For any $l \geq 0$ and $a \in \mathbb{R}$ and integer $m_{j \sigma \mu}\left(0 \leq m_{j \sigma \mu} \leq l+2 m-1\right)$, we introduce the following spaces of vector-valued functions:

$$
\begin{gathered}
\mathcal{H}_{a}^{l}(K)=\prod_{j=1}^{N} H_{a}^{l}\left(K_{j}\right), \quad \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)=\prod_{j=1}^{N} \prod_{\sigma=1,2} \prod_{\mu=1}^{m} H_{a}^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right) \\
\mathcal{H}_{a}^{l}(K, \gamma)=\mathcal{H}_{a}^{l}(K) \times \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)
\end{gathered}
$$

The spaces

$$
\mathcal{H}_{a}^{l}\left(K^{d}\right), \quad \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}\left(\gamma^{d}\right), \quad \mathcal{H}_{a}^{l}\left(K^{d}, \gamma^{d}\right), \quad d>0,
$$

are introduced similarly.

## 6. Statement of the Nonlocal Problem in Bounded Domains

6.1. Statement of the problem. Denote by $\mathbf{P}(y, D)$ and $B_{i \mu s}(y, D)$ linear differential operators with complex-valued coefficients of class $C^{\infty}$ of orders $2 m$ and $m_{i \mu}$, respectively, and by $\mathbf{P}^{0}(y, D)$ and $B_{i \mu s}^{0}(y, D)$ the principal homogeneous parts of the operators $\mathbf{P}(y, D)$ and $B_{i \mu s}(y, D)(i=1, \ldots, N$, $\left.\mu=1, \ldots, m, s=0, \ldots, S_{i}\right)$.

We formulate conditions for operators $\mathbf{P}(y, D)$ and $B_{i \mu 0}(y, D)$ that will correspond to a "local" elliptic problem (see, e.g., [57, Chap. 2, Sec. 1].

Condition 6.1. The operator $\mathbf{P}(y, D)$ is essentially elliptic on $\bar{G}$.
In particular, condition 6.1 means that the following estimate holds for all $\theta \in \mathbb{R}^{2}$ and $y \in \bar{G}$ :

$$
\begin{equation*}
A^{-1}|\theta|^{2 m} \leq\left|\mathbf{P}^{0}(y, \theta)\right| \leq A|\theta|^{2 m}, \quad A>0 . \tag{6.1}
\end{equation*}
$$

Condition 6.2. For every $y \in \overline{\overline{\Gamma_{i}}}$ and $i=1, \ldots, N$, the system $\left\{B_{i \mu 0}(y, D)\right\}_{\mu=1}^{m}$ satisfies the covering condition the (Lopatinskii condition) with respect to the operator $\mathbf{P}(y, D)$.

Condition 6.2 means the following. Let $y \in \overline{\Gamma_{i}}$. Assume, without loss of generality, that near a given point $y$, the curve $\overline{\Gamma_{i}}$ is defined by the equation $y_{2}=0$. Let the polynomial

$$
B_{i \mu 0}^{\prime}(y, \tau) \equiv \sum_{\nu=1}^{m} b_{i \mu \nu}(y) \tau^{\nu-1} \equiv B_{i \mu 0}^{0}(y, 1, \tau)\left(\bmod \mathbf{M}^{+}(y, \tau)\right)
$$

be the remainder after the division of $B_{i \mu 0}^{0}(y, 1, \tau)$ by $\mathbf{M}^{+}(y, \tau)$, where

$$
\mathbf{M}^{+}(y, \tau)=\prod_{\nu=1}^{m}\left(\tau-\tau_{\nu}^{+}(y)\right),
$$

$\tau_{1}^{+}(y), \ldots, \tau_{m}^{+}(y)$ are the roots of the polynomial $\mathbf{P}^{0}(y, 1, \tau)$ with a positive imaginary part. In this case, $\mathbf{P}^{0}(y, 1, \tau), B_{i \mu 0}^{0}(y, 1, \tau)$ and $\mathbf{M}^{+}(y, \tau)$ are considered as polynomials with respect to $\tau$. Then condition 6.2 means that

$$
d_{i}(y)=\operatorname{det}\left\|b_{i \mu \nu}(y)\right\|_{\mu, \nu=1}^{m} \neq 0 .
$$

Since every $\operatorname{arc} \overline{\Gamma_{i}}, i=1, \ldots, N$, is compact, we see that

$$
\begin{equation*}
D=\min _{i=1, \ldots, N} \inf _{y \in \overline{\Gamma_{i}}}\left|d_{i}(y)\right|>0 . \tag{6.2}
\end{equation*}
$$

Let us emphasize that, in the general case, we do not assume the normality of the operators $B_{i \mu 0}(y, D)$ on $\operatorname{arcs} \overline{\Gamma_{i}}$.

Consider operators

$$
\mathbf{P}: W^{l+2 m}(G) \rightarrow W^{l}(G), \quad \mathbf{B}_{i \mu}^{0}: W^{l+2 m}(G) \rightarrow W^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)
$$

defined by the formulas

$$
\mathbf{P} u=\mathbf{P}(y, D) u, \quad \mathbf{B}_{i \mu}^{0} u=\left.B_{i \mu 0}(y, D) u(y)\right|_{\Gamma_{i}} .
$$

In what follows, we assume that $l+2 m-m_{i \mu} \geq 1$.
Let us define operators that correspond to nonlocal conditions near the set $\mathcal{K}$. Let $\Omega_{i s}(i=1, \ldots, N$, $s=1, \ldots, S_{i}$ ) be infinitely differentiable nondegenerate transformations that map some neighborhood $\mathcal{O}_{i}$ of the curve $\overline{\Gamma_{i} \cap \mathcal{O}_{\varepsilon}(\mathcal{K})}$ to the set $\Omega_{i s}\left(\mathcal{O}_{i}\right)$ such that $\Omega_{i s}\left(\Gamma_{i} \cap \mathcal{O}_{\varepsilon}(\mathcal{K})\right) \subset G$ and

$$
\begin{equation*}
\Omega_{i s}(g) \in \mathcal{K} \quad \text { for } \quad g \in \overline{\Gamma_{i}} \cap \mathcal{K} . \tag{6.3}
\end{equation*}
$$

Thus, the transformations $\Omega_{i s}$ map $\operatorname{arcs} \Gamma_{i} \cap \mathcal{O}_{\varepsilon}(\mathcal{K})$ in the domain $G$, and their endpoints $\overline{\Gamma_{i}} \cap \mathcal{K}$ are mapped to endpoints.

Let us specify the structure of transformations $\Omega_{i s}$ near the set $\mathcal{K}$. Denote by $\Omega_{i s}^{+1}$ the transformation $\Omega_{i s}: \mathcal{O}_{i} \rightarrow \Omega_{i s}\left(\mathcal{O}_{i}\right)$ and the transformation inverse to $\Omega_{i s}$ by $\Omega_{i s}^{-1}: \Omega_{i s}\left(\mathcal{O}_{i}\right) \rightarrow \mathcal{O}_{i}$. The set of all points of the form $\Omega_{i_{q} s_{q}}^{ \pm 1}\left(\ldots \Omega_{i_{1} s_{1}}^{ \pm 1}(g)\right) \in \mathcal{K}\left(1 \leq s_{j} \leq S_{i_{j}}, j=1, \ldots, q\right)$ is called the orbit of a point $g \in \mathcal{K}$ and is denoted by $\operatorname{Orb}(g)$. We can obtain these points sequentially applying to $g$ the transforms $\Omega_{i_{j} s_{j}}^{+1}$ or $\Omega_{i_{j} s_{j}}^{-1}$ that map points of the set $\mathcal{K}$ to $\mathcal{K}$.

Obviously, for any $g, g^{\prime} \in \mathcal{K}$, either $\operatorname{Orb}(g)=\operatorname{Orb}\left(g^{\prime}\right)$ or $\operatorname{Orb}(g) \cap \operatorname{Orb}\left(g^{\prime}\right)=\varnothing$. In what follows, we assume that the set $\mathcal{K}$ consists of one orbit (all results are easily generalized to the general case where $\mathcal{K}$ consists of a finite number of nonintersecting orbits). For simplicity, we assume that the set (orbit) $\mathcal{K}$ consists of $N$ points $g_{1}, \ldots, g_{N}$.

Let us choose $\varepsilon$ (cf. Remark 2.2) so small that there exist neighborhoods $\mathcal{O}_{\varepsilon_{1}}\left(g_{j}\right)$ of the points $g_{j} \in \mathcal{K}$ such that $\mathcal{O}_{\varepsilon_{1}}\left(g_{j}\right) \supset \mathcal{O}_{\varepsilon}\left(g_{j}\right)$ and
(1) $\overline{G \cap \mathcal{O}_{\varepsilon_{1}}\left(g_{j}\right)}=\overline{K_{j}^{\varepsilon_{1}}}$;
(2) $\overline{\mathcal{O}_{\varepsilon_{1}}\left(g_{j}\right)} \cap \overline{\mathcal{O}_{\varepsilon_{1}}\left(g_{k}\right)}=\varnothing$ for any $g_{j}, g_{k} \in \mathcal{K}$ and $k \neq j$;
(3) if $g_{j} \in \overline{\Gamma_{i}}$ and $\Omega_{i s}\left(g_{j}\right)=g_{k}$, then $\mathcal{O}_{\varepsilon}\left(g_{j}\right) \subset \mathcal{O}_{i}$ and $\Omega_{i s}\left(\mathcal{O}_{\varepsilon}\left(g_{j}\right)\right) \subset \mathcal{O}_{\varepsilon_{1}}\left(g_{k}\right)$.

For any point $g_{j} \in \overline{\Gamma_{i}} \cap \mathcal{K}$, we fix a transform $Y_{j}: y \mapsto y^{\prime}\left(g_{j}\right)$, which is the composition of a shift by a vector $-\overrightarrow{O g_{j}}$ and a rotation by some angle such that

$$
\begin{gathered}
Y_{j}\left(\mathcal{O}_{\varepsilon_{1}}\left(g_{j}\right)\right)=\mathcal{O}_{\varepsilon_{1}}(0), \quad Y_{j}\left(G \cap \mathcal{O}_{\varepsilon_{1}}\left(g_{j}\right)\right)=K_{j} \cap \mathcal{O}_{\varepsilon_{1}}(0), \\
Y_{j}\left(\Gamma_{i} \cap \mathcal{O}_{\varepsilon_{1}}\left(g_{j}\right)\right)=\gamma_{j \sigma} \cap \mathcal{O}_{\varepsilon_{1}}(0), \quad \sigma=1 \text { or } 2 .
\end{gathered}
$$

Here the angles $K_{j}=\left\{y \in \mathbb{R}^{2}: r>0,|\omega|<\omega_{j}\right\}$ and their rays $\gamma_{j \sigma}=\left\{y \in \mathbb{R}^{2}: r>0, \omega=(-1)^{\sigma} \omega_{j}\right\}$ ( $\omega$ and $r$ are polar coordinates with pole at the origin, $0<\omega_{j}<\pi$ ) are the same ${ }^{3}$ as in Sec. 5.1.

Consider the following condition (see Fig. 6.1).
Condition 6.3. Let $g_{j} \in \overline{\Gamma_{i}} \cap \mathcal{K}$ and $\Omega_{i s}\left(g_{j}\right)=g_{k} \in \mathcal{K}$. Then the transformation

$$
Y_{k} \circ \Omega_{i s} \circ Y_{j}^{-1}: \mathcal{O}_{\varepsilon}(0) \rightarrow \mathcal{O}_{\varepsilon_{1}}(0)
$$

is the composition of operators of rotation and dilation centered at the origin.
Remark 6.1. Condition 6.3 and the assumption $\Omega_{i s}\left(\Gamma_{i}\right) \subset G$ imply that if $g \in \Omega_{i s}\left(\overline{\Gamma_{i}} \backslash \Gamma_{i}\right) \cap \overline{\Gamma_{j}} \cap \mathcal{K} \neq \varnothing$, then the curves $\Omega_{i s}\left(\overline{\overline{\Gamma_{i}}}\right)$ and $\overline{\Gamma_{j}}$ intersect at the point $g$ and make at this point a nonzero angle.

Choose a number $\varepsilon_{0}, 0<\varepsilon_{0} \leq \varepsilon$, satisfying the following condition: if $g_{j} \in \overline{\Gamma_{i}}$ and $\Omega_{i s}\left(g_{j}\right)=g_{k}$, then

$$
\mathcal{O}_{\varepsilon_{0}}\left(g_{k}\right) \subset \Omega_{i s}\left(\mathcal{O}_{\varepsilon}\left(g_{j}\right)\right) \subset \mathcal{O}_{\varepsilon_{1}}\left(g_{k}\right)
$$

Consider a function $\zeta \in C^{\infty}\left(\mathbb{R}^{2}\right)$ such that

$$
\begin{equation*}
\zeta(y)=1\left(y \in \mathcal{O}_{\varepsilon_{0} / 2}(\mathcal{K})\right), \quad \zeta(y)=0\left(y \notin \mathcal{O}_{\varepsilon_{0}}(\mathcal{K})\right) . \tag{6.4}
\end{equation*}
$$

[^2]

Fig. 6.1. Transformation $Y_{2} \circ \Omega_{11} \circ Y_{1}^{-1}: \mathcal{O}_{\varepsilon}(0) \rightarrow \mathcal{O}_{\varepsilon_{1}}(0)$ is the composition of a rotation and a dilation.

Introduce bounded operators $\mathbf{B}_{i \mu}^{1}: W^{l+2 m}(G) \rightarrow W^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ by the formula

$$
\begin{array}{ll}
\mathbf{B}_{i \mu}^{1} u=\left.\sum_{s=1}^{S_{i}}\left(B_{i \mu s}(y, D)(\zeta u)\right)\left(\Omega_{i s}(y)\right)\right|_{\Gamma_{i}}, & y \in \Gamma_{i} \cap \mathcal{O}_{\varepsilon}(\mathcal{K}), \\
\mathbf{B}_{i \mu}^{1} u=0, & y \in \Gamma_{i} \backslash \mathcal{O}_{\varepsilon}(\mathcal{K}) ;
\end{array}
$$

here

$$
\left(B_{i \mu s}(y, D) v\right)\left(\Omega_{i s}(y)\right)=\left.B_{i \mu s}\left(y^{\prime}, D_{y^{\prime}}\right) v\left(y^{\prime}\right)\right|_{y^{\prime}=\Omega_{i s}(y)} .
$$

If supp $u \subset \bar{G} \backslash \overline{\mathcal{O}_{\varepsilon_{0}}(\mathcal{K})}$, then $\mathbf{B}_{i \mu}^{1} u=0$; we say that operators $\mathbf{B}_{i \mu}^{1}$ correspond to nonlocal terms with supports near the set $\mathcal{K}$.

As earlier, we denote

$$
G_{\rho}=\{y \in G: \operatorname{dist}(y, \partial G)>\rho\} .
$$

Let us introduce bounded operators

$$
\mathbf{B}_{i \mu}^{2}: W^{l+2 m}(G) \rightarrow W^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)
$$

satisfying the following condition.
Condition 6.4. There exist numbers $\varkappa_{1}>\varkappa_{2}>0$ and $\rho>0$ such that the following inequalities hold:

$$
\begin{gather*}
\left\|\mathbf{B}_{i \mu}^{2} u\right\|_{W^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)} \leq c_{1}\|u\|_{W^{l+2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)} \quad \forall u \in W^{l+2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)  \tag{6.5}\\
\left\|\mathbf{B}_{i \mu}^{2} u\right\|_{W^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i} \backslash \overline{\left.\mathcal{O}_{\varkappa_{2}}(\mathcal{K})\right)}\right.} \leq c_{2}\|u\|_{W^{l+2 m}\left(G_{\rho}\right)} \quad \forall u \in W^{l+2 m}\left(G_{\rho}\right), \tag{6.6}
\end{gather*}
$$

where $i=1, \ldots, N, \mu=1, \ldots, m$, and $c_{1}$ and $c_{2}>0$ are independent of $u$.
Inequality (6.5) implies that if $\operatorname{supp} u \subset \mathcal{O}_{\varkappa_{1}}(\mathcal{K})$, then $\mathbf{B}_{i \mu}^{2} u=0$. Hence we say that the operators $\mathbf{B}_{i \mu}^{2}$ correspond to nonlocal terms with supports outside the set $\mathcal{K}$.

In Chaps. 2-5, we assume that conditions 6.1-6.4 hold; in Chaps. 4 and 5, condition 6.4 is considered for $l=0$. In Chap. 6, conditions $6.1-6.4$ will be replaced by their analogs.

Note that a priori we do not suppose any relation between the number $\varepsilon_{0}$ in the definition of the operators $\mathbf{B}_{i \mu}^{1}$ and the numbers $\varkappa_{1}, \varkappa_{2}$, and $\rho$ in condition 6.4.

We study the nonlocal elliptic problem

$$
\begin{array}{rlrlr}
\mathbf{P} u & \equiv \mathbf{P}(y, D) u=f_{0}(y), & & y \in G, & \\
\mathbf{B}_{i \mu} u & \equiv \mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u=f_{i \mu}(y), & y \in \Gamma_{i}, \quad i=1, \ldots, N, \quad \mu=1, \ldots, m . \tag{6.8}
\end{array}
$$

Let us introduce the following bounded operator corresponding to problem (6.7), (6.8) in the Sobolev spaces:

$$
\mathbf{L}=\left\{\mathbf{P}, \mathbf{B}_{i \mu}\right\}: W^{l+2 m}(G) \rightarrow \mathcal{W}^{l}(G, \partial G),
$$

where $\mathcal{W}^{l}(G, \partial G)$ is the space defined in Sec. 5.2.
Definition 6.1. A function $u \in W^{l+2 m}(G)$ is called a strong solution of problem (6.7), (6.8) with the right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{W}^{l}(G, \partial G)$ if the equality $\mathbf{L} u=\left\{f_{0}, f_{i \mu}\right\}$ holds.

In what follows, strong solutions of problem (6.7), (6.8) are called simply solutions.
6.2. Example of a nonlocal problem. We present an example of a problem with nonlocal conditions that satisfy the conditions of this section.

Let the operators $\mathbf{P}(y, D)$ and $B_{i \mu s}(y, D)$ be as above. Denote by $\Omega_{i s}\left(i=1, \ldots, N, s=1, \ldots, S_{i}\right)$ nondegenerate transformations of the class $C^{\infty}$ that map some neighborhood $\mathcal{O}_{i}$ of a curve $\Gamma_{i}$ to $\Omega_{i s}\left(\mathcal{O}_{i}\right)$ such that $\Omega_{i s}\left(\Gamma_{i}\right) \subset G$. Note that in this case, we do not assume that relations (6.3) hold.

Consider the nonlocal problem

$$
\begin{gather*}
\mathbf{P}(y, D) u=f_{0}(y), \quad y \in G,  \tag{6.9}\\
\left.B_{i \mu 0}(y, D) u(y)\right|_{\Gamma_{i}}+\left.\sum_{s=1}^{S_{i}}\left(B_{i \mu s}(y, D) u\right)\left(\Omega_{i s}(y)\right)\right|_{\Gamma_{i}}=f_{i \mu}(y)  \tag{6.10}\\
\left(y \in \Gamma_{i}, i=1, \ldots, N, \mu=1, \ldots, m\right) .
\end{gather*}
$$

Let us choose small $\varepsilon>0$ such that for any point $g \in \mathcal{K}$, the set $\overline{\mathcal{O}_{\varepsilon}(g)}$ intersects the curve $\overline{\Omega_{i s}\left(\Gamma_{i}\right)}$ if and only if $g \in \mathcal{K} \cap \overline{\Omega_{i s}\left(\Gamma_{i}\right)}$.

Assume that condition 6.3 holds. By Remark 6.1, condition 6.3 is a restriction on the geometric structure of the support of nonlocal terms near the set $\mathcal{K}$. However, if $\Omega_{i s}\left(\overline{\Gamma_{i}} \backslash \Gamma_{i}\right) \subset \partial G \backslash \mathcal{K}$, then no restrictions are imposed on the geometric structure of the curves $\Omega_{i s}\left(\overline{\Gamma_{i}}\right)$ near $\partial G$ (cf. [85, 89]).

Introduce the notation

$$
\begin{gathered}
\mathbf{P} u=\mathbf{P}(y, D) u, \quad \mathbf{B}_{i \mu}^{0} u=\left.B_{i \mu 0}(y, D) u(y)\right|_{\Gamma_{i}}, \\
\mathbf{B}_{i \mu}^{1} u=\left.\sum_{s=1}^{S_{i}}\left(B_{i \mu s}(y, D)(\zeta u)\right)\left(\Omega_{i s}(y)\right)\right|_{\Gamma_{i}}, \\
\mathbf{B}_{i \mu}^{2} u=\left.\sum_{s=1}^{S_{i}}\left(B_{i \mu s}(y, D)((1-\zeta) u)\right)\left(\Omega_{i s}(y)\right)\right|_{\Gamma_{i}},
\end{gathered}
$$

where the function $\zeta$ is defined in Eq. (6.4). Then problem (6.9), (6.10) has the form of Eqs. (6.7), (6.8).

Similarly to the proof of [89, Lemma 2.5] (where one must replace the weight spaces by the corresponding Sobolev spaces), we can show that the operators $\mathbf{B}_{i \mu}^{2}$ satisfy condition 6.4.
6.3. Reduction to model problems in infinite angles. As in Chap. 1, we pay special attention to the behavior of solutions near the set $\mathcal{K}$ consisting of conjugation points of the boundary conditions. Consider the corresponding model problem in plane angles. We set

$$
\begin{equation*}
\mathbf{B}_{i \mu}^{2}=0, \quad i=1, \ldots, N, \mu=1, \ldots, m . \tag{6.11}
\end{equation*}
$$

For $y \in \mathcal{O}_{\varepsilon_{1}}\left(g_{j}\right)$, we denote $u(y)$ by $u_{j}(y)$. If $g_{j} \in \overline{\Gamma_{i}}, y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right)$, and $\Omega_{i s}(y) \in \mathcal{O}_{\varepsilon_{1}}\left(g_{k}\right)$, then we denote $u\left(\Omega_{i s}(y)\right)$ by $u_{k}\left(\Omega_{i s}(y)\right)$. Then, by assumption (6.11), the nonlocal problem (6.7), (6.8) has the following form in a $\varepsilon$-neighborhood of the set $\mathcal{K}$ :

$$
\begin{gathered}
\mathbf{P}(y, D) u_{j}=f_{0}(y), \quad y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap G \\
\left.B_{i \mu 0}(y, D) u_{j}(y)\right|_{\mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap \Gamma_{i}}+\left.\sum_{s=1}^{S_{i}}\left(B_{i \mu s}(y, D)\left(\zeta u_{k}\right)\right)\left(\Omega_{i s}(y)\right)\right|_{\mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap \Gamma_{i}}=f_{i \mu}(y) \\
\left(y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap \Gamma_{i}, \quad i \in\left\{1 \leq i \leq N: g_{j} \in \overline{\Gamma_{i}}\right\}, \quad j=1, \ldots, N, \quad \mu=1, \ldots, m\right) .
\end{gathered}
$$

Let $y \mapsto y^{\prime}\left(g_{j}\right)$ be a transformation of coordinates described above. Let us introduce functions

$$
U_{j}\left(y^{\prime}\right)=u_{j}\left(y\left(y^{\prime}\right)\right), \quad f_{j}\left(y^{\prime}\right)=f_{0}\left(y\left(y^{\prime}\right)\right), \quad y^{\prime} \in K_{j}^{\varepsilon} ; \quad f_{j \sigma \mu}\left(y^{\prime}\right)=f_{i \mu}\left(y\left(y^{\prime}\right)\right), \quad y^{\prime} \in \gamma_{j \sigma}^{\varepsilon}
$$

where $\sigma=1(\sigma=2)$ if the transformation $y \mapsto y^{\prime}\left(g_{j}\right)$ maps $\Gamma_{i}$ to the ray $\gamma_{j 1}$ (respectively, $\gamma_{j 2}$ ) of the angle $K_{j}$; denote $y^{\prime}$ by $y$. Then, by condition 6.3 , problem (6.7), (6.8) has the following form:

$$
\begin{gather*}
\mathbf{P}_{j}(y, D) U_{j}=f_{j}(y), \quad y \in K_{j}^{\varepsilon}  \tag{6.12}\\
\left.\mathbf{B}_{j \sigma \mu}(y, D) U \equiv \sum_{k, s}\left(B_{j \sigma \mu k s}(y, D) U_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}}=f_{j \sigma \mu}(y), \quad y \in \gamma_{j \sigma}^{\varepsilon} \tag{6.13}
\end{gather*}
$$

where $j, k=1, \ldots, N, \sigma=1,2, \mu=1, \ldots, m$, and $s=0, \ldots, S_{j \sigma k}$;

$$
\mathbf{P}_{j}(y, D), \quad B_{j \sigma \mu k s}(y, D)
$$

are linear differential operators of orders $2 m$ and $m_{j \sigma \mu}\left(l+2 m-m_{j \sigma \mu} \geq 1\right)$ respectively with variables of class $C^{\infty}$ :

$$
\mathbf{P}_{j}(y, D)=\sum_{|\alpha| \leq 2 m} p_{j \alpha}(y) D_{y}^{\alpha}, \quad B_{j \sigma \mu k s}(y, D)=\sum_{|\alpha| \leq m_{j \sigma \mu}} b_{j \sigma \mu k s \alpha}(y) D_{y}^{\alpha} ;
$$

$\mathcal{G}_{j \sigma k s}$ is the operator of rotation by the angle $\omega_{j \sigma k s}$ and a dilation with scale factor $\chi_{j \sigma k s}>0$; moreover, $\left|(-1)^{\sigma} b_{j}+\omega_{j \sigma k s}\right|<b_{k}$ if $(k, s) \neq(j, 0)$ (see Remark 6.1), and $\omega_{j \sigma j 0}=0$ and $\chi_{j \sigma j 0}=1$ (i.e., $\left.\mathcal{G}_{j \sigma j 0} \equiv y\right)$.

Obviously,

$$
\begin{equation*}
b_{j \sigma \mu k s \alpha}(y)=0 \quad \text { as } \quad|y| \geq \varepsilon_{0}, \quad(k, s) \neq(j, 0) \tag{6.14}
\end{equation*}
$$

In Chaps. 2-6, we use the following notation:

$$
\begin{equation*}
d_{1}=\min \left\{\chi_{j \sigma k s}\right\} / 2, \quad d_{2}=2 \max \left\{\chi_{j \sigma k s}\right\} . \tag{6.15}
\end{equation*}
$$

We extend the coefficients of the operators $\mathbf{P}_{j}(y, D)$ and $\mathbf{B}_{j \sigma \mu}(y, D)$ to $\mathbb{R}^{2}$ such that we obtain compactly supported smooth functions. Consider the operator $\mathfrak{L}: \mathcal{W}^{l+2 m}(K) \rightarrow \mathcal{W}^{l}(K, \gamma)$ defined by the formula

$$
\mathfrak{L} U=\left\{\mathbf{P}_{j}(y, D) U_{j},\left.\mathbf{B}_{j \sigma \mu}(y, D) U\right|_{\gamma_{j \sigma}}\right\}
$$

where $\mathcal{W}^{l+2 m}(K)$ and $\mathcal{W}^{l}(K, \gamma)$ are the spaces defined in Sec. 5.2. The operator $\mathfrak{L}$ corresponds to problem (6.12), (6.13).

Denote the principal homogeneous parts of the operators $\mathbf{P}_{j}(0, D)$ and $B_{j \sigma \mu k s}(0, D)$ by $\mathcal{P}_{j}(D)$ and $B_{j \sigma \mu k s}(D)$, respectively. In addition to problem (6.12), (6.13), we consider the following nonlocal
model problem:

$$
\begin{gather*}
\mathcal{P}_{j}(D) U_{j}=f_{j}(y), \quad y \in K_{j},  \tag{6.16}\\
\left.\mathcal{B}_{j \sigma \mu}(D) U \equiv \sum_{k, s}\left(B_{j \sigma \mu k s}(D) U_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}}=f_{j \sigma \mu}(y), \quad y \in \gamma_{j \sigma} . \tag{6.17}
\end{gather*}
$$

Consider the operator $\mathcal{L}: \mathcal{W}^{l+2 m}(K) \rightarrow \mathcal{W}^{l}(K, \gamma)$ defined by the formula

$$
\mathcal{L} U=\left\{\mathcal{P}_{j}(D) U_{j},\left.\mathcal{B}_{j \sigma \mu}(D) U\right|_{\gamma_{j \sigma}}\right\}
$$

and corresponding to problem (6.16), (6.17) in Sobolev spaces.
Let us write the operators $\mathcal{P}_{j}(D)$ and $B_{j \sigma \mu k s}(D)$ in the polar coordinates as follows:

$$
\begin{gathered}
\mathcal{P}_{j}(D)=r^{-2 m} \tilde{\mathcal{P}}_{j}\left(\omega, D_{\omega}, r D_{r}\right), \\
B_{j \sigma \mu k s}(D)=r^{-m_{j \sigma \mu}} \tilde{B}_{j \sigma \mu k s}\left(\omega, D_{\omega}, r D_{r}\right),
\end{gathered}
$$

where

$$
D_{\omega}=-i \frac{\partial}{\partial \omega}, \quad D_{r}=-i \frac{\partial}{\partial r} .
$$

Consider an operator (an analytic operator-valued function depending on a parameter $\lambda \in \mathbb{C}$ )

$$
\tilde{\mathcal{L}}(\lambda): \mathcal{W}^{l+2 m}(-\bar{\omega}, \bar{\omega}) \rightarrow \mathcal{W}^{l}[-\bar{\omega}, \bar{\omega}]
$$

defined by the formula

$$
\begin{equation*}
\tilde{\mathcal{L}}(\lambda) \varphi=\left\{\tilde{\mathcal{P}}_{j}\left(\omega, D_{\omega}, \lambda\right) \varphi_{j}, \tilde{\mathcal{B}}_{j \sigma \mu}\left(\omega, D_{\omega}, \lambda\right) \varphi\right\}, \tag{6.18}
\end{equation*}
$$

where

$$
\tilde{\mathcal{B}}_{j \sigma \mu}\left(\omega, D_{\omega}, \lambda\right) \varphi=\left.\sum_{k, s}\left(\chi_{j \sigma k s}\right)^{i \lambda-m_{j \sigma \mu}} \tilde{B}_{j \sigma \mu k s}\left(\omega, D_{\omega}, \lambda\right) \varphi_{k}\left(\omega+\omega_{j \sigma k s}\right)\right|_{\omega=(-1)^{\sigma} \omega_{j}}
$$

and the spaces $\mathcal{W}^{l+2 m}(-\bar{\omega}, \bar{\omega})$ and $\mathcal{W}^{l}[-\bar{\omega}, \bar{\omega}]$ are introduced in Sec. 5.2.
The main definitions and facts on eigenvalues, eigenvectors, and adjoint vectors of analytic operatorvalued functions can be found in [23]. In the sequel, we extensively use the fact that the spectrum of the operator $\tilde{\mathcal{L}}(\lambda)$ is discrete; namely, the following lemma is valid.

Lemma 6.1 (see [88, Lemma 2.1]). For any $\lambda \in \mathbb{C}, \tilde{\mathcal{L}}(\lambda)$ is a Fredholm operator and ind $\tilde{\mathcal{L}}(\lambda)=0$. The spectrum of the operator $\tilde{\mathcal{L}}(\lambda)$ is discrete. For any numbers $c_{1}<c_{2}$, the strip $c_{1}<\operatorname{Im} \lambda<c_{2}$ contains no more than a finite number of eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$.

## 7. Nonlocal Problems in Plane Angles in Sobolev Spaces

7.1. Construction of a "solution" in the case of absence of eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ on the line $\operatorname{Im} \lambda=1-l-2 m$. In this subsection, we assume that the following condition holds.
Condition 7.1. The line $\operatorname{Im} \lambda=1-l-2 m$ does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$.
Let us consider the operators

$$
D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1} \mathcal{B}_{j \sigma \mu}(D) U \equiv D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1}\left(\sum_{k, s}\left(B_{j \sigma \mu k s}(D) U_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right)\right),
$$

where $\tau_{j \sigma}$ is a unit vector in the direction of the ray $\gamma_{j \sigma}$. Using the chain rule for differentiation, we write

$$
\begin{equation*}
D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1} \mathcal{B}_{j \sigma \mu}(D) U \equiv \sum_{k, s}\left(\hat{B}_{j \sigma \mu k s}(D) U_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right), \tag{7.1}
\end{equation*}
$$

where $\hat{B}_{j \sigma \mu k s}(D)$ are homogeneous differential operator of the order $l+2 m-1$ with constant coefficients. In particular,

$$
\hat{B}_{j \sigma \mu j 0}(D)=D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1} B_{j \sigma \mu j 0}(D)
$$

since $\mathcal{G}_{j \sigma j 0} y \equiv y$. Formally replacing nonlocal operators in Eq. (7.1) by the corresponding local operators, we introduce operators

$$
\begin{equation*}
\hat{\mathcal{B}}_{j \sigma \mu}(D) U \equiv \sum_{k, s} \hat{B}_{j \sigma \mu k s}(D) U_{k}(y) . \tag{7.2}
\end{equation*}
$$

If $l \geq 1$, then we consider the following operators (in addition to system (7.2)):

$$
\begin{equation*}
D^{\xi} \mathcal{P}_{j}(D) U_{j}(y), \quad|\xi|=l-1 . \tag{7.3}
\end{equation*}
$$

The system of operators (7.2) and (7.3) plays a key role in the proof of the following lemma that allows one to reduce problems in Sobolev spaces to problems in weight spaces.

Lemma 7.1. Let condition 7.1 hold. Then there exists a linear bounded operator

$$
\mathcal{A}:\left\{f \in \mathcal{S}^{l}(K, \gamma): \operatorname{supp} f \subset \mathcal{O}_{1}(0)\right\} \rightarrow \mathcal{S}^{l+2 m}(K)
$$

such that the function $V=\mathcal{A} f$ satisfies the following conditions: $V=0$ as $|y| \geq 1$,

$$
\begin{gather*}
\|\mathcal{L} V-f\|_{\mathcal{H}_{0}^{l}(K)} \leq c\|f\|_{\mathcal{W}^{l}(K, \gamma)},  \tag{7.4}\\
\|V\|_{\mathcal{H}_{a}^{l+2 m}(K)} \leq c_{a}\|f\|_{\mathcal{W}^{l}(K, \gamma)} \quad \forall a>0 \tag{7.5}
\end{gather*}
$$

for any function $f=\left\{f_{j}, f_{j \sigma \mu}\right\} \in \mathrm{D}(\mathcal{A})$.
Proof. 1. Introduce an operator

$$
\begin{equation*}
f_{j \sigma \mu} \mapsto \Phi_{j \sigma \mu}, \tag{7.6}
\end{equation*}
$$

that takes each function $f_{j \sigma \mu} \in W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)$ to its extension $\Phi_{j \sigma \mu} \in W^{l+2 m-m_{j \sigma \mu}}\left(\mathbb{R}^{2}\right)$ to $\mathbb{R}^{2}$ such that $\Phi_{j \sigma \mu}=0$ for $|y| \geq 2$. We also consider an extension of the function $f_{j}$ from $K_{j}$ to $\mathbb{R}^{2}$ such that the extended function (we denote it also by $f_{j}$ ) equals zero for $|y| \geq 2$. The corresponding extension operators can be taken linear and bounded (see [100, Chap. 6, Sec. 3]).

Consider the following algebraic system for partial derivatives $D^{\alpha} W_{j}$, where $|\alpha|=l+2 m-1$ and $j=1, \ldots, N$ :

$$
\begin{gather*}
\hat{\mathcal{B}}_{j \sigma \mu}(D) W=D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1} \Phi_{j \sigma \mu}  \tag{7.7}\\
D^{\xi} \mathcal{P}_{j}(D) W_{j}=D^{\xi} f_{j} \tag{7.8}
\end{gather*}
$$

where $j=1, \ldots, N, \sigma=1,2, \mu=1, \ldots, m$, and $|\xi|=l-1$. Each operator $\hat{\mathcal{B}}_{j \sigma \mu}(D)$ defined by formula (7.2) is the sum of "local" operators; therefore, system (7.7), (7.8) can be considered as an algebraic system. Assume that system (7.7), (7.8) has a unique solution for any right-hand side and denote this solution by $W_{j \alpha}$. It is obvious that $W_{j \alpha} \in W^{1}\left(\mathbb{R}^{2}\right)$ and $W_{j \alpha}=0$ for $|y| \geq 2$. By [53, Lemma 4.17], there exists a linear bounded operator

$$
\begin{equation*}
\left\{W_{j \alpha}\right\}_{|\alpha|=l+2 m-1} \mapsto V_{j}, \tag{7.9}
\end{equation*}
$$

which states the correspondence between the system $\left\{W_{j \alpha}\right\}_{|\alpha|=l+2 m-1} \in \prod_{|\alpha|=l+2 m-1} W^{1}\left(\mathbb{R}^{2}\right)$ and a function $V_{j} \in W^{l+2 m}\left(\mathbb{R}^{2}\right)$ such that $V_{j}(y)=0$ for $|y| \geq 1$ and

$$
\begin{gather*}
\left.D^{\alpha} V_{j}\right|_{y=0}=0, \quad|\alpha| \leq l+2 m-2,  \tag{7.10}\\
D^{\alpha} V_{j}-W_{j \alpha} \in H_{0}^{1}\left(\mathbb{R}^{2}\right), \quad|\alpha|=l+2 m-1 . \tag{7.11}
\end{gather*}
$$

2. Let us show that the function $V=\left(V_{1}, \ldots, V_{N}\right)$ is a required function. Inequality (7.5) follows from Eq. (7.10), Lemma 5.2, and the boundedness of operator (7.9).

Prove Eq. (7.4). Since the functions $W_{j \alpha}$ represent a solution of algebraic system (7.7), (7.8) and the functions $V_{j}$ satisfy Eq. (7.11), we have

$$
\begin{gather*}
\hat{\mathcal{B}}_{j \sigma \mu}(D) V-D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1} \Phi_{j \sigma \mu} \in H_{0}^{1}\left(\mathbb{R}^{2}\right),  \tag{7.12}\\
D^{l-1}\left(\mathcal{P}_{j}(D) V_{j}-f_{j}\right) \in H_{0}^{1}\left(\mathbb{R}^{2}\right) . \tag{7.13}
\end{gather*}
$$

Further, from Eqs. (7.10) and (5.6) we obtain that

$$
\left.D^{\alpha}\left(\mathcal{P}_{j}(D) V_{j}-f_{j}\right)\right|_{y=0}=0, \quad|\alpha| \leq l-2 .
$$

Combining these equalities with Eq. (7.13) and Lemma 5.2, we see that

$$
\left\{\mathcal{P}_{j}(D) V_{j}-f_{j}\right\} \in \mathcal{H}_{0}^{l}(K) .
$$

Now let us show that

$$
\begin{equation*}
\left\{\mathcal{B}_{j \sigma \mu}(D) V-f_{j \sigma \mu}\right\} \in \mathcal{H}_{0}^{l+2 m-\mathbf{m}-1 / 2}(\gamma) . \tag{7.14}
\end{equation*}
$$

We pass from the "local" operators $\hat{\mathcal{B}}_{j \sigma \mu}(D)$ to the nonlocal operators $D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1} \mathcal{B}_{j \sigma \mu}(D)$ in (7.12). Using Lemma 5.7, we obtain

$$
\begin{equation*}
D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1}\left(\mathcal{B}_{j \sigma \mu}(D) V-\Phi_{j \sigma \mu}\right) \in H_{0}^{1}\left(\mathbb{R}^{2}\right) \tag{7.15}
\end{equation*}
$$

from Eq. (7.12). Equation (7.15) and [53, Lemma 4.18] yield the following equality:

$$
\begin{array}{rl}
\int_{0}^{\infty} r^{-1}\left|D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1}\left(\mathcal{B}_{j \sigma \mu}(D) V-f_{j \sigma \mu}\right)\right|^{2} & d r \\
& \leq k_{1}\left\|D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1}\left(\mathcal{B}_{j \sigma \mu}(D) V-\Phi_{j \sigma \mu}\right)\right\|_{H_{0}^{1}\left(K_{j}\right)}^{2} \tag{7.16}
\end{array}
$$

Inequalities (7.16), Eqs. (5.6) and (7.10), and Lemma [53, Lemma 4.7] imply

$$
\begin{equation*}
\int_{0}^{\infty} r^{1-2\left(l+2 m-m_{j \sigma \mu}\right)}\left|\mathcal{B}_{j \sigma \mu}(D) V-f_{j \sigma \mu}\right|^{2} d r \leq k_{2}\left\|D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1}\left(\mathcal{B}_{j \sigma \mu}(D) V-\Phi_{j \sigma \mu}\right)\right\|_{H_{0}^{1}\left(K_{j}\right)}^{2} \tag{7.17}
\end{equation*}
$$

Combining this inequality with the relation

$$
\left\{\mathcal{B}_{j \sigma \mu}(D) V-f_{j \sigma \mu}\right\} \in \mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)
$$

we obtain (7.14) from Eq. (7.17) and [53, Lemma 4.16]. Using the boundedness of the operators in Eqs. (7.6) and (7.9), we can easily prove estimate (7.4).
3. It remains to prove that system (7.7), (7.8) has a unique solution for any right-hand side. Obviously, this system consists of $(l+2 m) N$ equations with respect to $(l+2 m) N$ variables. Hence, it suffices to show that the corresponding homogeneous system has only trivial solution. Assume the contrary: there exists a nontrivial set of numbers $\left\{q_{j \alpha}\right\}(j=1, \ldots, N$ and $|\alpha|=l+2 m-1)$, such that if we replace $D^{\alpha} W_{j}$ on the left-hand sides of system (7.7), (7.8) by the numbers $q_{j \alpha}$, then its right-hand sides vanish. Let us consider a homogeneous polynomial $Q_{j}(y)$ of degree $l+2 m-1$ such that $D^{\alpha} Q_{j}(y) \equiv q_{j \alpha}$. Then $\mathcal{P}_{j}(D) Q_{j}(y) \equiv 0\left(\right.$ since $D^{\xi} \mathcal{P}_{j}(D) Q_{j}(y) \equiv 0$ for all $\left.|\xi|=l-1\right)$ and

$$
\begin{equation*}
\hat{\mathcal{B}}_{j \sigma \mu}(D) Q(y) \equiv \sum_{k, s} \hat{B}_{j \sigma \mu k s}(D) Q_{k}(y) \equiv 0 \quad\left(Q=\left(Q_{1}, \ldots, Q_{N}\right)\right) . \tag{7.18}
\end{equation*}
$$

Note that $\hat{B}_{j \sigma \mu k s}(D) Q_{k}(y) \equiv$ const and the operators $\mathcal{G}_{j \sigma k s}$ map a constant to itself. Hence we have the following (along with (7.18)):

$$
\begin{equation*}
D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1}\left(\mathcal{B}_{j \sigma \mu}(D) Q(y)\right) \equiv \sum_{k, s}\left(\hat{B}_{j \sigma \mu k s}(D) Q_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right) \equiv 0 . \tag{7.19}
\end{equation*}
$$

Since $\mathcal{B}_{j \sigma \mu}(D) Q$ is a homogeneous polynomial of degree $l+2 m-m_{j \sigma \mu}-1$, we see that, by virtue of (7.19), $\left.\mathcal{B}_{j \sigma \mu}(D) Q\right|_{\gamma_{j \sigma}} \equiv 0$. Thus, we obtain that the vector-valued function $Q=\left(Q_{1}, \ldots, Q_{N}\right)$ is a solution of problem (6.16), (6.17) with zero right-hand side. Hence,

$$
\begin{align*}
& \tilde{\mathcal{P}}_{j}\left(\omega, D_{\omega}, r D_{r}\right)\left(r^{l+2 m-1} \tilde{Q}_{j}(\omega)\right) \equiv 0 \\
& \left.\sum_{k, s}\left(\chi_{j \sigma k s}\right)^{(l+2 m-1)-m_{j \sigma \mu}} \tilde{B}_{j \sigma \mu k s}\left(\omega, D_{\omega}, r D_{r}\right)\left(r^{l+2 m-1} \tilde{Q}_{k}\left(\omega+\omega_{j \sigma k s}\right)\right)\right|_{\omega=(-1)^{\sigma} \omega_{j}} \equiv 0, \tag{7.20}
\end{align*}
$$

where $Q_{j}(y) \equiv r^{l+2 m-1} \tilde{Q}_{j}(\omega)$. However, Eqs. (7.20) mean that

$$
\tilde{\mathcal{L}}(-i(l+2 m-1)) \tilde{Q}(\omega) \equiv 0
$$

where $\tilde{Q}=\left(\tilde{Q}_{1}, \ldots, \tilde{Q}_{N}\right)$. This contradicts the assumptions that the line $\operatorname{Im} \lambda=1-l-2 m$ does not contain any eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$.

Corollary 7.1. The function $V$ constructed in Lemma 7.1 satisfies the inequality

$$
\begin{equation*}
\|\mathfrak{L} V-f\|_{\mathcal{H}_{0}^{l}(K)} \leq c\|f\|_{\mathcal{W}^{l}(K, \gamma)} . \tag{7.21}
\end{equation*}
$$

Proof. By virtue of Eq. (7.4), it suffices to estimate the differences

$$
\left(\mathbf{P}_{j}(y, D)-\mathcal{P}_{j}(D)\right) V_{j}, \quad\left(\mathbf{B}_{j \sigma \mu}(y, D)-\mathcal{B}_{j \sigma \mu}(D)\right) V
$$

The first of these differences contains terms of the form

$$
\left(a_{\alpha}(y)-a_{\alpha}(0)\right) D^{\alpha} V_{j}(|\alpha|=2 m), \quad a_{\beta}(y) D^{\beta} V_{j}(|\beta| \leq 2 m-1),
$$

where $a_{\alpha}$ and $a_{\beta}$ are infinitely differentiable functions. Fix a number $a$ such that $0<a<1$. Taking into account the fact that $V=0$ for $|y| \geq 1$ and using Lemma 5.5 and Eq. (7.5), we obtain

$$
\begin{aligned}
&\left\|\left(a_{\alpha}(y)-a_{\alpha}(0)\right) D^{\alpha} V_{j}\right\|_{H_{0}^{l}\left(K_{j}\right)} \leq k_{1}\left\|\left(a_{\alpha}(y)-a_{\alpha}(0)\right) D^{\alpha} V_{j}\right\|_{H_{a-1}^{l}\left(K_{j}\right)} \\
& \leq k_{2}\left\|D^{\alpha} V_{j}\right\|_{H_{a}^{l}\left(K_{j}\right)} \leq k_{3}\|f\|_{\mathcal{W}^{l}(K, \gamma)} .
\end{aligned}
$$

Similarly, using the definition of weight spaces and Eq. (7.5), we obtain

$$
\left\|a_{\beta}(y) D^{\beta} V_{j}\right\|_{H_{0}^{l}\left(K_{j}\right)} \leq k_{4}\left\|a_{\beta}(y) D^{\beta} V_{j}\right\|_{H_{a}^{l+1}\left(K_{j}\right)} \leq k_{5}\left\|V_{j}\right\|_{H_{a}^{l+2 m}\left(K_{j}\right)} \leq k_{6}\|f\|_{\mathcal{W}^{l}(K, \gamma)} .
$$

The relation $\left(\mathbf{B}_{j \sigma \mu}(y, D)-\mathcal{B}_{j \sigma \mu}(D)\right) V$ can be proved similarly.
7.2. Spaces $\hat{\mathcal{S}}^{l}(K, \gamma)$. Consider the case where the line $\operatorname{Im} \lambda=1-l-2 m$ contains eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Let $\lambda_{0}$ be one of these eigenvalue.
Definition 7.1. We say that $\lambda_{0}$ is a regular eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$ if
(1) none of the eigenvectors $\varphi(\omega)=\left(\varphi_{1}(\omega), \ldots, \varphi_{N}(\omega)\right)$ corresponding to $\lambda_{0}$ has adjoint vectors;
(2) for any eigenvector $\varphi(\omega)=\left(\varphi_{1}(\omega), \ldots, \varphi_{N}(\omega)\right)$ corresponding to $\lambda_{0}$, the functions $r^{i \lambda_{0}} \varphi_{j}(\omega)$, $j=1, \ldots, N$, are polynomials with respect in the variables $y_{1}$ and $y_{2}$.
Eigenvalues that are not regular are said to be irregular.
Remark 7.1. The notion of a regular eigenvalue was first introduced by V. A. Kondrat'ev in [53] for "local" elliptic boundary-value problems in domains with angular or conical points at the boundary.

Obviously, if $\lambda_{0}$ is a regular eigenvalue, then $\operatorname{Re} \lambda_{0}=0$. Hence, the line $\operatorname{Im} \lambda=1-l-2 m$ can contain no more than one eigenvalue. In this case, the functions $r^{i \lambda_{0}} \varphi_{j}(\omega)$ are homogeneous polynomials of degree $i \lambda_{0}$.

In Secs. 7.2-7.4, we assume that the following condition holds.
Condition 7.2. The line $\operatorname{Im} \lambda=1-l-2 m$ contains a unique regular eigenvalue $\lambda_{0}=i(1-l-2 m)$.

If this condition holds, Lemma 7.1 is invalid since the algebraic system (7.7), (7.8) can be insolvable for some right-hand side and the system of operators $(7.2),(7.3)$ is not linearly independent. Indeed, let $\varphi(\omega)=\left(\varphi_{1}(\omega), \ldots, \varphi_{N}(\omega)\right)$ be an eigenvector corresponding to a regular eigenvalue $\lambda_{0}=i(1-l-2 m)$. Then, by the definition of regular eigenvalues, the function $Q_{j}(y)=r^{l+2 m-1} \varphi_{j}(\omega)$ is a homogeneous polynomial of degree $l+2 m-1$ with respect to the variables $y=\left(y_{1}, y_{2}\right)$. Repeating the reasoning of the proof of Lemma 7.1, item 3, we see that if we substitute numbers $q_{j \alpha}=D^{\alpha} Q_{j}$ instead of $D^{\alpha} W_{j}$ on the left-hand sides of system (7.7), (7.8), then its right-hand sides vanish. Hence, system (7.2), (7.3) is linearly dependent.

In this case, we use the spaces $\hat{\mathcal{S}}^{l}(K, \gamma)$ (to be defined below) instead of the spaces $\mathcal{S}^{l}(K, \gamma)$. We note that the set $\hat{\mathcal{S}}^{l}(K, \gamma)$ is not closed in topology of the space $\mathcal{W}^{l}(K, \gamma)$.

Choose the maximal number of linearly independent operators from system (7.2) that consists of homogeneous differential operators of order $l+2 m-1$ and denote them by

$$
\begin{equation*}
\hat{\mathcal{B}}_{j^{\prime} \sigma^{\prime} \mu^{\prime}}(D) U \tag{7.22}
\end{equation*}
$$

Any operator $\hat{\mathcal{B}}_{j \sigma \mu}(D)$ that does not belong to system (7.22) can be written in the form

$$
\begin{equation*}
\hat{\mathcal{B}}_{j \sigma \mu}(D) U=\sum_{j^{\prime}, \sigma^{\prime}, \mu^{\prime}} \beta_{j \sigma \mu}^{j^{\prime} \sigma^{\prime} \mu^{\prime}} \hat{\mathcal{B}}_{j^{\prime} \sigma^{\prime} \mu^{\prime}}(D) U \tag{7.23}
\end{equation*}
$$

where $\beta_{j \sigma \mu}^{j^{\prime} \sigma^{\prime} \mu^{\prime}}$ are some constants.
Let us consider functions $f=\left\{f_{j}, f_{j \sigma \mu}\right\} \in \mathcal{W}^{l}(K, \gamma)$ satisfying the conditions

$$
\begin{equation*}
\mathcal{T}_{j \sigma \mu} f \equiv D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1} \Phi_{j \sigma \mu}-\sum_{j^{\prime}, \sigma^{\prime}, \mu^{\prime}} \beta_{j \sigma \mu}^{j^{\prime} \sigma^{\prime} \mu^{\prime}} D_{\tau_{j^{\prime} \sigma^{\prime}}^{l}}^{l+2 m-m_{j^{\prime} \sigma^{\prime} \mu^{\prime}}-1} \Phi_{j^{\prime} \sigma^{\prime} \mu^{\prime}} \in H_{0}^{1}\left(\mathbb{R}^{2}\right), \tag{7.24}
\end{equation*}
$$

where the indices $j^{\prime}, \sigma^{\prime}$, and $\mu^{\prime}$ correspond to operators (7.22) and the indices $j, \sigma$, and $\mu$ correspond to operators from system (7.2) that do not belong to system (7.22); $\Phi_{j \sigma \mu}$ are some fixed extensions of the functions $f_{j \sigma \mu}$ to $\mathbb{R}^{2}$ determined by operator (7.6); $\beta_{j \sigma \mu}^{j^{\prime} \sigma^{\prime} \mu^{\prime}}$ are constants from Eq. (7.23). If system (7.2) is linearly dependent, then the set of conditions (7.24) is empty.

Note that the fulfillment of conditions (7.24) is independent of the choice of extensions of the functions $f_{j \sigma \mu}$ to $\mathbb{R}^{2}$. Indeed, let $\hat{\Phi}_{j \sigma \mu}$ be an extension that differs from $\Phi_{j \sigma \mu}$. Then $\left.\left(\Phi_{j \sigma \mu}-\hat{\Phi}_{j \sigma \mu}\right)\right|_{\gamma_{j \sigma}}=0$; hence, by [53, Theorem 4.8],

$$
D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1}\left(\Phi_{j \sigma \mu}-\hat{\Phi}_{j \sigma \mu}\right) \in H_{0}^{1}\left(\mathbb{R}^{2}\right)
$$

Now we supplement system (7.22) by differential operators of order $l+2 m-1$ taken from system (7.3) such that the resulting system consists of linearly independent operators:

$$
\begin{equation*}
\hat{\mathcal{B}}_{j^{\prime} \sigma^{\prime} \mu^{\prime}}(D) U, \quad D^{\xi^{\prime}} \mathcal{P}_{j^{\prime}}(D) U_{j^{\prime}} \tag{7.25}
\end{equation*}
$$

and any operator $D^{\xi} \mathcal{P}_{j}(D) U_{j}$ that does not belong to (7.25) can be represented in the form

$$
\begin{equation*}
D^{\xi} \mathcal{P}_{j}(D) U_{j}=\sum_{j^{\prime}, \sigma^{\prime}, \mu^{\prime}} p_{j \xi}^{j^{\prime} \sigma^{\prime} \mu^{\prime}} \hat{\mathcal{B}}_{j^{\prime} \sigma^{\prime} \mu^{\prime}}(D) U+\sum_{j^{\prime}, \xi^{\prime}} p_{j \xi}^{j^{\prime} \xi^{\prime}} D^{\xi^{\prime}} \mathcal{P}_{j^{\prime}}(D) U_{j^{\prime}} \tag{7.26}
\end{equation*}
$$

where $p_{j \xi}^{j^{\prime}, \sigma^{\prime}, \mu^{\prime}}$ and $p_{j \xi}^{j^{\prime}, \xi^{\prime}}$ are some constants.
Now we expand the components $f_{j} \in W^{l}\left(K_{j}\right)$ of the vector $f$ to the whole $\mathbb{R}^{2}$. We denote the expanded functions by $f_{j} \in W^{l}\left(\mathbb{R}^{2}\right)$. Consider the functions $f$ satisfying the conditions
where the indices $j^{\prime}, \sigma^{\prime}$, and $\mu^{\prime}$ and the indices $j^{\prime}$ and $\xi^{\prime}$ correspond to operators (7.25), and the indices $j$ and $\xi$ correspond to the operators from system (7.3) that do not belong to (7.25); $p_{j \xi}^{j^{\prime} \sigma^{\prime} \mu^{\prime}}$ and $p_{j \xi}^{j^{\prime} \xi^{\prime}}$
are constants from Eq. (7.26). Similarly to the previous reasoning, we can show that the fulfillment of conditions (7.27) is independent of the choice of the functions $f_{j}$ and $f_{j \sigma \mu}$ in $\mathbb{R}^{2}$.

Note that the set of conditions (7.27) is empty if either $l=0$ or $l \geq 1$ and system (7.25) contains all operators from Eq. (7.3).

Let us introduce an analog of the set $\mathcal{S}^{l}(K, \gamma)$ in the case where Eq. 7.2 is valid. Denote the set of functions $f \in \mathcal{W}^{l}(K, \gamma)$ satisfying conditions (5.6), (5.7), (7.24), and (7.27) by $\hat{\mathcal{S}}^{l}(K, \gamma)$. The space $\hat{\mathcal{S}}^{l}(K, \gamma)$ with the norm

$$
\begin{equation*}
\|f\|_{\hat{\mathcal{S}}^{l}(K, \gamma)}=\left(\|f\|_{\mathcal{W}^{l}(K, \gamma)}^{2}+\sum_{j, \sigma, \mu}\left\|\mathcal{T}_{j \sigma \mu} f\right\|_{H_{0}^{1}\left(\mathbb{R}^{2}\right)}^{2}+\sum_{j, \xi}\left\|\mathcal{T}_{j \xi} f\right\|_{H_{0}^{1}\left(\mathbb{R}^{2}\right)}^{2}\right)^{1 / 2} \tag{7.28}
\end{equation*}
$$

is complete. (In the definition of norm (7.28), the indices $j, \sigma$, and $\mu$ and the indices $j$ and $\xi$ correspond to operators that do not belong to system (7.25).) Introduce the space

$$
\hat{\mathcal{S}}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)=\left\{f^{\prime} \in \mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}(\gamma):\left(0, f^{\prime}\right) \in \hat{\mathcal{S}}^{l}(K, \gamma)\right\} .
$$

Obviously, the following embeddings are valid:

$$
\begin{gather*}
\hat{\mathcal{S}}^{l+2 m-\mathbf{m}-1 / 2}(\gamma) \subset \mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}(\gamma) \subset W^{l+2 m-\mathbf{m}-1 / 2}(\gamma) \\
\hat{\mathcal{S}}^{l}(K, \gamma) \subset \mathcal{S}^{l}(K, \gamma) \subset \mathcal{W}^{l}(K, \gamma) \tag{7.29}
\end{gather*}
$$

Let us prove some important properties of the space $\hat{\mathcal{S}}^{l}(K, \gamma)$. The following lemma shows that if a compactly supported function $U \in \mathcal{W}^{l+2 m}(K)$ satisfies finitely many orthogonality conditions of the form

$$
\begin{equation*}
\left.D^{\alpha} U\right|_{y=0}=0, \quad|\alpha| \leq l+2 m-2, \tag{7.30}
\end{equation*}
$$

then the right-hand side of the corresponding nonlocal problem belongs to $\hat{\mathcal{S}}^{l}(K, \gamma)$.
Lemma 7.2. Let condition 7.2 hold. Assume that $U \in \mathcal{S}^{l+2 m}(K)$ and $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon d_{1}}(0)$ (the number $d_{1}$ is defined in Eq. (6.15)). Then

$$
\begin{equation*}
\|\mathcal{L} U\|_{\hat{\mathcal{S}}^{l}(K, \gamma)} \leq c\|U\|_{\mathcal{W}^{l+2 m}(K)}, \quad\|\mathfrak{L} U\|_{\hat{\mathcal{S}}^{l}(K, \gamma)} \leq c\|U\|_{\mathcal{W}^{l+2 m}(K)} . \tag{7.31}
\end{equation*}
$$

Proof. 1. Let $f=\left\{f_{j}, f_{j \sigma \mu}\right\}=\mathcal{L} U$. It follows from assumptions of the lemma that $f \in \mathcal{W}^{l}(K, \gamma)$, $\operatorname{supp} f \subset \mathcal{O}_{\varepsilon}(0)$, and the functions $f_{j}$ and $f_{j \sigma \mu}$ satisfy relations (5.6) and (5.7), respectively.

Denote by $\Phi_{j \sigma \mu} \in W^{l+2 m-m_{j \sigma \mu}}\left(\mathbb{R}^{2}\right)$ the extension of the function $f_{j \sigma \mu}$ defined by operator (7.6). Let us show that

$$
\begin{equation*}
\hat{\mathcal{B}}_{j \sigma \mu}(D) U-D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1} \Phi_{j \sigma \mu} \in H_{0}^{1}\left(\mathbb{R}^{2}\right) \tag{7.32}
\end{equation*}
$$

By lemma 5.7,

$$
\hat{\mathcal{B}}_{j \sigma \mu}(D) U-D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1} \mathcal{B}_{j \sigma \mu}(D) U \in H_{0}^{1}\left(\mathbb{R}^{2}\right) ;
$$

thus, to prove (7.32), it suffices to show that

$$
\begin{equation*}
D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1}\left(\mathcal{B}_{j \sigma \mu}(D) U-\Phi_{j \sigma \mu}\right) \in H_{0}^{1}\left(\mathbb{R}^{2}\right) . \tag{7.33}
\end{equation*}
$$

But

$$
\begin{gathered}
D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1}\left(\mathcal{B}_{j \sigma \mu}(D) U-\Phi_{j \sigma \mu}\right) \in W^{1}\left(\mathbb{R}^{2}\right), \\
\left.D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1}\left(\mathcal{B}_{j \sigma \mu}(D) U-\Phi_{j \sigma \mu}\right)\right|_{\gamma_{j \sigma}}=0 .
\end{gathered}
$$

Thus, relations (7.33) follow from [53, Lemma 4.8]. Relation (7.32) is proved.
The operators $\hat{\mathcal{B}}_{j \sigma \mu}(D) U$ satisfy relations (7.23); hence, by (7.32), the functions $\Phi_{j \sigma \mu}$ satisfy relations (7.24).

Similarly, it follows from Eq. (7.32), the equalities $\mathcal{P}_{j}(D) U_{j}-f_{j}=0$, and relations (7.26) that the function $f$ satisfies relations (7.27). Hence $f \in \hat{\mathcal{S}}^{l}(K, \gamma)$ and we can easily verify that the first inequality in Eq. (7.31) holds.
2. Now, to prove that $\mathfrak{L} U \in \hat{\mathcal{S}}^{l}(K, \gamma)$, it suffices to show that

$$
\begin{gathered}
D^{l-1}\left(\mathbf{P}_{j}(y, D)-\mathcal{P}_{j}(D)\right) U_{j} \in H_{0}^{1}\left(\mathbb{R}^{2}\right) \\
D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1}\left(\mathbf{B}_{j \sigma \mu}(y, D) U-\mathcal{B}_{j \sigma \mu}(D) U\right) \in H_{0}^{1}\left(\mathbb{R}^{2}\right)
\end{gathered}
$$

where $U_{j} \in W^{l+2 m}\left(\mathbb{R}^{2}\right)$ is an extension of the function $U_{j} \in W^{l+2 m}\left(K_{j}\right)$ to $\mathbb{R}^{2}$ (we denote this extension also by $U_{j}$ ). These relations consist of terms of the form

$$
\left(a_{\alpha}(y)-a_{\alpha}(0)\right) D^{\alpha} U_{j}, \quad|\alpha|=l+2 m-1, \quad a_{\beta}(y) D^{\beta} U_{j}, \quad|\beta| \leq l+2 m-2,
$$

where $a_{\alpha}$ and $a_{\beta}$ are infinitely differentiable functions.
Since $U_{j} \in W^{l+2 m}\left(\mathbb{R}^{2}\right)$, we see that $D^{\alpha} U_{j} \in H_{1}^{1}\left(\mathbb{R}^{2}\right)$. This and Lemma 5.5 imply that

$$
\left(a_{\alpha}(y)-a_{\alpha}(0)\right) D^{\alpha} U_{j} \in H_{0}^{1}\left(\mathbb{R}^{2}\right) .
$$

The function $a_{\beta} D^{\beta} U_{j}(|\beta| \leq l+2 m-2)$ belongs to $W^{2}\left(\mathbb{R}^{2}\right)$. This and relations (7.30) and Lemma 5.2 imply that

$$
a_{\beta} D^{\beta} U_{j} \in H_{a}^{2}\left(\mathbb{R}^{2}\right) \subset H_{a-1}^{1}\left(\mathbb{R}^{2}\right), \quad a>0
$$

Let $0<a<1$; then, by virtue of the compactness of supports of functions $U_{j}$, we obtain an embedding $a_{\beta} D^{\beta} U_{j} \in H_{0}^{1}\left(\mathbb{R}^{2}\right)$. Hence, as we can easily verify, the second inequality in Eq. (7.31) is valid.

The following lemma shows that the set $\hat{\mathcal{S}}^{l}(K, \gamma)$ is not closed in the topology of the space $\mathcal{W}^{l}(K, \gamma)$.
Lemma 7.3. Let condition 7.2 hold. Then there exists a family of functions $f^{\delta} \in \hat{\mathcal{S}}^{l}(K, \gamma), \delta>0$, such that $\operatorname{supp} f^{\delta} \subset \mathcal{O}_{\varepsilon}(0)$ and $f^{\delta}$ converges to a function $f^{0} \notin \hat{\mathcal{S}}^{l}(K, \gamma)$ in $\mathcal{W}^{l}(K, \gamma)$ as $\delta \rightarrow 0$.
Proof. 1. As was shown above, if a number $\lambda_{0}=i(1-l-2 m)$ is a regular eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$, then system (7.2), (7.3) is linearly dependent. The two cases are possible: (a) either system (7.2) is linearly dependent, (b) or system (7.2) is linearly independent, but system (7.2), (7.3) is linearly dependent.
2. First, we assume that system (7.2) is linearly dependent. Then the set of conditions (7.24) is not empty. In this case, norm (7.28) contains the corresponding term $\left\|\mathcal{T}_{j \sigma \mu} f\right\|_{H_{0}^{1}\left(\mathbb{R}^{2}\right)}$ for some $j, \sigma$, and $\mu$; we fix these values of $j, \sigma$, and $\mu$. Without loss of generality, we assume that $\gamma_{j \sigma}$ corresponds to the axis $O y_{1}$. Introduce functions $f^{\delta}=\left\{0, f_{j_{1} \sigma_{1} \mu_{1}}^{\delta}\right\}(0 \leq \delta \leq 1)$ such that $f_{j_{1} \sigma_{1} \mu_{1}}^{\delta}=0$ for $\left(j_{1}, \sigma_{1}, \mu_{1}\right) \neq(j, \sigma, \mu)$ and

$$
f_{j \sigma \mu}^{\delta}\left(y_{1}\right)=\psi\left(y_{1}\right) y_{1}^{l+2 m-m_{j \sigma \mu}-1+\delta}
$$

where $\psi \in C_{0}^{\infty}([0, \infty))$ and $\psi\left(y_{1}\right)=1$ for $0 \leq y_{1} \leq \varepsilon / 2$ and $\psi\left(y_{1}\right)=0$ for $y_{1} \geq 2 \varepsilon / 3$. Obviously,

$$
\hat{\Phi}_{j \sigma \mu}^{\delta}(y)=\psi(r) y_{1}^{l+2 m-m_{j \sigma \mu}-1} r^{\delta}
$$

is an extension of the function $f_{j \sigma \mu}^{\delta}$ to $\mathbb{R}^{2}$. Moreover, the extension operator defined on functions $f_{j \sigma \mu}^{\delta}$ $(0 \leq \delta \leq 1)$ is a bounded operator from the space $W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)$ to the space $W^{l+2 m-m_{j \sigma \mu}}\left(\mathbb{R}^{2}\right)$. This follows from the estimates $\left\|f_{j \sigma \mu}^{\delta}\right\|_{W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)} \geq c_{1}$ and $\left\|\hat{\Phi}_{j \sigma \mu}^{\delta}\right\|_{W^{l+2 m-m_{j \sigma \mu}\left(\mathbb{R}^{2}\right)}} \leq c_{2}$. Here $c_{1}, c_{2}>0$ are independent of $0 \leq \delta \leq 1$.

Thus, for $0<\delta \leq 1$ we have

$$
\begin{align*}
\left\|f^{\delta}\right\|_{\mathcal{W}^{l}(K, \gamma)}^{2} & =\left\|f_{j \sigma \mu}^{\delta}\right\|_{W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)}^{2} \\
\left\|f^{\delta}\right\|_{\hat{\mathcal{S}}^{l}(K, \gamma)}^{2} & \approx\left\|f_{j \sigma \mu}^{\delta}\right\|_{W^{l+2 m-m_{j \sigma \mu^{-1 / 2}}\left(\gamma_{j \sigma}\right)}}^{2}+\left\|D_{y_{1}}^{l+2 m-m_{j \sigma \mu}-1} \hat{\Phi}_{j \sigma \mu}^{\delta}\right\|_{H_{0}^{1}\left(\mathbb{R}^{2}\right)}^{2} \tag{7.34}
\end{align*}
$$

(a direct calculation shows that norms (7.34) are finite for $\delta>0$ ). Here the symbol " $\approx$ " means that the corresponding norms are equivalent. Further, we verify directly that $\hat{\Phi}_{j \sigma \mu}^{\delta} \rightarrow \hat{\Phi}_{j \sigma \mu}^{0}$ in $W^{l+2 m-m_{j \sigma \mu}}\left(\mathbb{R}^{2}\right)$ as $\delta \rightarrow 0$. Hence, $f_{j \sigma \mu}^{\delta} \rightarrow f_{j \sigma \mu}^{0}$ in $W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)$ as $\delta \rightarrow 0$. However, the corresponding function $f^{0}=\left\{0, f_{j \sigma \mu}^{0}\right\}$ does not belong to $\hat{\mathcal{S}}^{l}(K, \gamma)$. Indeed, let us assume the contrary. Then by Eq. (7.34)
we have that $D_{y_{1}}^{l+2 m-m_{j \sigma \mu}-1} \hat{\Phi}_{j \sigma \mu}^{0} \in H_{0}^{1}\left(\mathbb{R}^{2}\right)$. This is invalid because the function $D_{y_{1}}^{l+2 m-m_{j \sigma \mu}-1} \hat{\Phi}_{j \sigma \mu}^{0}$ equals a nonzero constant near the origin.
3. Now we assume that system (7.2) is linearly independent. Then system (7.2), (7.3) is linearly dependent. In this case, there are no conditions (7.24), but the set of conditions (7.27) is not empty. Hence, norm (7.28) contains the corresponding term $\left\|\mathcal{T}_{j \xi} f\right\|_{H_{0}^{1}\left(\mathbb{R}^{2}\right)}$ for some $j$ and $\xi$. We fix these values of $j$ and $\xi$ and introduce functions $f^{\delta}=\left\{f_{j_{1}}^{\delta}, 0\right\}(0 \leq \delta \leq 1)$ such that $f_{j_{1}}^{\delta}=0$ for $j_{1} \neq j$ and $f_{j}^{\delta}=\psi(r) y^{\xi} r^{\delta}$. It can be directly verified that $f_{j}^{\delta} \rightarrow f_{j}^{0}$ in $W^{l}\left(\mathbb{R}^{2}\right)$ as $\delta \rightarrow 0$, but $f^{0}=\left\{f_{j}^{0}, f_{j \sigma \mu}^{0}\right\} \notin$ $\hat{\mathcal{S}}^{l}(K, \gamma)$ since $D^{\xi} f_{j}^{0} \notin H_{0}^{1}\left(\mathbb{R}^{2}\right)$.
7.3. Construction of a "solution" in the case of a regular eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$ on the line $\operatorname{Im} \lambda=1-l-2 m$. Prove an analog of Lemma 7.1 in the case where condition 7.2 holds.

Lemma 7.4. Let condition 7.2 hold. Then there exists a bounded operator

$$
\hat{\mathcal{A}}:\left\{f \in \hat{\mathcal{S}}^{l}(K, \gamma): \operatorname{supp} f \subset \mathcal{O}_{1}(0)\right\} \rightarrow \mathcal{S}^{l+2 m}(K)
$$

such that for any $f=\left\{f_{j}, f_{j \sigma \mu}\right\} \in \mathrm{D}(\hat{\mathcal{A}})$, the function $V=\hat{\mathcal{A}} f$ satisfies the following conditions: $V=0$ for $|y| \geq 1$,

$$
\begin{equation*}
\|\mathcal{L} V-f\|_{\mathcal{H}_{0}^{l}(K)} \leq c\|f\|_{\hat{\mathcal{S}}^{l}(K, \gamma)} \tag{7.35}
\end{equation*}
$$

and inequality (7.5) holds.
Proof. 1. Similarly to the proof of Lemma 7.1, we consider the following algebraic system with respect to all partial derivatives $D^{\alpha} W_{j},|\alpha|=l+2 m-1, j=1, \ldots, N$ :

$$
\begin{align*}
\hat{\mathcal{B}}_{j^{\prime} \sigma^{\prime} \mu^{\prime}}(D) W & =D_{j_{j^{\prime} \sigma^{\prime}}^{l+m_{j^{\prime} \sigma^{\prime} \mu^{\prime}}-1} \Phi_{j^{\prime} \sigma^{\prime} \mu^{\prime}},},  \tag{7.36}\\
D^{\xi^{\prime}} \mathcal{P}_{j^{\prime}}(D) W_{j^{\prime}} & =D^{\xi^{\prime}} f_{j^{\prime}},
\end{align*}
$$

where $\Phi_{j^{\prime} \sigma^{\prime} \mu^{\prime}}$ and $f_{j^{\prime}}$ are the extensions of the functions $f_{j^{\prime} \sigma^{\prime} \mu^{\prime}}$ and $f_{j^{\prime}}$ to $\mathbb{R}^{2}$ described in the proof of Lemma 7.1. Now the left-hand side of system (7.36) contains only operators from system (7.25). The matrix of system (7.36) consists of $(l+2 m) N$ columns and $q(q<(l+2 m) N)$ linearly independent rows. Choose $q$ linearly independent columns and assume that the unknowns $D^{\alpha} W_{j}$ corresponding to the remaining $(l+2 m) N-q$ columns are equal to zero; then we obtain the system of $q$ equations with respect to $q$ variables, which has a unique solution. Thus, we define a linear bounded operator

Moreover, $W_{j \alpha}(y)=0$ for $|y| \geq 2$. Using the functions $D^{\alpha} W_{j}$ and operators (7.9), we obtain functions $V_{j}, j=1, \ldots, N$, satisfying relations (7.10) and (7.11). We show that the function $V=\left(V_{1}, \ldots, V_{N}\right)$ is as required.
2. Similarly to the proof of Lemma 7.1, we prove estimate (7.5) for all functions $V$. Prove inequality (7.35). Since $\left\{W_{j \alpha}\right\}$ is a solution of system (7.36) and the functions $V_{j}$ satisfy conditions (7.11), we have

$$
\begin{gather*}
\hat{\mathcal{B}}_{j^{\prime} \sigma^{\prime} \mu^{\prime}}(D) V-D_{\tau_{j^{\prime} \sigma^{\prime}}^{l+2 m-m_{j^{\prime} \sigma^{\prime} \mu^{\prime}}-1} \Phi_{j^{\prime} \sigma^{\prime} \mu^{\prime}}^{l+} \in H_{0}^{1}\left(\mathbb{R}^{2}\right),}^{D^{\xi^{\prime}}\left(\mathcal{P}_{j^{\prime}}(D) V_{j^{\prime}}-f_{j^{\prime}}\right) \in H_{0}^{1}\left(\mathbb{R}^{2}\right) .} . \tag{7.38}
\end{gather*}
$$

Consider an arbitrary operator $\hat{\mathcal{B}}_{j \sigma \mu}(D)$, which does not belong to system (7.25). Using (7.23), we have

$$
\begin{align*}
& \hat{\mathcal{B}}_{j \sigma \mu}(D) V-D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1} \Phi_{j \sigma \mu} \\
&=\sum_{j^{\prime}, \sigma^{\prime}, \mu^{\prime}} \beta_{j \sigma \mu}^{j^{\prime} \sigma^{\prime} \mu^{\prime}}\left(\hat{\mathcal{B}}_{j^{\prime} \sigma^{\prime} \mu^{\prime}}(D) V-D_{\tau_{j^{\prime} \sigma^{\prime}}}^{l+2 m-m_{j^{\prime} \sigma^{\prime} \mu^{\prime}}-1} \Phi_{j^{\prime} \sigma^{\prime} \mu^{\prime}}\right) \\
& \quad+\sum_{j^{\prime}, \sigma^{\prime}, \mu^{\prime}} \beta_{j \sigma \mu}^{j^{\prime} \sigma^{\prime} \mu^{\prime}} D_{\tau_{j^{\prime} \sigma^{\prime}}^{l+2 m-m_{j^{\prime} \sigma^{\prime} \mu^{\prime}}-1} \Phi_{j^{\prime} \sigma^{\prime} \mu^{\prime}}-D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1} \Phi_{j \sigma \mu} .} \tag{7.40}
\end{align*}
$$

However, $f \in \hat{\mathcal{S}}^{l}(K, \gamma)$; hence, conditions (7.24) hold. This and Eqs. (7.38) and (7.40) imply that the following relations hold for all $j, \sigma$, and $\mu$ :

$$
\begin{equation*}
\hat{\mathcal{B}}_{j \sigma \mu}(D) V-D_{\tau_{j \sigma}}^{l+2 m-m_{j \sigma \mu}-1} \Phi_{j \sigma \mu} \in H_{0}^{1}\left(\mathbb{R}^{2}\right) . \tag{7.41}
\end{equation*}
$$

We can similarly consider the operators $D^{\xi} P_{j}(D)$ that do not belong to system (7.25). Using relations (7.23) and (7.24), (7.26) and(7.27), and (7.38) and (7.39), we prove the relations

$$
\begin{equation*}
D^{\xi}\left(\mathcal{P}_{j}(D) V_{j}-f_{j}\right) \in H_{0}^{1}\left(\mathbb{R}^{2}\right) \tag{7.42}
\end{equation*}
$$

for all $j$ and $\xi$. Equations (7.41) and (7.42) and the proof of Lemma 7.1 yield estimate (7.35).
Similarly to Corollary 7.1, we prove the following corollary of Lemma 7.4.
Corollary 7.2. The function $V$ constructed in Lemma 7.4 satisfies the inequality

$$
\begin{equation*}
\|\mathfrak{L} V-f\|_{\mathcal{H}_{0}^{l}(K)} \leq c\|f\|_{\hat{\mathcal{S}}^{l}(K, \gamma)} . \tag{7.43}
\end{equation*}
$$

Consider a bounded operator $\mathcal{L}_{a}: \mathcal{H}_{a}^{l+2 m}(K) \rightarrow \mathcal{H}_{a}^{l}(K, \gamma)$ defined by the formula

$$
\begin{equation*}
\mathcal{L}_{a} U=\left\{\mathcal{P}_{j}(D) U_{j}, \mathcal{B}_{j \sigma \mu}(D) U\right\} \tag{7.44}
\end{equation*}
$$

where $\mathcal{H}_{a}^{l+2 m}(K)$ and $\mathcal{H}_{a}^{l}(K, \gamma)$ are the spaces introduced in Sec. 5.3. The operator $\mathcal{L}_{a}$ corresponds to problem (6.16), (6.17) in weight spaces. It follows from [88, Theorem 2.1] that the operator $\mathcal{L}_{a}$ has a bounded inverse operator if and only if the $\operatorname{line} \operatorname{Im} \lambda=a+1-l-2 m$ does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Using the invertibility of the operator $\mathcal{L}_{a}$ and the following lemma, we obtain a solution of problem (6.16), (6.17) in Sobolev spaces.

Lemma 7.5. Let $W \in \mathcal{H}_{a}^{l+2 m}(K)$ for some $a>0$ and $f=\mathcal{L}_{a} W \in \mathcal{H}_{0}^{l}(K, \gamma)$. Assume that the closed strip $1-l-2 m \leq \operatorname{Im} \lambda \leq a+1-l-2 m$ contains a unique eigenvalue $\lambda_{0}=i(1-l-2 m)$ of the operator $\tilde{\mathcal{L}}(\lambda)$ and this is a regular eigenvalue. Then

$$
\begin{equation*}
\left\|D^{l+2 m} W\right\|_{\mathcal{H}_{0}^{0}(K)} \leq c\|f\|_{\mathcal{H}_{0}^{l}(K, \gamma)} . \tag{7.45}
\end{equation*}
$$

Lemma 7.5 will be proved in Sec. 7.4. Now we study the solvability of problems (6.16), (6.17) and (6.12), (6.13).

Lemma 7.6. Let condition 7.2 hold. Then for any function $f \in \hat{\mathcal{S}}^{l}(K, \gamma)$ such that $\operatorname{supp} f \subset \mathcal{O}_{\varepsilon}(0)$, there exists a solution $U$ of problem (6.16), (6.17) such that $U \in \mathcal{S}^{l+2 m}\left(K^{d}\right)$ for any $d>0$ and the following inequalities hold:

$$
\begin{align*}
\|U\|_{\mathcal{W}^{l+2 m}\left(K^{d}\right)} & \leq c_{d}\|f\|_{\hat{\mathcal{S}}^{l}(K, \gamma)}  \tag{7.46}\\
\|U\|_{\mathcal{H}_{0}^{l+2 m-1}\left(K^{d}\right)} & \leq c_{d}\|f\|_{\mathcal{W}^{l}(K, \gamma)} \tag{7.47}
\end{align*}
$$

Proof. 1. Fix a number $a, 0<a<1$, for which the line

$$
1-l-2 m<\operatorname{Im} \lambda \leq a+1-l-2 m
$$

does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. The existence of such a number $a$ follows from the discontinuity of the spectrum of the operator $\tilde{\mathcal{L}}(\lambda)$ (see Lemma 6.1). By the definition of the space
$\hat{\mathcal{S}}^{l}(K, \gamma)$, relations (5.6) and (5.7) hold for functions $f=\left\{f_{j}, f_{j \sigma \mu}\right\}$ that satisfy the conditions of the lemma. Therefore, using Lemma 5.2, we have

$$
\begin{equation*}
\|f\|_{\mathcal{H}_{a}^{l}(K, \gamma)} \leq k_{1}\|f\|_{\mathcal{W}^{l}(K, \gamma)} \tag{7.48}
\end{equation*}
$$

Consider the function $f-\mathcal{L} V$, where $V=\hat{\mathcal{A}} f \in \mathcal{W}^{l+2 m}(K) \cap \mathcal{H}_{a}^{l+2 m}(K)$ is a function defined in Lemma 7.4. By inequalities (7.5) and (7.48), we see that

$$
\begin{equation*}
\|f-\mathcal{L} V\|_{\mathcal{H}_{a}^{l}(K, \gamma)} \leq k_{2}\|f\|_{\mathcal{W}^{l}(K, \gamma)} . \tag{7.49}
\end{equation*}
$$

Hence, the function $f-\mathcal{L} V \in \mathcal{H}_{a}^{l}(K, \gamma)$ belongs to the domain of the operator $\mathcal{L}_{a}^{-1}$. Introducing the notation $W=\mathcal{L}_{a}^{-1}(f-\mathcal{L} V)$, we see that $U=V+W$ is a solution of problem (6.16), (6.17).
2. Prove inequality (7.47). By virtue of the boundedness of the operator $\mathcal{L}_{a}^{-1}$ and inequality (7.49), we have

$$
\begin{equation*}
\|W\|_{H_{a}^{l+2 m}(K)} \leq k_{3}\|f\|_{\mathcal{W}^{l}(K, \gamma)} \tag{7.50}
\end{equation*}
$$

Now estimate (7.47) follows from inequalities (7.50) and (7.5) and the boundedness of the embedding $\mathcal{H}_{a}^{l+2 m}(K) \subset \mathcal{H}_{0}^{l+2 m-1}\left(K^{d}\right)$.
3. Prove (7.46). By virtue of the boundedness of the operator

$$
\hat{\mathcal{A}}: \hat{\mathcal{S}}^{l}(K, \gamma) \rightarrow \mathcal{W}^{l+2 m}(K)
$$

and inequality (7.50), it suffices to estimate the functions $D^{l+2 m} W$. Lemma 7.4 implies that

$$
f-\mathcal{L} V \in \mathcal{H}_{0}^{l}(K, \gamma)
$$

and estimate (7.35) holds. Hence, applying Lemma 7.5 to the function $W=\mathcal{L}_{a}^{-1}(f-\mathcal{L} V)$ and using Eq. (7.35), we obtain

$$
\left\|D^{l+2 m} W\right\|_{\mathcal{H}_{0}^{0}(K)} \leq k_{4}\|f-\mathcal{L} V\|_{\mathcal{H}_{0}^{l}(K, \gamma)} \leq k_{5}\|f\|_{\hat{\mathcal{S}}^{l}(K, \gamma)} .
$$

Since $H_{0}^{0}\left(K_{j}\right)=L_{2}\left(K_{j}\right)$, inequality (7.46) is proved.
7.4. Proof of Lemma 7.5. First, we assume that $W \in \prod_{j=1}^{N} C_{0}^{\infty}\left(\overline{K_{j}} \backslash\{0\}\right)$. Then $f_{j} \in C_{0}^{\infty}\left(\overline{K_{j}} \backslash\{0\}\right)$ and $f_{j \sigma \mu} \in C_{0}^{\infty}\left(\gamma_{j \sigma}\right)$, where $f=\left\{f_{j}, f_{j \sigma \mu}\right\}=\mathcal{L} W$. Denote the functions $W_{j}(y)$ and $f_{j}(y)$ written in the polar coordinates by $W_{j}(\omega, r)$ and $f_{j}(\omega, r)$, respectively. We denote the Fourier transformations of the functions $W_{j}\left(\omega, e^{\tau}\right), e^{2 m \tau} f_{j}\left(\omega, e^{\tau}\right)$, and $e^{m_{j \sigma \mu} \tau} f_{j \sigma \mu}\left(e^{\tau}\right)$ with respect to the variable $\tau$ by $\tilde{W}_{j}(\omega, \lambda)$, $\tilde{f}_{j}(\omega, \lambda)$, and $\tilde{f}_{j \sigma \mu}(\lambda)$. Assume that $\tilde{f}=\left\{\tilde{f}_{j}, \tilde{f}_{j \sigma \mu}\right\}$. Then $\lambda \mapsto \tilde{f}(\lambda)$ is a function analytic on the whole complex plane; moreover, if $|\operatorname{Re} \lambda| \rightarrow \infty$ in the strip $|\operatorname{Im} \lambda| \leq$ const, then this function tends to zero uniformly with respect to $\omega$ and $\lambda$ and faster than any power of $|\lambda|$.

By [88, Lemma 2.1], there exists a finitely meromorphic operator-valued function $\tilde{\mathcal{R}}(\lambda)$ such that $\tilde{\mathcal{R}}(\lambda)=(\tilde{\mathcal{L}}(\lambda))^{-1}$ for any number $\lambda$, which is not an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$. Moreover, if the line $\operatorname{Im} \lambda=a+1-l-2 m$ does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$, then, as was shown in the proof of [88, Theorem 2.1], the solution $W$ can be represented in the form

$$
\begin{equation*}
W\left(\omega, e^{\tau}\right)=\int_{-\infty+i(a+1-l-2 m)}^{+\infty+i(a+1-l-2 m)} e^{i \lambda \tau} \tilde{\mathcal{R}}(\lambda) \tilde{f}(\lambda) d \lambda \tag{7.51}
\end{equation*}
$$

Consider the derivative $D^{l+2 m} W(y)$ of order $l+2 m$ of the function $W$ with respect to the variables $y_{1}$ and $y_{2}$. Let us write the operator $D^{l+2 m}$ in the polar coordinates as follows: $r^{-(l+2 m)} \tilde{M}\left(\omega, D_{\omega}, r D_{r}\right)$.

After the substitution $r=e^{\tau}$, the operator $D^{l+2 m}$ has the form $e^{-(l+2 m) \tau} \tilde{M}\left(\omega, D_{\omega}, D_{\tau}\right)$, where $D_{\tau}=$ $-i \frac{\partial}{\partial \tau}$. It follows from Eq. (7.51) that the function $D^{l+2 m} W(y)$ can be obtained from the function

$$
\begin{equation*}
e^{-(l+2 m) \tau} \int_{-\infty+i(a+1-l-2 m)}^{+\infty+i(a+1-l-2 m)} e^{i \lambda \tau} \tilde{M}\left(\omega, D_{\omega}, \lambda\right) \tilde{\mathcal{R}}(\lambda) \tilde{f}(\lambda) d \lambda \tag{7.52}
\end{equation*}
$$

by the substitution $\tau=\ln r$ and the passage to Cartesian coordinates. Let us show that the operatorvalued function $\tilde{M}\left(\omega, D_{\omega}, \lambda\right) \tilde{\mathcal{R}}(\lambda)$ is analytic near the point $\lambda_{0}=i(1-l-2 m)$. Since $\lambda_{0}$ is a (regular) eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$, by [23] we have the following:

$$
\tilde{\mathcal{R}}(\lambda)=\frac{A_{-1}}{\lambda-\lambda_{0}}+\Gamma(\lambda)
$$

where $\Gamma(\lambda)$ is an analytic near $\lambda_{0}$ operator-valued function and the image of the operator $A_{-1}$ coincides with the span of eigenvectors corresponding to $\lambda_{0}$. Hence, for any function $\tilde{f} \in \mathcal{W}^{l}[-\bar{\omega}, \bar{\omega}]$ we have the following:

$$
\tilde{M}\left(\omega, D_{\omega}, \lambda\right) \tilde{\mathcal{R}}(\lambda) \tilde{f}=\frac{\tilde{M}\left(\omega, D_{\omega}, \lambda\right) A_{-1} \tilde{f}}{\lambda-\lambda_{0}}+\tilde{M}\left(\omega, D_{\omega}, \lambda\right) \Gamma(\lambda) \tilde{f}
$$

By the definition of regular eigenvalues, the function $r^{l+2 m-1} A_{-1} \tilde{f}$ is a vector $Q(y)=$ $\left(Q_{1}(y), \ldots, Q_{N}(y)\right)$, where $Q_{j}(y)$ are polynomials of order $l+2 m-1$ with respect to the variables $y_{1}$ and $y_{2}$. Hence,

$$
\tilde{M}\left(\omega, D_{\omega}, \lambda\right) A_{-1} \tilde{f}=r^{1-l-2 m} \tilde{M}\left(\omega, D_{\omega}, r D_{r}\right)\left(r^{l+2 m-1} A_{-1} \tilde{f}\right)=r D^{l+2 m} Q(y)=0
$$

Thus, the operator-valued function $\tilde{M}\left(\omega, D_{\omega}, \lambda\right) \tilde{\mathcal{R}}(\lambda)$ is analytic near $\lambda_{0}=i(1-l-2 m)$ and, hence, in the closed strip $1-l-2 m \leq \operatorname{Im} \lambda \leq a+1-l-2 m$.

Further, for $|\operatorname{Im} \lambda| \leq$ const, if $|\operatorname{Re} \lambda| \rightarrow \infty$, then the norm $\left\|\tilde{M}\left(\omega, D_{\omega}, \lambda\right) \tilde{\mathcal{R}}(\lambda)\right\|_{\mathcal{W}^{l}[-\bar{\omega}, \bar{\omega}] \rightarrow \mathcal{W}^{0}(-\bar{\omega}, \bar{\omega})}$ does not increase faster than some power of $|\lambda|$ (see [88, Lemma 2.1]), while $\|\tilde{f}(\lambda)\|_{\mathcal{W}^{l}[-\bar{\omega}, \bar{\omega}]}$ tends to zero faster than any power of $|\lambda|$. Hence, we can integrate Eq. (7.52) over the line $\operatorname{Im} \lambda=1-l-2 m$ instead of $\operatorname{Im} \lambda=a+1-l-2 m$. Thus, the function $D^{l+2 m} W(y)$ can be obtained from the function

$$
e^{-(l+2 m) \tau} \int_{-\infty+i(1-l-2 m)}^{+\infty+i(1-l-2 m)} e^{i \lambda \tau} \tilde{M}\left(\omega, D_{\omega}, \lambda\right) \tilde{\mathcal{R}}(\lambda) \tilde{f}(\lambda) d \lambda
$$

by the substitution $\tau=\ln r$ and the passage to the Cartesian coordinates. Estimate the norm of the function $D^{l+2 m} W$ :

$$
\begin{aligned}
\left\|D^{l+2 m} W\right\|_{\mathcal{H}_{0}^{0}(K)}^{2}= & \sum_{j} \int_{K_{j}}\left|D^{l+2 m} W_{j}\right|^{2} d y \\
& =\sum_{j} \int_{-\omega_{j}}^{\omega_{j}} d \omega \int_{-\infty}^{+\infty} e^{-2(l+2 m-1) \tau}\left|\int_{-\infty+i(1-l-2 m)}^{+\infty+i(1-l-2 m)} e^{i \lambda \tau} \tilde{M}\left(\omega, D_{\omega}, \lambda\right) \tilde{\mathcal{R}}(\lambda) \tilde{f}(\lambda) d \lambda\right|^{2} d \tau
\end{aligned}
$$

The following equality can be obtained from here and the complex analog of the Parseval inequality:

$$
\begin{equation*}
\left\|D^{l+2 m} W\right\|_{\mathcal{H}_{0}^{0}(K)}^{2}=\int_{-\infty+i(1-l-2 m)}^{+\infty+i(1-l-2 m)}\left\|\tilde{M}\left(\omega, D_{\omega}, \lambda\right) \tilde{\mathcal{R}}(\lambda) \tilde{f}(\lambda)\right\|_{\mathcal{W}^{0}(-\bar{\omega}, \bar{\omega})}^{2} d \lambda \tag{7.54}
\end{equation*}
$$

Estimate the norm in the integral on the right-hand side. For this, we introduce equivalent norms depending on a parameter $\lambda \neq 0$ :

$$
\begin{gathered}
\left\|\left|\left|\tilde{U}_{j}\| \|_{W^{k}\left(-\omega_{j}, \omega_{j}\right)}^{2}=\left\|\tilde{U}_{j}\right\|_{W^{k}\left(-\omega_{j}, \omega_{j}\right)}^{2}+|\lambda|^{2 k}\left\|\tilde{U}_{j}\right\|_{L_{2}\left(-\omega_{j}, \omega_{j}\right)}^{2},\right.\right.\right. \\
\mid\|\tilde{f}\| \|_{\mathcal{W}^{l}[-\bar{\omega}, \bar{\omega}]}^{2}=\sum_{j}\left(| |\left|\tilde{f}_{j}\right| \|_{W^{l}\left(-\omega_{j}, \omega_{j}\right)}^{2}+\sum_{\sigma, \mu}|\lambda|^{2\left(l+2 m-m_{j \sigma \mu}-1 / 2\right)}\left|\tilde{f}_{j \sigma \mu}\right|^{2}\right) .
\end{gathered}
$$

By the interpolation inequality (see. [1, Chap. 1])

$$
|\lambda|^{l+2 m-k}\left\|\tilde{U}_{j}\right\|_{W^{k}\left(-\omega_{j}, \omega_{j}\right)} \leq c_{k} \mid\left\|\tilde{U}_{j}\right\| \|_{W^{l+2 m}\left(-\omega_{j}, \omega_{j}\right)}, \quad 0<k<l+2 m,
$$

and [88, Lemma 2.1], there exists a constant $C>0$ such that the following estimate holds for any $\lambda \in \mathbb{C}$ satisfying the conditions $\operatorname{Im} \lambda=1-l-2 m$ and $|\operatorname{Re} \lambda|>C$ :

$$
\begin{equation*}
\left\|\tilde{M}\left(\omega, D_{\omega}, \lambda\right) \tilde{\mathcal{R}}(\lambda) \tilde{f}(\lambda)\right\|_{\mathcal{W}^{0}(-\bar{\omega}, \bar{\omega})}^{2} \leq k_{1} \mid\|\tilde{f}(\lambda)\|_{\mathcal{W}^{l}[-\bar{\omega}, \bar{\omega}]}^{2} . \tag{7.55}
\end{equation*}
$$

Since the operator-valued function

$$
\tilde{M}\left(\omega, D_{\omega}, \lambda\right) \tilde{\mathcal{R}}(\lambda): \mathcal{W}^{l}[-\bar{\omega}, \bar{\omega}] \rightarrow \mathcal{W}^{0}(-\bar{\omega}, \bar{\omega})
$$

is analytic in the interval $\{\lambda \in \mathbb{C}: \operatorname{Im} \lambda=1-l-2 m,|\operatorname{Re} \lambda| \leq C\}$, we see that inequality (7.55) holds on the whole line $\operatorname{Im} \lambda=1-l-2 m$. Equations (7.54) and (7.55) yield the inequality

$$
\left\|D^{l+2 m} W\right\|_{\mathcal{H}_{0}^{0}(K)}^{2} \leq k_{1} \int_{-\infty+i(1-l-2 m)}^{+\infty+i(1-l-2 m)}\||\tilde{f}(\lambda)|\|_{\mathcal{W}^{l}[-\bar{\omega}, \bar{\omega}]}^{2} d \lambda .
$$

Estimate (7.45) follows from here and [53, (1.9), (1.10)]. Since the set $C_{0}^{\infty}\left(\overline{K_{j}} \backslash\{0\}\right)$ is dense in $H_{a}^{k}\left(K_{j}\right)$ for any $a$ and $k$, we see that estimate (7.45) holds for $W \in \mathcal{H}_{a}^{l+2 m}(K)$ and $f \in \mathcal{H}_{0}^{l}(K, \gamma)$.

## Chapter 3

## STRONG SOLUTIONS OF NONLOCAL ELLIPTIC PROBLEMS IN BOUNDED DOMAINS IN SOBOLEV SPACES

8. Absence of Eigenvalues of the Operator $\tilde{\mathcal{L}}(\lambda)$ on the Line $\operatorname{Im} \lambda=1-l-2 m$

In this section, we use results of Sec. 7.1 (see Chap. 2) for the construction of a right regularizer for the operator

$$
\mathbf{L}=\left\{\mathbf{P}, \mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+\mathbf{B}_{i \mu}^{2}\right\}: W^{l+2 m}(G) \rightarrow \mathcal{W}^{l}(G, \partial G)
$$

(see Sec. 6.1, Chap. 2), corresponding to problem (6.7), (6.8) in Sobolev spaces. It follows from the existence of a right regularizer that the operator $\mathbf{L}$ is closed and its co-kernel is finite-dimensional. To prove the fact that the kernel of the operator $\mathbf{L}$ has a finite dimension, we reduce the operator $\mathbf{L}$ to an operator acting in weight spaces and having a finite-dimensional kernel.

Introduce the notation $\mathbf{B}^{k}=\left\{\mathbf{B}_{i \mu}^{k}\right\}_{i, \mu}, k=0, \ldots, 2 ; \mathbf{B}=\mathbf{B}^{0}+\mathbf{B}^{1}+\mathbf{B}^{2}, \mathbf{C}=\mathbf{B}^{0}+\mathbf{B}^{1}$. In addition to the operator $\mathbf{L}=\{\mathbf{P}, \mathbf{B}\}$ ), we consider the following bounded operators:

$$
\begin{aligned}
& \mathbf{L}^{0}=\left\{\mathbf{P}, \mathbf{B}^{0}\right\}: W^{l+2 m}(G) \rightarrow \mathcal{W}^{l}(G, \partial G), \\
& \mathbf{L}^{1}=\{\mathbf{P}, \mathbf{C}\}: W^{l+2 m}(G) \rightarrow \mathcal{W}^{l}(G, \partial G)
\end{aligned}
$$

First, we study the operator $\mathbf{L}^{1}$ (i.e., we assume that $\mathbf{B}_{i \mu}^{2}=0$ ); then we study the operator $\mathbf{L}$ in the general case $\mathbf{B}_{i \mu}^{2} \neq 0$. In this section, we assume that condition 7.1 holds.
8.1. Construction of a right regularizer in the case where $\mathbf{B}_{i \mu}^{2}=0$. Here we consider the case $\mathbf{B}_{i \mu}^{2}=0$, i.e., we assume that the support of nonlocal terms is located near the set $\mathcal{K}$.

Assume that $\delta=\varepsilon_{0} / d_{1}$, where $\varepsilon_{0}$ determines the diameter of the support of the function $\zeta$ (see Eq. (6.4)), from the definition of nonlocal operators $\mathbf{B}_{i \mu}^{1}$, and the number $d_{1}$ is defined in Eq. (6.15). Consider functions $\psi, \xi \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ such that $0 \leq \psi, \xi \leq 1, \psi(y)=1$ for $y \in \mathcal{O}_{\delta}(\mathcal{K})$, $\operatorname{supp} \psi \subset \mathcal{O}_{2 \delta}(\mathcal{K})$; $\xi(y)=1$ for $y \in \mathcal{O}_{2 \delta}(\mathcal{K}), \operatorname{supp} \xi \subset \mathcal{O}_{4 \delta}(\mathcal{K})$.
Lemma 8.1. Let condition 7.1 hold. Then for any sufficiently small $\varepsilon_{0}>0$, there exist bounded operators

$$
\mathbf{R}_{\mathcal{K}}: \mathcal{S}^{l}(G, \partial G) \rightarrow S^{l+2 m}(G), \quad \mathbf{M}_{\mathcal{K}}, \mathbf{S}_{\mathcal{K}}: \mathcal{S}^{l}(G, \partial G) \rightarrow \mathcal{S}^{l}(G, \partial G)
$$

such that

$$
\begin{equation*}
\mathbf{L}^{1} \mathbf{R}_{\mathcal{K}} f=\psi f+\mathbf{M}_{\mathcal{K}} f+\mathbf{S}_{\mathcal{K}} f \tag{8.1}
\end{equation*}
$$

$\left\|\mathbf{M}_{\mathcal{K}}\right\| \leq c_{1} \varepsilon_{0},\left\|\mathbf{S}_{\mathcal{K}}\right\| \leq c_{2}$, and the squared operator $\mathbf{S}_{\mathcal{K}}$ is compact; moreover, the operator $\mathbf{S}_{\mathcal{K}}$ can be represented in the form $\mathbf{S}_{\mathcal{K}}=\mathbf{U}_{\mathcal{K}}+\mathbf{F}_{\mathcal{K}}$, where $\left\|\mathbf{U}_{\mathcal{K}}\right\| \leq c_{3}$, and the operator $\mathbf{F}_{\mathcal{K}}$ is compact; the constants $c_{1}, c_{2}, c_{3}>0$ are independent of $\varepsilon_{0}$.
Proof. Let us perform the change of variables $y \rightarrow y^{\prime}$ described in Sec. 6.1 in a neighborhood of the set $\mathcal{K}$ and denote $y^{\prime}$ by $y$. We also denote the functions $\psi, \xi$, and $f$ written in the new coordinates by the same symbols.

By Lemma 7.1, we see that $\psi f-\mathcal{L} \mathcal{A}(\psi f) \in \mathcal{H}_{0}^{l}(K, \gamma)$. Hence,

$$
\mathcal{L}_{0}^{-1}(\psi f-\mathcal{L} \mathcal{A}(\psi f)) \in \mathcal{H}_{0}^{l+2 m}(K),
$$

where $\mathcal{L}_{0}: \mathcal{H}_{0}^{l+2 m}(K) \rightarrow \mathcal{H}_{0}^{l}(K, \gamma)$ is an operator defined in Eq. (7.44) for $a=0$. Assume that

$$
\mathbf{R}_{\mathcal{K}} f=\xi U, \quad U=\mathcal{L}_{0}^{-1}(\psi f-\mathcal{L} \mathcal{A}(\psi f))+\mathcal{A}(\psi f)
$$

Let us show that the operator $\mathbf{R}_{\mathcal{K}}$ is as required. Using the boundedness of the embedding operator $\mathcal{H}_{0}^{l+2 m}(K) \subset \mathcal{W}^{l+2 m}(K)$ defined on compactly supported functions, inequality (7.4), and the boundedness of the operator $\mathcal{A}$, we see that the operator $\mathbf{R}_{\mathcal{K}}$ is bounded.

Prove relation (8.1). Since $\mathcal{P}_{j}(D) U_{j}=\psi f_{j}$ and $\xi \psi f_{j}=\psi f_{j}$, we have

$$
\begin{equation*}
\mathbf{P}_{j}(y, D)\left(\xi U_{j}\right)-\psi f_{j}=\left[\mathbf{P}_{j}(y, D), \xi\right] U_{j}+\xi(y)\left(\mathbf{P}_{j}(y, D)-\mathcal{P}_{j}(D)\right) U_{j}, \tag{8.2}
\end{equation*}
$$

where $[\cdot, \cdot]$ denotes the commutator. This and Lemma 1.1 imply that

$$
\begin{equation*}
\mathbf{P} \mathbf{R}_{\mathcal{K}} f-\psi f_{0}=\left(M_{0}+T_{0}\right) f \tag{8.3}
\end{equation*}
$$

where the operator $M_{0}$ is "small" (i.e., $\left\|M_{0}\right\| \leq k_{1} \varepsilon_{0}$ ), the operator $T_{0}: \mathcal{S}^{l}(G, \partial G) \rightarrow W^{l}(G)$ is compact, and $k_{1}, k_{2}, \ldots>0$ are independent of $\varepsilon_{0}$.

Taking into account the fact that $\mathcal{B}_{j \sigma \mu}(D) U=\psi f_{j \sigma \mu}$ and $\xi \psi f_{j \sigma \mu}=\psi f_{j \sigma \mu}$, we obtain

$$
\begin{equation*}
\mathbf{B}_{j \sigma \mu}(y, D)(\xi U)-\psi f_{j \sigma \mu}=\left[\mathbf{B}_{j \sigma \mu}(y, D), \xi\right] U+\xi(y)\left(\mathbf{B}_{j \sigma \mu}(y, D)-\mathcal{B}_{j \sigma \mu}(D)\right) U \tag{8.4}
\end{equation*}
$$

Let $B(y)$ be an arbitrary coefficient of the operator $B_{j \sigma \mu k s}(y, D),(k, s) \neq(j, 0)$. By Eq. (6.14), according to the choice of the function $\psi$, we have

$$
\begin{array}{rlll}
B\left(\mathcal{G}_{j \sigma k s} y\right)=0 & \text { for } & |y| \geq \varepsilon_{0} / \chi_{j \sigma k s} \\
\xi\left(\mathcal{G}_{j \sigma k s} y\right)=\xi(y)=1 & \text { for } & |y| \leq \varepsilon_{0} / \chi_{j \sigma k s}
\end{array}
$$

Thus, for any function $v$, we have

$$
\begin{equation*}
(B v \xi)\left(\mathcal{G}_{j \sigma k s} y\right) \equiv \xi(y)(B v)\left(\mathcal{G}_{j \sigma k s} y\right) \tag{8.5}
\end{equation*}
$$

Obviously, if $(k, s)=(j, 0)$, then Eq. (8.5) also holds. Hence, the commutator on the right-hand side of formula (8.4) does not contain higher derivatives; this means that it is a compact operator. Minor derivatives from the second term of the right-hand side of formula (8.4) also form a compact operator.

Consider an arbitrary term containing a higher derivative of order $\alpha,|\alpha|=m_{j \sigma \mu}$, as $(k, s) \neq(j, 0)$; it has the form

$$
\begin{align*}
& \left.\xi(y)\left(\zeta\left(G_{j \sigma k s} y\right) b\left(G_{j \sigma k s} y\right)-b(0)\right)\left(D^{\alpha} U\right)\left(G_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}} \\
& =\left.\xi(y) \zeta\left(G_{j \sigma k s} y\right)\left(b\left(G_{j \sigma k s} y\right)-b(0)\right)\left(D^{\alpha} U\right)\left(G_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}} \\
& \quad+\left.\xi(y)\left(\zeta\left(G_{j \sigma k s} y\right)-1\right) b(0)\left(D^{\alpha} U\right)\left(G_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}} \tag{8.6}
\end{align*}
$$

where $b(y)$ is an infinitely smooth function. Using Lemma 1.1, we can represent the first difference on the right-hand side of Eq. (8.6) as the sum of "small" and compact operators. Moreover, if $(k, s)=(j, 0)$, then there is no second differences on the right-hand side of Eq. (8.6). Thus,

$$
\begin{equation*}
\left(\mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}\right) \mathbf{R}_{\mathcal{K}} f-\psi f_{i \mu}=\left(M_{i \mu}+T_{i \mu}+S_{i \mu}\right) f \tag{8.7}
\end{equation*}
$$

where $\left\|M_{i \mu}\right\| \leq k_{2} \varepsilon_{0}$, the operator $T_{i \mu}$ is compact, and the operator $S_{i \mu}$ consists of terms that have the following form in the coordinates $y^{\prime}$ :

$$
S_{j \sigma \mu} f=\left.\xi\left(y^{\prime}\right)\left(\zeta\left(G_{j \sigma k s} y^{\prime}\right)-1\right) b(0)\left(D^{\alpha} U\right)\left(G_{j \sigma k s} y^{\prime}\right)\right|_{\gamma_{j \sigma}}, \quad(k, s) \neq(j, 0)
$$

Let $\mathbf{S}=\left\{0, S_{i \mu}\right\}$. The representation of the operator $\mathbf{S}$ in the form of

$$
\mathbf{S}=\mathbf{U}_{\mathcal{K}}+\mathbf{F},
$$

where $\left\|\mathbf{U}_{\mathcal{K}}\right\| \leq k_{3}$, and the operator $\mathbf{F}$ is compact, follows from Lemma 1.1. Let us prove that the squared operator $\left\{0, S_{j \sigma \mu}\right\}: \mathcal{W}^{l}(K, \gamma) \rightarrow \mathcal{W}^{l}(K, \gamma)$ is compact.

Introduce the notation

$$
\Phi=\left\{\Phi_{0}, \Phi_{j \sigma \mu}\right\}=\left\{0, S_{j \sigma \mu} f\right\}, \quad V=\mathcal{L}_{0}^{-1}(\psi \Phi-\mathcal{L A}(\psi \Phi))+\mathcal{A}(\psi \Phi) .
$$

Since the function $\zeta$ has a compact support and is equal to 1 near the origin, we see that the support of the function $\zeta\left(\mathcal{G}_{j \sigma k s} y^{\prime}\right)-1$ is bounded and separated from the origin. Hence, using [57, Chap. 2, Theorem 4.3] (about an a priori estimate of solutions of elliptic equations) and taking into account the fact that $\mathcal{P}_{j}(D) V_{j}=\psi \Phi_{0}=0$, we obtain the inequality

$$
\begin{equation*}
\left\|S_{j \sigma \mu} \Phi\right\|_{W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)} \leq k_{4}\left\|V_{k}\right\|_{W^{l+2 m-1}\left(Q_{k}\right)}, \tag{8.8}
\end{equation*}
$$

where $Q_{k}$ are bounded domains such that $\overline{Q_{k}} \subset K_{k}$. Equation (8.8) and the compactness of the embedding $W^{l+2 m}\left(Q_{k}\right) \subset W^{l+2 m-1}\left(Q_{k}\right)$ imply that the operator $\left\{0, S_{j \sigma \mu}\right\}^{2}$ is compact. This means that the operator $\mathbf{S}^{2}=\left\{0, S_{i \mu}\right\}^{2}$ is also compact. The statement of the lemma with operators $\mathbf{M}_{\mathcal{K}}=$ $\left\{M_{0}, M_{i \mu}\right\}, \mathbf{S}_{\mathcal{K}}=\left\{T_{0}, T_{i \mu}\right\}+\mathbf{S}$, and $\mathbf{F}_{\mathcal{K}}=\left\{T_{0}, T_{i \mu}\right\}+\mathbf{F}$ follows from here and Eqs. (8.3) and (8.7).
Lemma 8.2. Let condition 7.1 hold. Then for sufficiently small $\varepsilon_{0}$, there exist a bounded operator $\mathbf{R}_{1}: \mathcal{S}^{l}(G, \partial G) \rightarrow S^{l+2 m}(G)$ and a compact operator $\mathbf{T}_{1}: \mathcal{S}^{l}(G, \partial G) \rightarrow \mathcal{S}^{l}(G, \partial G)$ such that

$$
\begin{equation*}
\mathbf{L}^{1} \mathbf{R}_{1}=\mathbf{I}_{1}+\mathbf{T}_{1} \tag{8.9}
\end{equation*}
$$

where $\mathbf{I}_{1}$ is the identity operator in $\mathcal{S}^{l}(G, \partial G)$.
Proof. 1. According to the general theory of elliptic boundary-value problems in domains with a smooth boundary (see, e.g., [108]), there exist a bounded operator

$$
\mathbf{R}_{0}: \mathcal{W}^{l}(G, \partial G) \rightarrow\left\{u \in W^{l+2 m}(G): \operatorname{supp} u \subset \bar{G} \backslash \mathcal{O}_{\delta / 2}(\mathcal{K})\right\} \subset S^{l+2 m}(G)
$$

and a compact operator $\mathbf{T}_{0}: \mathcal{W}^{l}(G, \partial G) \rightarrow \mathcal{W}^{l}(G, \partial G)$ such that

$$
\begin{equation*}
\mathbf{L}^{0} \mathbf{R}_{0}(1-\psi) f=(1-\psi) f+\mathbf{T}_{0} f \tag{8.10}
\end{equation*}
$$

For any $f \in \mathcal{S}^{l}(G, \partial G)$, let $\hat{\mathbf{R}}_{1} f=\mathbf{R}_{\mathcal{K}}(\psi f)+\mathbf{R}_{0}(1-\psi) f$, where $\mathbf{R}_{\mathcal{K}}$ is the operator defined in Lemma 8.1. Then by virtue of Lemma 8.1 and Eq. (8.10), we have

$$
\begin{equation*}
\mathbf{L}^{1} \hat{\mathbf{R}}_{1} f=f+\mathbf{M}_{\mathcal{K}} f+\left(\mathbf{S}_{\mathcal{K}}+\mathbf{T}_{0}\right) f+\left\{0, \mathbf{B}^{1} \mathbf{R}^{0}(1-\psi) f\right\} \tag{8.11}
\end{equation*}
$$

Since the embedding supp $\mathbf{R}_{0}(1-\psi) f \subset \bar{G} \backslash \mathcal{O}_{\varepsilon_{0}}(\mathcal{K})$ is valid, from the definition of the operator $\mathbf{B}^{1}$ we obtain $\mathbf{B}^{1} \mathbf{R}_{0}(1-\psi) f=0$. Then Eq. (8.11) and Lemma 1.2 imply that $\mathbf{L}^{1} \hat{\mathbf{R}}_{1}: \mathcal{S}^{l}(G, \partial G) \rightarrow \mathcal{S}^{l}(G, \partial G)$ is a Fredholm operator and, therefore, it has a right regularizer, i.e., there exist a bounded operator $\mathbf{R}_{1}^{\prime}: \mathcal{S}^{l}(G, \partial G) \rightarrow \mathcal{S}^{l}(G, \partial G)$ and a compact operator $\mathbf{T}_{1}: \mathcal{S}^{l}(G, \partial G) \rightarrow \mathcal{S}^{l}(G, \partial G)$ such that

$$
\mathbf{L}^{1} \hat{\mathbf{R}}_{1} \mathbf{R}_{1}^{\prime}=\mathbf{I}_{1}+\mathbf{T}_{1}
$$

where $\mathbf{I}_{1}$ is the identity operator in the space $\mathcal{S}^{l}(G, \partial G)$. Denoting $\mathbf{R}_{1}=\hat{\mathbf{R}}_{1} \mathbf{R}_{1}^{\prime}$, we complete the proof.
8.2. Construction of a right regularizer in the case where $\mathbf{B}_{i \mu}^{2} \neq 0$. Here we assume that the number $\varepsilon_{0}$ is fixed and consider the operator $\mathbf{L}$ in the case where $\mathbf{B}_{i \mu}^{2} \neq 0$. In other words, we assume that the support of nonlocal terms is located not only near the set $\mathcal{K}$ but also outside it.

Let $\xi$ and $\psi$ be functions defined before Lemma 8.1; now we assume that $\delta>0$ is arbitrary (in particular, it is independent of $\varepsilon_{0}$ ).

To construct a right regularizer for the operator $\mathbf{L}$, we need a "right regularizer" $\mathbf{R}_{\mathcal{K}}^{\prime}$ for the operator $\mathbf{L}^{1}$, which is defined on functions $\left.f^{\prime}=\left\{f_{i \mu}\right\} \in \mathcal{S}_{1}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)\right)$ and for which the diameter of the support of the function $\mathbf{R}_{\mathcal{K}}^{\prime} f^{\prime}$, which is located near the set $\mathcal{K}_{1}$, can be made arbitrary small.
Lemma 8.3. Let condition 7.1 hold. Then for any $\delta, 0<\delta<1$, there exist bounded operators

$$
\mathbf{R}_{\mathcal{K}}^{\prime}: \mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) \rightarrow S^{l+2 m}(G), \quad \mathbf{M}_{\mathcal{K}}^{\prime}, \mathbf{T}_{\mathcal{K}}^{\prime}: \mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) \rightarrow \mathcal{S}^{l}(G, \partial G)
$$

such that

$$
\begin{equation*}
\mathbf{L}^{1} \mathbf{R}_{\mathcal{K}}^{\prime} f^{\prime}=\psi\left\{0, f^{\prime}\right\}+\mathbf{M}_{\mathcal{K}}^{\prime} f^{\prime}+\mathbf{T}_{\mathcal{K}}^{\prime} f^{\prime}, \tag{8.12}
\end{equation*}
$$

where $\left\|\mathbf{M}_{\mathcal{K}}^{\prime}\right\| \leq c \delta$, the operator $\mathbf{T}_{\mathcal{K}}^{\prime}$ is compact, and $c>0$ is independent of $\delta$.
Proof. Let us perform the change of variables $y \rightarrow y^{\prime}$ from Sec. 6.1 in a neighborhood of the set $\mathcal{K}$ and denote $y^{\prime}$ by $y$. We denote the functions $\psi, \xi$, and $f^{\prime}$ written in the new coordinates by the same symbols.

Let $f=\left(0, f^{\prime}\right) \in \mathcal{S}^{l}(G, \partial G)$. By virtue of Lemma 7.1, we see that $\psi f-\mathcal{L} \mathcal{A}(\psi f) \in \mathcal{H}_{0}^{l}(K, \gamma)$. Hence,

$$
\mathcal{L}_{0}^{-1}(\psi f-\mathcal{L A}(\psi f)) \in \mathcal{H}_{0}^{l+2 m}(K),
$$

where $\mathcal{L}_{0}: \mathcal{H}_{0}^{l+2 m}(K) \rightarrow \mathcal{H}_{0}^{l}(K, \gamma)$ is the operator defined in Eq. (7.44) for $a=0$. Let

$$
\mathbf{R}_{\mathcal{K}}^{\prime} f^{\prime}=\xi U, \quad U=\mathcal{L}_{0}^{-1}(\psi f-\mathcal{L} \mathcal{A}(\psi f))+\mathcal{A}(\psi f)
$$

We show that the operator $\mathbf{R}_{\mathcal{K}}^{\prime}$ is as required. Using the boundedness of the embedding operator $\mathcal{H}_{0}^{l+2 m}(K) \subset \mathcal{W}^{l+2 m}(K)$ defined on compactly supported functions, inequality (7.4), and the boundedness of the operator $\mathcal{A}$, we obtain that the operator $\mathbf{R}_{\mathcal{K}}^{\prime}$ is bounded.

Prove relation (8.12). Similarly to the proof of Lemma 8.1, we have

$$
\begin{gather*}
\mathbf{P}_{j}(y, D)\left(\xi U_{j}\right)=\left[\mathbf{P}_{j}(y, D), \xi\right] U_{j}+\xi(y)\left(\mathbf{P}_{j}(y, D)-\mathcal{P}_{j}(D)\right) U_{j}  \tag{8.13}\\
\mathbf{B}_{j \sigma \mu}(y, D)(\xi U)-\psi f_{j \sigma \mu}=\left[\mathbf{B}_{j \sigma \mu}(y, D), \xi\right] U+\xi(y)\left(\mathbf{B}_{j \sigma \mu}(y, D)-\mathcal{B}_{j \sigma \mu}(D)\right) U . \tag{8.14}
\end{gather*}
$$

Equation (8.13) and Lemma 1.1 yield the equality

$$
\begin{equation*}
\mathbf{P R}_{\mathcal{K}}^{\prime} f^{\prime}=\left(M_{0}^{\prime}+T_{0}^{\prime}\right) f^{\prime} \tag{8.15}
\end{equation*}
$$

where the operator $M_{0}^{\prime}$ is "small" (i.e., $\left\|M_{0}^{\prime}\right\| \leq k_{1} \delta$ ), the operator $T_{0}: \mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) \rightarrow W^{l}(G)$ is compact, and $k_{1}, k_{2}, \ldots>0$ are independent of $\delta$.

The commutator on the right-hand side of (8.4) contains minor derivatives of the function $U$ and operators of the form

$$
\begin{equation*}
U_{k} \mapsto J_{j \sigma \mu k s}=\left.\left(\xi\left(\mathcal{G}_{j \sigma k s} y\right)-\xi(y)\right)\left(B_{j \sigma \mu k s}(y, D) U_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}}, \quad(k, s) \neq(j, 0) \tag{8.16}
\end{equation*}
$$

Since the function $\xi$ has a compact support and is equal to 1 near the origin, we see that the support of the function $\xi\left(\mathcal{G}_{j \sigma k s} y\right)-\xi(y)$ is bounded and separated from the origin. Therefore, using [57, Chap. 2,

Theorem 4.3] (on a priori estimates of solutions of elliptic equations) and taking into account the relation $\mathcal{P}_{j}(D) U_{j}=0$, we obtain the inequality

$$
\begin{equation*}
\left\|J_{j \sigma \mu k s}\right\|_{W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)} \leq k_{2}\left\|U_{k}\right\|_{W^{l+2 m-1}\left(Q_{k}\right)} \tag{8.17}
\end{equation*}
$$

where $Q_{k}$ is a bounded domain such that $\overline{Q_{k}} \subset K_{k}$. Equation (8.17) and the compactness of the embedding $W^{l+2 m}\left(Q_{k}\right) \subset W^{l+2 m-1}\left(Q_{k}\right)$ imply that operator (8.16) is compact. This means that the commutator in (8.4) is also compact.

Minor derivatives from the second term on the right-hand side of formula (8.4) also form a compact operator. Consider an arbitrary term containing a higher derivative of order $\alpha,|\alpha|=m_{j \sigma \mu}$, as $(k, s) \neq(j, 0)$; it has the following form:

$$
\begin{align*}
& \left.\xi(y)\left(\zeta\left(G_{j \sigma k s} y\right) b\left(G_{j \sigma k s} y\right)-b(0)\right)\left(D^{\alpha} U\right)\left(G_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}} \\
& =\left.\xi(y) \zeta\left(G_{j \sigma k s} y\right)\left(b\left(G_{j \sigma k s} y\right)-b(0)\right)\left(D^{\alpha} U\right)\left(G_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}} \\
& \quad+\left.\xi(y)\left(\zeta\left(G_{j \sigma k s} y\right)-1\right) b(0)\left(D^{\alpha} U\right)\left(G_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}} . \tag{8.18}
\end{align*}
$$

Using Lemma 1.1, we can represent the first difference on the right-hand side in Eq. (8.18) as the sum of "small" and compact operators. Moreover, if $(k, s)=(j, 0)$, then there is no second difference on the right-hand side on Eq. (8.18). Thus,

$$
\begin{equation*}
\left(\mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}\right) \mathbf{R}_{\mathcal{K}}^{\prime} f^{\prime}-\psi f^{\prime}=\left(M_{i \mu}^{\prime}+T_{i \mu}^{\prime}+S_{i \mu}^{\prime}\right) f^{\prime} \tag{8.19}
\end{equation*}
$$

where $\left\|M_{i \mu}^{\prime}\right\| \leq k_{3} \delta$, the operator $T_{i \mu}^{\prime}$ is compact, and the operator $S_{i \mu}^{\prime}$ consists of terms that can be written in the following form in the coordinates $y^{\prime}$ :

$$
S_{j \sigma \mu}^{\prime} f^{\prime}=\left.\xi\left(y^{\prime}\right)\left(\zeta\left(G_{j \sigma k s} y^{\prime}\right)-1\right) b(0)\left(D^{\alpha} U_{k}\right)\left(G_{j \sigma k s} y^{\prime}\right)\right|_{\gamma_{j \sigma}}
$$

Since the function $\zeta$ has a compact support and is equal to 1 near the origin, we see that the support of the function $\zeta\left(\mathcal{G}_{j \sigma k s} y^{\prime}\right)-1$ is bounded and is separated from the origin. Using [57, Chap. 2, Theorem 4.3] (on a priori estimates of solutions of elliptic equations) again and taking into account the fact that $\mathcal{P}_{j}(D) U_{j}=0$, we obtain the inequality

$$
\begin{equation*}
\left\|S_{j \sigma \mu}^{\prime} f^{\prime}\right\|_{W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)} \leq k_{3}\left\|U_{k}\right\|_{W^{l+2 m-1}\left(Q_{k}\right)} \tag{8.20}
\end{equation*}
$$

where $Q_{k}$ are bounded domains such that $\overline{Q_{j}} \subset K_{k}$. Equation (8.20) and the compactness of the embedding $W^{l+2 m}\left(Q_{k}\right) \subset W^{l+2 m-1}\left(Q_{k}\right)$ imply that the operator $S_{j \sigma \mu}^{\prime}$ is compact. This means that the operator $S_{i \mu}^{\prime}$ is compact. From here and Eqs. (8.15) and (8.19), we obtain the statement of the lemma.

Consider a $\delta / 2$-neighborhood $\mathcal{O}_{\delta / 2}(g)$ of every point $g \in \partial G \backslash \mathcal{O}_{2 \delta}(\mathcal{K})$. All these neighborhoods with the set $\mathcal{O}_{2 \delta}(\mathcal{K})$ form a covering $\partial G$. Let us choose a finite sub-covering $\mathcal{O}_{2 \delta}(\mathcal{K}), \mathcal{O}_{\delta / 2}\left(g_{j}\right), j=1, \ldots, J$, $J=J(\delta)$, of the boundary $\partial G$. Let $\psi, \psi_{j}^{\prime} \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right), j=1, \ldots, J$, be a partition of unit subordinated to this covering (we assume that the function $\psi$ from Lemma 8.3 coincides with the function $\psi$ from this partition of unit).

According to the general theory of elliptic boundary-value problems in domains with smooth boundaries (see, e.g., [108]), there exist bounded operators

$$
\mathbf{R}_{0 j}^{\prime}:\left\{f^{\prime} \in \mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}(\partial G), \quad \operatorname{supp} f \subset \mathcal{O}_{\delta / 2}\left(g_{j}\right)\right\} \rightarrow\left\{u \in W^{l+2 m}(G): \operatorname{supp} u \subset \mathcal{O}_{\delta}\left(g_{j}\right)\right\}
$$

and compact operators

$$
\mathbf{T}_{0 j}^{\prime}:\left\{f^{\prime} \in \mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}(\partial G): \operatorname{supp} f \subset \mathcal{O}_{\delta / 2}\left(g_{j}\right)\right\} \rightarrow\left\{f \in \mathcal{W}^{l}(G, \partial G): \operatorname{supp} f \subset \mathcal{O}_{\delta}\left(g_{j}\right)\right\}
$$

such that

$$
\mathbf{L}^{0} \mathbf{R}_{0 j}^{\prime} f^{\prime}=\left\{0, f^{\prime}\right\}+\mathbf{T}_{0 j}^{\prime} f^{\prime}
$$

For any function $f^{\prime} \in \mathcal{S}^{l+2 m-\mathbf{m - 1 / 2}}(\partial G)$, we assume that

$$
\begin{equation*}
\mathbf{R}_{1}^{\prime} f^{\prime}=\mathbf{R}_{\mathcal{K}}^{\prime} f^{\prime}+\sum_{j=1}^{J} \mathbf{R}_{0 j}^{\prime}\left(\psi_{j}^{\prime} f^{\prime}\right) \tag{8.21}
\end{equation*}
$$

Similarly to the proof of Lemma 8.2 (applying Lemma 8.3 instead of Lemma 8.1), we can show that

$$
\begin{equation*}
\mathbf{L}^{1} \mathbf{R}_{1}^{\prime} f^{\prime}=\left\{0, f^{\prime}\right\}+\mathbf{M}_{1}^{\prime} f^{\prime}+\mathbf{T}_{1}^{\prime} f^{\prime} \tag{8.22}
\end{equation*}
$$

where $\mathbf{M}_{1}^{\prime}, \mathbf{T}_{1}^{\prime}: \mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) \rightarrow \mathcal{S}^{l}(G, \partial G)$ are bounded operators such that $\left\|\mathbf{M}_{1}^{\prime}\right\| \leq c \delta$, where $c>0$ is independent of $\delta$, and the operator $\mathbf{T}_{1}^{\prime}$ is compact.

Using the operators $\mathbf{R}_{1}$ (see Lemma 8.2) and $\mathbf{R}_{1}^{\prime}$ (see (8.21)), we construct a right regularizer for the operator $\mathbf{L}$ in the case where $\mathbf{B}_{i \mu}^{2} \neq 0$.

Let us introduce the set

$$
\mathcal{S}_{B}^{l}(G, \partial G)=\left\{f \in \mathcal{S}^{l}(G, \partial G): \Phi=\mathbf{B}^{2} \mathbf{R}_{1} f \quad \text { and } \quad \mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi \quad \text { belong } \quad \mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)\right\}
$$

The Sobolev embedding theorem and the Riesz theorem on the general form of a linear continuous functional in a Hilbert space imply that $\mathcal{S}_{B}^{l}(G, \partial G)$ is a closed finite-dimensional subspace in $\mathcal{W}^{l}(G, \partial G)$. Obviously, $\mathcal{S}_{B}^{l}(G, \partial G) \subset \mathcal{S}^{l}(G, \partial G)$.
Lemma 8.4. Let condition 7.1 hold. Then there exist a bounded operator

$$
\mathbf{R}: \mathcal{W}^{l}(G, \partial G) \rightarrow W^{l+2 m}(G)
$$

and a compact operator $\mathbf{T}: \mathcal{W}^{l}(G, \partial G) \rightarrow \mathcal{W}^{l}(G, \partial G)$ such that

$$
\begin{equation*}
\mathbf{L R}=\mathbf{I}+\mathbf{T}, \tag{8.23}
\end{equation*}
$$

where $\mathbf{I}$ is the identity operator in $\mathcal{W}^{l}(G, \partial G)$.
Proof. 1. Let $\Phi=\mathbf{B}^{2} \mathbf{R}_{1} f$, where $f=\left\{f_{0}, f^{\prime}\right\} \in \mathcal{S}_{B}^{l}(G, \partial G)$. Then, by the definition of the space $\mathcal{S}_{B}^{l}(G, \partial G)$, the functions $\Phi$ and $\mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi$ belong to the domain of the operator $\mathbf{R}_{1}^{\prime}$. Hence we can introduce a bounded operator $\mathbf{R}_{\mathcal{S}}: \mathcal{S}_{B}^{l}(G, \partial G) \rightarrow W^{l+2 m}(G)$ by the formula

$$
\mathbf{R}_{\mathcal{S}} f=\mathbf{R}_{1} f-\mathbf{R}_{1}^{\prime} \Phi+\mathbf{R}_{1}^{\prime} \mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi
$$

We show that the operator $\mathbf{R}_{\mathcal{S}}$ is the inverse operator for $\mathbf{L}$ with accuracy up to the sum of small and compact perturbations. For simplicity, we denote different operators (acting in the corresponding spaces) whose norms do not exceed $c \delta$ by the same letter $M$. Similarly, we denote different compact operators by the same letter $T$.

By Eqs. (8.9) and (8.22), we have

$$
\begin{align*}
\mathbf{P R}_{\mathcal{S}} f=\mathbf{P R}_{1} f-\mathbf{P} \mathbf{R}_{1}^{\prime}( & \left.\Phi-\mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi\right) \\
& =f_{0}+T f_{0}-M\left(\Phi-\mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi\right)-T\left(\Phi-\mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi\right)=f_{0}+M f+T f \tag{8.24}
\end{align*}
$$

$$
\begin{align*}
& \mathbf{C R}_{\mathcal{S}} f=\mathbf{C R}_{1} f-\mathbf{C R}_{1}^{\prime} \Phi+\mathbf{C} \mathbf{R}_{1}^{\prime} \mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi \\
& \qquad=\left(f^{\prime}+T f^{\prime}\right)-(\Phi+M \Phi+T \Phi)+\left(\mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi+M \mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi+T \mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi\right) \\
&  \tag{8.25}\\
& =f^{\prime}-\Phi+\mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi+M f+T f
\end{align*}
$$

Applying the operator $\mathbf{B}^{2}$ to the function $\mathbf{R}_{\mathcal{S}} f$, we obtain the equality

$$
\begin{equation*}
\mathbf{B}^{2} \mathbf{R}_{\mathcal{S}} f=\Phi-\mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi+\mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi \tag{8.26}
\end{equation*}
$$

Adding Eqs. (8.25) and (8.26), we obtain the inequality

$$
\begin{equation*}
\mathbf{B R}_{\mathcal{S}} f=f^{\prime}+M f+T f+\mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi \tag{8.27}
\end{equation*}
$$

Let us show that

$$
\begin{equation*}
\mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi=0 \tag{8.28}
\end{equation*}
$$

for sufficiently small $\delta=\delta\left(\varkappa_{1}, \varkappa_{2}, \rho\right)$, where $\varkappa_{1}, \varkappa_{2}$, and $\rho$ are constants from condition 6.4. Note that $\delta$ is independent of $\varepsilon_{0}$.

By (8.21), we have $\operatorname{supp} \mathbf{R}_{1}^{\prime} \Phi \subset \bar{G} \backslash \bar{G}_{4 \delta}$. Let $\delta$ be so small that $4 \delta<\rho$. Then Eq. (6.6) implies that $\operatorname{supp} \mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi \subset \mathcal{O}_{\varkappa_{2}}(\mathcal{K})$.

Further, let $\delta$ be small that $4 \delta<\varkappa_{1}$ and $\varkappa_{2}+3 \delta / 2<\varkappa_{1}$. Then, applying (8.21) again, we see that $\operatorname{supp} \mathbf{R}_{1}^{\prime} \mathbf{B}^{2} \mathbf{R}_{1}^{\prime} \Phi \subset \mathcal{O}_{\varkappa_{1}}(\mathcal{K})$. This relation and Eq. (6.5) yield Eq. (8.28).

It follows from Eqs. (8.24), (8.27), and (8.28) that

$$
\mathbf{L R}_{\mathcal{S}}=\mathbf{I}_{\mathcal{S}}+M+T
$$

where $\mathbf{I}_{\mathcal{S}}, M, T: \mathcal{S}_{B}^{l}(G, \partial G) \rightarrow \mathcal{W}^{l}(G, \partial G)$ are bounded operators such that $\mathbf{I}_{\mathcal{S}} f=f,\|M\| \leq c \delta(c>0$ are independent of $\delta$ ), and the operator $T$ is compact.
3. Since the subspace $\mathcal{S}_{B}^{l}(G, \partial G)$ is finite-dimensional in $\mathcal{W}^{l}(G, \partial G)$, we see that $\mathbf{I}_{\mathcal{S}}$ is a Fredholm operator. Hence, by [56, Theorems 16.2, 16.4], $\mathbf{I}_{\mathcal{S}}+M+T$ is also a Fredholm operator for sufficiently small $\delta$. Now it follows from [56, Theorem 15.2] that there exist a bounded operator $\tilde{\mathbf{R}}$ and a compact operator $\mathbf{T}$ such that they act from $\mathcal{W}^{l}(G, \partial G)$ to $\mathcal{S}_{B}^{l}(G, \partial G)$ and to $\mathcal{W}^{l}(G, \partial G)$, respectively, and $\left(\mathbf{I}_{\mathcal{S}}+M+T\right) \tilde{\mathbf{R}}=\mathbf{I}+\mathbf{T}$. Denoting $\mathbf{R}=\mathbf{R}_{\mathcal{S}} \tilde{\mathbf{R}}: \mathcal{W}^{l}(G, \partial G) \rightarrow W^{l+2 m}(G)$, we obtain Eq. (8.23).
8.3. Fredholm solvability of nonlocal problems. Here we prove the following result on the solvability of problem (6.7), (6.8) in bounded domains in Sobolev spaces.

Theorem 8.1. Let condition 7.1 hold. Then $\mathbf{L}: W^{l+2 m}(G) \rightarrow \mathcal{W}^{l}(G, \partial G)$ is a Fredholm operator and $\operatorname{ind} \mathbf{L}=\operatorname{ind} \mathbf{L}^{1}$.

Conversely, let $\mathbf{L}: W^{l+2 m}(G) \rightarrow \mathcal{W}^{l}(G, \partial G)$ be a Fredholm operator. Then condition 7.1 holds.
Below, we show that if condition 7.1 is violated, then the image of the operator $\mathbf{L}$ is not closed (see Lemma 8.7). This statement, Theorem 8.1, and [56, Theorem 7.1] yield the following result.

Corollary 8.1. Condition 7.1 is necessary and sufficient for the fulfilment of the following a priori estimate:

$$
\|u\|_{W^{l+2 m}(G)} \leq c\left(\|\mathbf{L} u\|_{\mathcal{W}^{l}(G, \partial G)}+\|u\|_{L_{2}(G)}\right)
$$

where $c>0$ is independent of $u$.
8.3.1. Proof of Theorem 8.1. Sufficiency. Let us show that the kernel of the operator $\mathbf{L}$ is finitedimensional. For this, we consider problem (6.7), (6.8) in weight spaces.

Introduce the operator

$$
\mathbf{L}_{a}=\{\mathbf{P}, \mathbf{B}\}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G), \quad a>l+2 m-1,
$$

which corresponds to Problem (6.7), (6.8) in weight spaces. Note that, by Eq. (6.5) and Lemma 5.3, for $a>l+2 m-1$, we have

$$
\mathbf{B}_{i \mu}^{2} u \in W^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right) \subset H_{a}^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)
$$

for any function $u \in H_{a}^{l+2 m}(G) \subset W^{l+2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$. Since the functions $\mathbf{B}_{i \mu}^{0} u$ and $\mathbf{B}_{i \mu}^{1} u$ belong to $H_{a}^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$, we see that the operator $\mathbf{L}_{a}$ is well defined.

Thus, the operators $\mathbf{L}$ and $\mathbf{L}_{a}$ correspond to the same nonlocal problem (6.7), (6.8), which is considered in Sobolev spaces and in weight spaces respectively.
Lemma 8.5. The kernel of the operator $\mathbf{L}$ is finite-dimensional.

Proof. Lemma 6.1 and [89, Theorem 3.2] ${ }^{4}$ imply that $\mathbf{L}_{a}$ is a Fredholm operator for almost all $a>$ $l+2 m-1$. Let us fix $a>l+2 m-1$ for which $\mathbf{L}_{a}$ is a Fredholm operator. By Lemma 5.2, we have $W^{l+2 m}(G) \subset H_{a}^{l+2 m}(G)$; hence, $\operatorname{ker} \mathbf{L} \subset \operatorname{ker} \mathbf{L}_{a}$. Since $\operatorname{ker} \mathbf{L}_{a}$ is finite-dimensional for the chosen $a$, we see that $\operatorname{ker} \mathbf{L}$ is also finite-dimensional.

Remark 8.1. Emphasize that the kernel of the operator $\mathbf{L}$ is finite-dimensional independently of the location of eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$.

By virtue of [56, Theorem 15.2] and Lemma 8.4, the image of the operator $\mathbf{L}$ is closed and has a finite codimension. This and Lemma 8.5 imply that $\mathbf{L}$ is a Fredholm operator.

We show that ind $\mathbf{L}=\operatorname{ind} \mathbf{L}^{1}$. Introduce an operator

$$
\mathbf{L}_{t} u=\left\{\mathbf{P} u, \mathbf{C} u+(1-t) \mathbf{B}^{2} u\right\} .
$$

Obviously, $\mathbf{L}_{0}=\mathbf{L}$ and $\mathbf{L}_{1}=\mathbf{L}^{1}$.
As was proved above, the operators $\mathbf{L}_{t}$ are Fredholm operators for all $t$. Further, for all $t_{0}$ and $t$, the following estimate holds:

$$
\left\|\mathbf{L}_{t} u-\mathbf{L}_{t_{0}} u\right\|_{\mathcal{W}^{l}(G, \partial G)} \leq k_{t_{0}}\left|t-t_{0}\right| \cdot\|u\|_{W^{l+2 m}(G)},
$$

where $k_{t_{0}}>0$ is independent of $t$. Hence, by [56, Theorem 16.2], we have ind $\mathbf{L}_{t}=\operatorname{ind} \mathbf{L}_{t_{0}}$ for all $t$ from a sufficiently small neighborhood of the point $t_{0}$. Since $t_{0}$ is arbitrary, we see that these neighborhoods cover the interval $[0,1]$. Choosing a finite subcovering, we obtain

$$
\operatorname{ind} \mathbf{L}=\operatorname{ind} \mathbf{L}_{0}=\operatorname{ind} \mathbf{L}_{1}=\operatorname{ind} \mathbf{L}^{1} .
$$

The sufficiency of condition 7.1 in Theorem 8.1 is proved.

### 8.3.2. Proof of Theorem 8.1. Necessity. Let $d=d(\varrho)=2 d_{2} \varrho$, where $d_{2}$ is defined in (6.15).

Lemma 8.6. Let the image of the operator $\mathbf{L}$ be closed. Then for all sufficiently small $\varrho>0$ and $U \in \mathcal{W}^{l+2 m}\left(K^{d}\right)$, the following estimate holds:

$$
\begin{equation*}
\|U\|_{\mathcal{W}^{l+2 m}\left(K^{g}\right)} \leq c\left(\|\mathcal{L} U\|_{\mathcal{W}^{l}\left(K^{2 \varrho}, \gamma^{2 \varrho}\right)}+\sum_{j=1}^{N}\left\|\mathcal{P}_{j}(D) U_{j}\right\|_{W^{l}\left(K_{j}^{d}\right)}+\|U\|_{\mathcal{W}^{l+2 m-1}\left(K^{d}\right)}\right) \tag{8.29}
\end{equation*}
$$

Proof. 1. Since the image of the operator $\mathbf{L}$ is closed, it follows from Lemma 8.5, the compactness of the embedding operator $W^{l+2 m}(G) \subset W^{l+2 m-1}(G)$, and [56, Theorem 7.1] that

$$
\begin{equation*}
\|u\|_{W^{l+2 m}(G)} \leq c\left(\|\mathbf{L} u\|_{\mathcal{W}^{l}(G, \partial G)}+\|u\|_{W^{l+2 m-1}(G)}\right) . \tag{8.30}
\end{equation*}
$$

Let us substitute in Eq. (8.30) a function $u \in W^{l+2 m}(G)$ such that $\operatorname{supp} u \subset \mathcal{O}_{2 \varrho}(\mathcal{K}), 2 \varrho<\min \left\{\varepsilon_{0}, \varkappa_{1}\right\}$. By virtue of Eq. (6.5) we have that $\mathbf{B}^{2} u=0$ for such a function $u$. Hence, using [57, Chap. 2, Lemma 3.2], we obtain the estimate

$$
\begin{equation*}
\|U\|_{\mathcal{W}^{l+2 m}(K)} \leq c\left(\|\mathcal{L} U\|_{\mathcal{W}^{l}(K, \gamma)}+\|U\|_{\mathcal{W}^{l+2 m-1}(K)}\right) \tag{8.31}
\end{equation*}
$$

which is valid for $U \in \mathcal{W}^{l+2 m}(K)$ such that $\operatorname{supp} U \subset \mathcal{O}_{2 \varrho}(0)$, where $\varrho$ is sufficiently small.
2. Now we omit the restriction $\operatorname{supp} U \subset \mathcal{O}_{2 \varrho}(0)$ and show that estimate (8.29) holds for any function $U \in \mathcal{W}^{l+2 m}\left(K^{d}\right)$.

[^3]Let us introduce a function $\psi \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ such that $\psi(y)=1$ for $|y| \leq \varrho, \operatorname{supp} \psi \subset \mathcal{O}_{2 \varrho}(0)$ and $\psi$ is independent of the polar angle $\omega$. Applying inequality (8.31) and the Leibnitz formula to the functions $U \in \mathcal{W}^{l+2 m}\left(K^{d}\right)$, we obtain the inequality

$$
\begin{align*}
& \|U\|_{\mathcal{W}^{l+2 m}\left(K^{\varrho}\right)} \leq\|\psi U\|_{\mathcal{W}^{l+2 m}(K)} \leq k_{1}\left(\|\mathcal{L}(\psi U)\|_{\mathcal{W}^{l}(K, \gamma)}+\|\psi U\|_{\mathcal{W}^{l+2 m-1}(K)}\right) \\
& \quad \leq k_{2}\left(\|\psi \mathcal{L} U\|_{\mathcal{W}^{l}(K, \gamma)}+\sum_{j, \sigma, \mu} \sum_{(k, s) \neq(j, 0)}\left\|J_{j \sigma \mu k s}\right\|_{W^{l+2 m-m_{j \sigma \mu^{-1 / 2}}\left(\gamma_{j \sigma}\right)}}+\|U\|_{\mathcal{W}^{l+2 m-1}\left(K^{2 \varrho}\right)}\right), \tag{8.32}
\end{align*}
$$

where

$$
J_{j \sigma \mu k s}=\left.\left(\psi\left(\mathcal{G}_{j \sigma k s} y\right)-\psi(y)\right)\left(B_{j \sigma \mu k s}(D) U_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}} .
$$

Similarly to Eq. (8.17), we obtain the inequality

$$
\begin{align*}
& \left.\left\|J_{j \sigma \mu k s}\right\|_{W^{l+2 m-m_{j \sigma \mu^{-1 / 2}}\left(\gamma_{j \sigma}\right)}{ }^{\leq} k_{4}\left(\left\|\mathcal{P}_{k}(D) U_{k}\right\|_{W^{l}\left(\left\{d_{1} \varrho / 2<|y|<2 d_{2} \varrho\right\}\right)}\right.} \quad+\left\|U_{k}\right\|_{W^{l+2 m-1}\left(\left\{d_{1} \varrho / 2<|y|<2 d_{2} \varrho\right\}\right)}\right) .
\end{align*}
$$

Now estimate (8.29) follows from (8.32) and (8.33).
Lemma 8.7. Assume that there exists an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$ lying on the line $\operatorname{Im} \lambda=$ $1-l-2 m$. Then the image of the operator $\mathbf{L}$ is not closed.
Proof. 1. Assume that the image of the operator $\mathbf{L}$ is closed. The following two cases are possible: (a) either the line $\operatorname{Im} \lambda=1-l-2 m$ contains an irregular eigenvalue, (b) or the $\operatorname{line} \operatorname{Im} \lambda=1-l-2 m$ contains only a regular eigenvalue $\lambda_{0}=i(1-l-2 m)$ (see Definition 7.1).
2. First, we assume that there is an irregular eigenvalue $\lambda=\lambda_{0}$ at the line. Let us show that estimate (8.29) is violated in this case. Let us denote an eigenvector and adjoint vectors (a Jordan chain of the length $\varkappa \geq 1$ ), corresponding to the eigenvalue $\lambda_{0}$ (see [23]) by $\varphi^{(0)}(\omega), \ldots, \varphi^{(\varkappa-1)}(\omega)$. The vectors $\varphi^{(k)}(\omega)$ belong to $\mathcal{W}^{l+2 m}(-\bar{\omega}, \bar{\omega})$, and, by [26, Lemma 2.1], we have

$$
\begin{equation*}
\mathcal{L} V^{k}=0, \tag{8.34}
\end{equation*}
$$

where

$$
V^{k}=r^{i \lambda_{0}} \sum_{s=0}^{k} \frac{1}{s!}(i \ln r)^{k} \varphi^{(k-s)}(\omega), \quad k=0, \ldots, \varkappa-1
$$

Since $\lambda_{0}$ is an irregular eigenvalue, we see that the function $V^{k}(y)$ is not a vector-valued polynomial for some $k \geq 0$. For simplicity, assume that $V^{0}=r^{i \lambda_{0}} \varphi^{(0)}(\omega)$ is not a vector-valued polynomial (the case $k>0$ is similar).

Introduce a sequence $U^{\delta}=r^{\delta} V^{0} /\left\|r^{\delta} V^{0}\right\|_{\mathcal{W}^{l+2 m}\left(K^{g}\right)}$. For any $\delta>0$, the denominator is finite, but $\left\|r^{\delta} V^{0}\right\|_{\mathcal{W}^{l+2 m}\left(K^{e}\right)} \rightarrow \infty$ as $\delta \rightarrow 0$ since $V^{0}$ is not a vector-valued polynomial. However,

$$
\left\|r^{\delta} V^{0}\right\|_{\mathcal{W}^{l+2 m-1}\left(K^{d}\right)} \leq c
$$

where $c>0$ is independent of $\delta \geq 0$; hence,

$$
\begin{equation*}
\left\|U^{\delta}\right\|_{\mathcal{W}^{l+2 m-1}\left(K^{d}\right)} \rightarrow 0 \quad \text { as } \quad \delta \rightarrow 0 \tag{8.35}
\end{equation*}
$$

Moreover, from relation (8.34) we have

$$
\mathcal{P}_{j}(D) U^{\delta}=\frac{r^{\delta} \mathcal{P}_{j}(D) V^{0}+\sum_{|\alpha|+|\beta|=2 m,|\alpha| \geq 1} p_{j \alpha \beta} D^{\alpha} r^{\delta} \cdot D^{\beta} V_{j}^{0}}{\left\|r^{\delta} V^{0}\right\|_{\mathcal{W}^{l+2 m}\left(K^{g}\right)}}=\frac{\sum_{|\alpha|+|\beta|=2 m,|\alpha| \geq 1} p_{j \alpha \beta} D^{\alpha} r^{\delta} \cdot D^{\beta} V_{j}^{0}}{\left\|r^{\delta} V^{0}\right\|_{\mathcal{W}^{l+2 m}\left(K^{g}\right)}},
$$

where $p_{j \alpha \beta}$ are some complex constants. Hence,

$$
\left|D^{\xi} \mathcal{P}_{j}(D) U^{\delta}\right| \leq c_{j \xi} \delta r^{l-1-|\xi|+\delta} /\left\|r^{\delta} V^{0}\right\|_{\mathcal{W}^{l+2 m}\left(K^{\varrho}\right)}, \quad|\xi| \leq l
$$

and we obtain the relation

$$
\begin{equation*}
\left\|\mathcal{P}_{j}(D) U^{\delta}\right\|_{W^{l}\left(K_{j}^{d}\right)} \rightarrow 0 \quad \text { as } \quad \delta \rightarrow 0 \tag{8.36}
\end{equation*}
$$

Similarly, using (8.34), we can prove that

$$
\begin{equation*}
\left\|\left.\mathcal{B}_{j \sigma \mu}(D) U^{\delta}\right|_{\gamma_{j \sigma}}\right\|_{W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}^{2 \rho}\right)} \rightarrow 0 \quad \text { as } \quad \delta \rightarrow 0 . \tag{8.37}
\end{equation*}
$$

To prove Eq. (8.37), we additionally need to estimate the relation

$$
\frac{\sum_{(k, s) \neq(j, 0)}\left\|\left.\left(\chi_{j \sigma k s}^{\delta}-1\right) r^{\delta}\left(B_{j \sigma \mu k s}(y, D) V^{0}\right)\left(\mathcal{G}_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}}\right\|_{W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}^{2 \varrho}\right)}}{\left\|r^{\delta} V^{0}\right\|_{\mathcal{W}^{l+2 m}\left(K^{\varrho}\right)}},
$$

which converges to zero as $\delta \rightarrow 0$ (according to inequality $\left|\chi_{j \sigma k s}^{\delta}-1\right| \leq k_{6} \delta$ ).
However, statements (8.35)-(8.37) contradict estimate (8.29) since $\left\|U^{\delta}\right\|_{\mathcal{W}^{l+2 m}\left(K^{e}\right)}=1$.
3. It remains to prove the case where the line $\operatorname{Im} \lambda=1-l-2 m$ contains only a regular eigenvalue $\lambda_{0}=i(1-l-2 m)$ of the operator $\tilde{\mathcal{L}}(\lambda)$. In this case, we cannot repeat the above reasonings since $V^{0}$ is a vector-valued polynomial and the norms $\left\|r^{\delta} V^{0}\right\|_{\mathcal{W}^{l+2 m}\left(K^{\varrho}\right)}$ are homogeneously bounded as $\delta \rightarrow 0$.

We use the results from Sec. 7.3 (see Chap. 2). According to Lemma 7.3, there exists a sequence $f^{\delta} \in \hat{\mathcal{S}}^{l}(K, \gamma), \delta>0$, such that $\operatorname{supp} f^{\delta} \subset \mathcal{O}_{\varrho}(0)$ and $f^{\delta}$ converges to the function $f^{0} \notin \hat{\mathcal{S}}^{l}(K, \gamma)$ as $\delta \rightarrow 0$ in $\mathcal{W}^{l}(K, \gamma)$. By Lemma 7.6, for every function $f^{\delta}$, there exists a function $U^{\delta} \in \mathcal{W}^{l+2 m}\left(K^{d}\right)$ such that

$$
\begin{gather*}
\mathcal{L} U^{\delta}=f^{\delta},  \tag{8.38}\\
\left\|U^{\delta}\right\|_{\mathcal{W}^{l+2 m-1}\left(K^{d}\right)} \leq c\left\|f^{\delta}\right\|_{\mathcal{W}^{l}(K, \gamma)} \tag{8.39}
\end{gather*}
$$

( $c>0$ is independent of $\delta$ ) and $U^{\delta}$ satisfies relations (7.30). Inequalities (8.29) and (8.39), relation (8.38), and the convergence of $f^{\delta}$ in the space $\mathcal{W}^{l}(K, \gamma)$ imply that the sequence $U^{\delta}$ is a Cauchy sequence in $\mathcal{W}^{l+2 m}\left(K^{\varrho}\right)$. Hence, the sequence $U^{\delta}$ converges in $\mathcal{W}^{l+2 m}\left(K^{\varrho}\right)$ as $\delta \rightarrow 0$ to some function $U$. Moreover, the limit function $U$ also satisfies relations (7.30). According to the boundedness of the operator

$$
\mathcal{L}: \mathcal{W}^{l+2 m}\left(K^{\varrho}\right) \rightarrow \mathcal{W}^{l}\left(K^{2 d_{1} \varrho}, \gamma^{2 d_{1} \varrho}\right),
$$

the following equality holds:

$$
\mathcal{L} U=f^{0} \quad \text { as } \quad y \in \mathcal{O}_{2 d_{1} \varrho}(0) .
$$

Consider a function $\psi \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ such that $\psi(y)=1$ for $|y| \leq d_{1}^{2} \varrho$ and $\operatorname{supp} \psi \subset \mathcal{O}_{2 d_{1}^{2} \varrho}(0)$. Obviously, $\psi U \in \mathcal{W}^{l+2 m}(K), \psi U$ satisfies relations (7.30), and $\operatorname{supp} \mathfrak{L}(\psi U) \subset \mathcal{O}_{2 d_{1} \varrho}(0)$. Hence,

$$
\mathcal{L}(\psi U)=\psi f^{0}+\hat{f}
$$

where $\hat{f} \in \mathcal{W}^{l}(K, \gamma)$ and the support $\hat{f}$ is compact and is separated from the origin. Hence, the function $\psi f^{0}+\hat{f}$, as well as $f^{0}$, does not belong to $\hat{\mathcal{S}}^{l}(K, \gamma)$. However, this contradicts Lemma 7.2.

Now the necessity of condition 7.1 of Theorem 8.1 follows from Lemma 8.7.

## 9. Nonlocal Problems in Weight Spaces with a Small Weight Index

9.1. Statement of the main result. In Sec. 8.3, we introduced the operator

$$
\begin{equation*}
\mathbf{L}_{a}=\{\mathbf{P}, \mathbf{B}\}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G), \quad a>l+2 m-1 . \tag{9.1}
\end{equation*}
$$

As was stated above, by virtue of Lemma 6.1 and [89, Theorem 3.2], $\mathbf{L}_{a}$ is a Fredholm operator for almost all $a>l+2 m-1$.

Here, we study problem (6.7), (6.8) in weight spaces with weight index $a>0$. As earlier, in this case, $\mathbf{B}_{i \mu}^{2} u \in W^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ for all $u \in H_{a}^{l+2 m}(G) \subset W^{l+2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$. However, now the function
$\mathbf{B}_{i \mu}^{2} u$ may not belong to the space $H_{a}^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$. Then the operator $\mathbf{L}_{a}$ defined in Eq. (9.1) is not well defined.

Let us introduce the set

$$
S_{a}^{l+2 m}(G)=\left\{u \in H_{a}^{l+2 m}(G): \text { functions } \mathbf{B}_{i \mu}^{2} u \text { satisfy conditions (5.5) }\right\}
$$

Using Eq. (6.5), we obtain the inequality

$$
\left\|\mathbf{B}_{i \mu}^{2} u\right\|_{W^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)} \leq k_{1}\|u\|_{W^{l+2 m}\left(G \backslash \overline{\mathcal{O}_{1}}(\mathcal{K})\right)} \leq k_{2}\|u\|_{H_{a}^{l+2 m}(G)}
$$

for all $u \in H_{a}^{l+2 m}(G)$. This, the Sobolev inequality, and the Riesz theorem on the general form of linear continuous functionals in a Hilbert space imply that $S_{a}^{l+2 m}(G)$ is a closed finite-dimensional subspace in $H_{a}^{l+2 m}(G)$.

On the other hand, by Lemma 5.2, we have $\mathbf{B}_{i \mu}^{2} u \in H_{a}^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ for any function $u \in$ $S_{a}^{l+2 m}(G), a>0$. Since the functions $\mathbf{B}_{i \mu}^{0} u$ and $\mathbf{B}_{i \mu}^{1} u$ belong to the space $H_{a}^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ for all $a \in \mathbb{R}$ and $u \in S_{a}^{l+2 m}(G)$ (and even for $u \in H_{a}^{l+2 m}(G)$ ), we see that

$$
\{\mathbf{P} u, \mathbf{B} u\} \in \mathcal{H}_{a}^{l}(G, \partial G) \quad \forall u \in S_{a}^{l+2 m}(G), \quad a>0
$$

Thus, there exists a finite-dimensional space $\mathcal{R}_{a}^{l}(G, \partial G)$ (contained in $\{0\} \times \prod_{i, \mu} H_{a^{\prime}}^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$, $\left.a^{\prime}>l+2 m-1\right)$ such that

$$
\{\mathbf{P} u, \mathbf{B} u\} \in \mathcal{H}_{a}^{l}(G, \partial G)+\mathcal{R}_{a}^{l}(G, \partial G) \quad \forall u \in H_{a}^{l+2 m}(G), \quad a>0 .
$$

Hence we can define a bounded operator

$$
\mathbf{L}_{a}=\{\mathbf{P}, \mathbf{B}\}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G) \dot{+} \mathcal{R}_{a}^{l}(G, \partial G), \quad a>0 .
$$

Obviously, we can assume that $\mathcal{R}_{a}^{l}(G, \partial G)=\{0\}$ for $a>l+2 m-1$.
Theorem 9.1. Let $a>0$, and let the line $\operatorname{Im} \lambda=a+1-l-2 m$ not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then $\mathbf{L}_{a}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G) \dot{+} \mathcal{R}_{a}^{l}(G, \partial G)$ is a Fredholm operator.

Conversely, let $\mathbf{L}_{a}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G) \dot{+} \mathcal{R}_{a}^{l}(G, \partial G)$ be a Fredholm operator. Then the line $\operatorname{Im} \lambda=a+1-l-2 m$ does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$.

Note that if $f \in \mathcal{H}_{a}^{l}(G, \partial G)$, then

$$
\|f\|_{\mathcal{H}_{a}^{l}(G, \partial G)+\mathcal{R}_{a}^{l}(G, \partial G)}=\|f\|_{\mathcal{H}_{a}^{l}(G, \partial G)} .
$$

This relation, Theorem 9.1, and the Riesz theorem on the general form of linear continuous functionals in a Hilbert space yield the following result.

Corollary 9.1. Let $a>0$, and let the line $\operatorname{Im} \lambda=a+1-l-2 m$ not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then there exist functions $f^{q} \in \mathcal{H}_{a}^{l}(G, \partial G), q=1, \ldots, q_{1}$, such that if the right-hand side $f$ of problem (6.7), (6.8) belongs to $\mathcal{H}_{a}^{l}(G, \partial G)$ and

$$
\left(f, f^{q}\right)_{\mathcal{H}_{a}^{l}(G, \partial G)}=0, \quad q=1, \ldots, q_{1},
$$

then problem (6.7), (6.8) has a solution $u \in H_{a}^{l+2 m}(G)$.
Corollary 9.1 shows that the inclusion $u \in H_{a}^{l+2 m}(G)$, generally speaking, does not yield the inclusion $\mathbf{L}_{a} u \in \mathcal{H}_{a}^{l}(G, \partial G)$ for $0<a \leq l+2 m-1$; however, if we impose a finite number of orthogonality conditions on the right-hand side $f \in \mathcal{H}_{a}^{l}(G, \partial G)$, then problem (6.7), (6.8) has a solution $u \in H_{a}^{l+2 m}(G)$.

### 9.2. Proof of the main result.

### 9.2.1. Proof of Theorem 9.1. Sufficiency.

Lemma 9.1. The kernel of the operator $\mathbf{L}_{a}$ is finite-dimensional.
Proof. Since $H_{a}^{l+2 m}(G) \subset H_{a^{\prime}}^{l+2 m}(G)$ for $a \leq a^{\prime}$, we see that the proof of this lemma is similar to the proof of Lemma 8.5.

Now we pass to the construction of a right regularizer for the operator $\mathbf{L}_{a}$.
As was noted above, the functions $\mathbf{B}_{i \mu}^{0} u$ and $\mathbf{B}_{i \mu}^{1} u$ belong to $H_{a}^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ for all $u \in H_{a}^{l+2 m}(G)$ and $a \in \mathbb{R}$. Hence, we can define a bounded operator

$$
\mathbf{L}_{a}^{1}=\{\mathbf{P}, \mathbf{C}\}: H_{a}^{l+2 m}(G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G) .
$$

It is proved in $[89, \mathrm{Sec} .3]$ that there exist a bounded operator

$$
\mathbf{R}_{a, 1}: \mathcal{H}_{a}^{l}(G, \partial G) \rightarrow H_{a}^{l+2 m}(G)
$$

and a compact operator

$$
\mathbf{T}_{a, 1}: \mathcal{H}_{a}^{l}(G, \partial G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G)
$$

such that

$$
\begin{equation*}
\mathbf{L}_{a}^{1} \mathbf{R}_{a, 1}=\mathbf{I}_{a}+\mathbf{T}_{a} \tag{9.2}
\end{equation*}
$$

where $\mathbf{I}_{a}$ is the identity operator in $\mathcal{H}_{a}^{l}(G, \partial G)$.
Let us formulate an analog of Lemma 8.3 in wight spaces.
Let $\xi$ and $\psi$ be the functions defined before Lemma 8.1, but now we assume that $\delta>0$ is arbitrary (in particular, it is independent of $\varepsilon_{0}$ ).

Lemma 9.2. Let the line $\operatorname{Im} \lambda=a+1-l-2 m$ not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then for any $\delta, 0<\delta<1$, there exist bounded operators

$$
\begin{aligned}
\mathbf{R}_{a, \mathcal{K}}^{\prime}: \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) & \rightarrow H_{a}^{l+2 m}(G), \\
\mathbf{M}_{a, \mathcal{K}}^{\prime}, \mathbf{T}_{a, \mathcal{K}}^{\prime}: \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) & \rightarrow \mathcal{H}_{a}^{l}(G, \partial G),
\end{aligned}
$$

such that $\left\|\mathbf{M}_{a, \mathcal{K}}^{\prime}\right\| \leq c \delta$, where $c>0$ is independent of $\delta$, the operator $\mathbf{T}_{a, \mathcal{K}}^{\prime}$ is compact, and

$$
\mathbf{L}_{a}^{1} \mathbf{R}_{a, \mathcal{K}}^{\prime} f^{\prime}=\psi\left\{0, f^{\prime}\right\}+\mathbf{M}_{a, \mathcal{K}}^{\prime} f^{\prime}+\mathbf{T}_{a, \mathcal{K}}^{\prime} f^{\prime}
$$

Proof. It follows from [88, Theorem 2.1] that the operator $\mathcal{L}_{a}$ defined in (7.44) has a bounded inverse operator. Let

$$
\mathbf{R}_{a, \mathcal{K}}^{\prime} f^{\prime}=\xi U, \quad U=\mathcal{L}_{a}^{-1}\left(\psi\left\{0, f^{\prime}\right\}\right),
$$

where $\xi$ and $\psi$ are the same functions as in the proof of Lemma 8.3. The further proof is similar to the proof of Lemma 8.3.

For any function $f^{\prime} \in \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)$, we set

$$
\begin{equation*}
\mathbf{R}_{a, 1}^{\prime} f^{\prime}=\mathbf{R}_{a, \mathcal{K}}^{\prime} f^{\prime}+\sum_{j=1}^{J} \mathbf{R}_{0 j}^{\prime}\left(\psi_{j}^{\prime} f^{\prime}\right) \tag{9.3}
\end{equation*}
$$

where the functions $\psi_{j}^{\prime}$ and the operators $\mathbf{R}_{0 j}^{\prime}$ are the same as in Sec. 8.2.
It can be directly verified by Lemma 9.2 that

$$
\begin{equation*}
\mathbf{L}_{a}^{1} \mathbf{R}_{a, 1}^{\prime} f^{\prime}=\left\{0, f^{\prime}\right\}+\mathbf{M}_{a, 1}^{\prime} f^{\prime}+\mathbf{T}_{a, 1}^{\prime} f^{\prime} \tag{9.4}
\end{equation*}
$$

where

$$
\mathbf{M}_{a, 1}^{\prime}, \mathbf{T}_{a, 1}^{\prime}: \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G)
$$

are bounded operators such that $\left\|\mathbf{M}_{a, 1}^{\prime}\right\| \leq c \delta$, where $c>0$ is independent of $\delta$, and the operator $\mathbf{T}_{a, 1}^{\prime}$ is compact.

Using the operators $\mathbf{R}_{a, 1}$ and $\mathbf{R}_{a, 1}^{\prime}$, we construct a right regularizer for problem (6.7), (6.8) in weight spaces in the case where $\mathbf{B}_{i \mu}^{2} \neq 0$.

Let us introduce the following space for $a>0$ :

$$
\mathcal{S}_{a}^{l}(G, \partial G)=\left\{f \in \mathcal{H}_{a}^{l}(G, \partial G): \Phi=\mathbf{B}^{2} \mathbf{R}_{a, 1} f \text { and } \mathbf{B}^{2} \mathbf{R}_{a, 1}^{\prime} \Phi \text { belong to } \mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)\right\} .
$$

First, we show that $\mathcal{S}_{a}^{l}(G, \partial G)$ is a closed subspace in $\mathcal{H}_{a}^{l}(G, \partial G)$ with finite codimension. Indeed, using inequality (6.5), we obtain that

$$
\begin{equation*}
\left\|\Phi_{i \mu}\right\|_{W^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)} \leq k_{1}\left\|\mathbf{R}_{a, 1} f\right\|_{W^{l+2 m}\left(G \backslash \overline{\left.\mathcal{O}_{1}(\mathcal{K})\right)}\right.} \leq k_{2}\left\|\mathbf{R}_{a, 1} f\right\|_{H_{a}^{l+2 m}(G)} \leq k_{3}\|f\|_{\mathcal{H}_{a}^{l}(G, \partial G)} . \tag{9.5}
\end{equation*}
$$

Since the function $\Phi_{i \mu}$ satisfies condition (5.5), we see from Eq. (9.5) and Lemma 5.2 that

$$
\Phi \in \mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)
$$

and

$$
\begin{equation*}
\left\|\Phi_{i \mu}\right\|_{H_{a}^{l+2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)} \leq k_{4}\|f\|_{\mathcal{H}_{a}^{l}(G, \partial G)} . \tag{9.6}
\end{equation*}
$$

Hence, the expression $\mathbf{B}^{2} \mathbf{R}_{a, 1}^{\prime} \Phi$ is well defined. Similarly, using Eqs. (9.6) and (5.5), we obtain inequalities

$$
\begin{align*}
\left\|\mathbf{B}^{2} \mathbf{R}_{a, 1}^{\prime} \Phi\right\|_{\mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)} & \leq k_{5}\|f\|_{\mathcal{H}_{a}^{l}(G, \partial G)}  \tag{9.7}\\
\left\|\mathbf{B}^{2} \mathbf{R}_{a, 1}^{\prime} \Phi\right\|_{\mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)} & \leq k_{6}\|f\|_{\mathcal{H}_{a}^{l}(G, \partial G)} . \tag{9.8}
\end{align*}
$$

It follows from Eqs. (9.5) and (9.7), the Sobolev embedding theorem, and the Riesz theorem on the general form of linear continuous functionals in a Hilbert space that $\mathcal{S}_{a}^{l}(G, \partial G)$ is a subspace of a finite co-dimension in $\mathcal{H}_{a}^{l}(G, \partial G)$. Hence,

$$
\begin{equation*}
\mathcal{H}_{a}^{l}(G, \partial G) \dot{+} \mathcal{R}_{a}^{l}(G, \partial G)=\mathcal{S}_{a}^{l}(G, \partial G) \dot{+} \hat{\mathcal{R}}_{a}^{l}(G, \partial G), \tag{9.9}
\end{equation*}
$$

where $\hat{\mathcal{R}}_{a}^{l}(G, \partial G)$ is a finite-dimensional space.
Now we prove the following result.
Lemma 9.3. Let $a>0$, and let the line $\operatorname{Im} \lambda=a+1-l-2 m$ not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then there exist a bounded operator

$$
\mathbf{R}_{a}: \mathcal{H}_{a}^{l}(G, \partial G) \dot{+} \mathcal{R}_{a}^{l}(G, \partial G) \rightarrow H_{a}^{l+2 m}(G)
$$

and a compact operator

$$
\mathbf{T}_{a}: \mathcal{H}_{a}^{l}(G, \partial G)+\mathcal{R}_{a}^{l}(G, \partial G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G)+\mathcal{R}_{a}^{l}(G, \partial G)
$$

such that

$$
\begin{equation*}
\mathbf{L}_{a} \mathbf{R}_{a}=\hat{\mathbf{I}}_{a}+\mathbf{T}_{a}, \tag{9.10}
\end{equation*}
$$

where $\hat{\mathbf{I}}_{a}$ is the identity operator in $\mathcal{H}_{a}^{l}(G, \partial G)+\mathcal{R}_{a}^{l}(G, \partial G)$.
Proof. 1. Let $\Phi=\mathbf{B}^{2} \mathbf{R}_{a, 1} f$, where $f \in \mathcal{S}_{a}^{l}(G, \partial G)$. It follows from Eqs. (9.6) and (9.8) that the functions $\{0, \Phi\}$ and $\left\{0, \mathbf{B}^{2} \mathbf{R}_{a, 1}^{\prime} \Phi\right\}$ belong to $\mathcal{H}_{a}^{l}(G, \partial G)$. Hence, the functions $\Phi$ and $\mathbf{B}^{2} \mathbf{R}_{a, 1}^{\prime} \Phi$ belong to the domain of the operator $\mathbf{R}_{a, 1}^{\prime}$, and we can introduce the operator

$$
\mathbf{R}_{a, \mathcal{S}}: \mathcal{S}_{a}^{l}(G, \partial G) \rightarrow H_{a}^{l+2 m}(G)
$$

by the formula

$$
\mathbf{R}_{a, \mathcal{S}} f=\mathbf{R}_{a, 1} f-\mathbf{R}_{a, 1}^{\prime} \Phi+\mathbf{R}_{a, 1}^{\prime} \mathbf{B}^{2} \mathbf{R}_{a, 1}^{\prime} \Phi .
$$

Similarly to the proof of Lemma 8.4, applying inequalities (9.2) and (9.4), we can show that

$$
\mathbf{L}_{a} \mathbf{R}_{a, \mathcal{S}}=\mathbf{I}_{a, \mathcal{S}}+M+T,
$$

where

$$
\mathbf{I}_{a, \mathcal{S}}, M, T: \mathcal{S}_{a}^{l}(G, \partial G) \rightarrow \mathcal{H}_{a}^{l}(G, \partial G)+\mathcal{R}_{a}^{l}(G, \partial G)
$$

are bounded operators such that $\mathbf{I}_{a, \mathcal{S}} f=f$ and $\|M\| \leq c \delta(c>0$ is independent of $\delta)$, and the operator $T$ is compact.
2.By Eq. (9.9), the subspace $\mathcal{S}_{a}^{l}(G, \partial G)$ has finite codimension in the space $\mathcal{H}_{a}^{l}(G, \partial G) \dot{+} \mathcal{R}_{a}^{l}(G, \partial G)$; hence, $\mathbf{I}_{a, \mathcal{S}}$ is a Fredholm operator. By [56, Theorems 16.2, 16.4], $\mathbf{I}_{a, \mathcal{S}}+M+T$ is also a Fredholm operator for a sufficiently small $\delta$. It follows from [56, Theorem 15.2] that there exist a bounded operator $\tilde{\mathbf{R}}_{a}$ and a compact operator $\mathbf{T}_{a}$ acting from $\mathcal{H}_{a}^{l}(G, \partial G) \dot{+} \mathcal{R}_{a}^{l}(G, \partial G)$ to $\mathcal{S}_{a}^{l}(G, \partial G)$ and to $\mathcal{H}_{a}^{l}(G, \partial G) \dot{+} \mathcal{R}_{a}^{l}(G, \partial G)$, respectively. In addition, these operators are such that

$$
\left(\mathbf{I}_{a, \mathcal{S}}+M+T\right) \tilde{\mathbf{R}}_{a}=\hat{\mathbf{I}}_{a}+\mathbf{T}_{a} .
$$

Introducing the notation

$$
\mathbf{R}_{a}=\mathbf{R}_{a, \mathcal{S}} \tilde{\mathbf{R}}_{a}: \mathcal{H}_{a}^{l}(G, \partial G)+\mathcal{R}_{a}^{l}(G, \partial G) \rightarrow H_{a}^{l+2 m}(G),
$$

we obtain Eq. (9.10).
By [56, Theorem 15.2] and Lemma 9.3, the image of the operator $\mathbf{L}_{a}, a>0$, is closed and has finite codimension. The first item of Theorem 9.1 follows from here and Lemma 9.1.

### 9.2.2. Proof of Theorem 9.1. Necessity.

Lemma 9.4. Let $a>0$, and let the line $\operatorname{Im} \lambda=a+1-l-2 m$ contain an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$. Then the image of the operator $\mathbf{L}_{a}$ is not closed.

Proof. 1. Let $d=d(\varrho)=2 d_{2} \varrho$, where $d_{2}$ is defined in Eq. (6.15). Assume that the image of the operator $\mathbf{L}_{a}$ is closed. Then, similarly to the proof of Lemma 8.6, we use Lemma 9.1, the compactness of the embedding operator $H_{a}^{l+2 m}(G) \subset H_{a}^{l+2 m-1}(G)$, and [56, Theorem 7.1] and show that

$$
\begin{equation*}
\|U\|_{\mathcal{H}_{a}^{l+2 m}\left(K^{e}\right)} \leq c\left(\|\mathcal{L} U\|_{\mathcal{H}_{a}^{l}\left(K^{2 \varrho}, \gamma^{2 \varrho}\right)}+\sum_{j=1}^{N}\left\|\mathcal{P}_{j}(D) U_{j}\right\|_{H_{a}^{l}\left(K_{j}^{d}\right)}+\|U\|_{\mathcal{H}_{a}^{l+2 m-1}\left(K^{d}\right)}\right) \tag{9.11}
\end{equation*}
$$

for all $U \in \mathcal{H}_{a}^{l+2 m}\left(K^{d}\right)$ and sufficiently small $\varrho$.
2. Let $\lambda_{0}$ be an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$ belonging to the line $\operatorname{Im} \lambda=a+1-l-2 m$, and $\varphi^{(0)}(\omega)$ be a corresponding eigenvector. The vector $\varphi^{(0)}(\omega)$ belongs to $\mathcal{W}^{l+2 m}(-\bar{\omega}, \bar{\omega})$, and, by $[26$, Lemma 2.1], we have

$$
\begin{equation*}
\mathcal{L} V^{0}=0, \tag{9.12}
\end{equation*}
$$

where $V^{0}=r^{i \lambda_{0}} \varphi^{(0)}(\omega)$.
Let us substitute the sequence $U^{\delta}=r^{\delta} V^{0} /\left\|r^{\delta} V^{0}\right\|_{\mathcal{H}_{a}^{l+2 m}\left(K^{\rho}\right)}, \delta>0$, to Eq. (9.11) and let $\delta \rightarrow 0$. Similarly to the proof of Lemma 8.7, it is easy to verify (using Eq. (9.12)) that the right-hand side of inequality (9.11) tends to zero when the left-hand side is equal to 1 . The contradiction obtained proves the lemma.

Now the second item of Theorem 9.1 follows from Lemma 9.4.
10. Regular Eigenvalues of the Operator $\tilde{\mathcal{L}}(\lambda)$ on the Line $\operatorname{Im} \lambda=1-l-2 m$

In the previous sections, we have proved the Fredholm solvability of problem (6.7), (6.8) in the cases where there are no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ on the corresponding line on the complex plane. In this section, we use results of Sec. 7.3 (see Chap. 2) and study the situation where only a regular eigenvalue $\lambda_{0}=i(1-l-2 m)$ of the operator $\tilde{\mathcal{L}}(\lambda)$ lies on the line $\operatorname{Im} \lambda=1-l-2 m$. In other words, we assume that condition 7.2 holds. In this case, by virtue of Theorem $8.1, \mathbf{L}: W^{l+2 m}(G) \rightarrow \mathcal{W}^{l}(G, \partial G)$ is not a Fredholm operator (its image is not closed). Hence we introduce an operator corresponding to problem (6.7), (6.8) but acting in other spaces and prove that it is a Fredholm operator.
10.1. Construction of the right regularizer in the case where $\mathbf{B}_{i \mu}^{2}=0$. Introduce functions $\hat{\psi} \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ such that $\hat{\psi}(y)=1$ for $y \in \mathcal{O}_{\varepsilon / 2}(\mathcal{K})$ and $\operatorname{supp} \hat{\psi} \subset \mathcal{O}_{\varepsilon}(\mathcal{K})$. Let the vector $F=\left\{f_{j}, f_{j \sigma \mu}\right\}$ of the right-hand sides of problem (6.12), (6.13) correspond to the vector $\hat{\psi} f=\left\{\hat{\psi} f_{0}, \hat{\psi} f_{i \mu}\right\}$ of the right-hand sides of problem (6.7), (6.8). Obviously, $\operatorname{supp} F \subset \mathcal{O}_{\varepsilon}(0)$.

Consider the space $\hat{\mathcal{S}}^{l}(G, \partial G)$ with the norm

$$
\begin{equation*}
\|f\|_{\hat{\mathcal{S}}^{l}(G, \partial G)}=\left(\|(1-\hat{\psi}) f\|_{\mathcal{W}^{l}(G, \partial G)}^{2}+\|F\|_{\hat{\mathcal{S}}^{l}(K, \gamma)}^{2}\right)^{1 / 2} \tag{10.1}
\end{equation*}
$$

Introduce the space

$$
\hat{\mathcal{S}}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)=\left\{f^{\prime} \in \mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}(\partial G):\left(0, f^{\prime}\right) \in \hat{\mathcal{S}}^{l}(G, \partial G)\right\} .
$$

Obviously, the following embeddings hold (cf. (7.29)):

$$
\begin{gathered}
\hat{\mathcal{S}}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) \subset \mathcal{S}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) \subset W^{l+2 m-\mathbf{m}-1 / 2}(\partial G), \\
\hat{\mathcal{S}}^{l}(G, \partial G) \subset \mathcal{S}^{l}(G, \partial G) \subset \mathcal{W}^{l}(G, \partial G)
\end{gathered}
$$

By Lemma 7.3, the set $\hat{\mathcal{S}}^{l}(G, \partial G)$ is not closed in the topology of the space $\mathcal{W}^{l}(G, \partial G)$.
On the other hand, by Lemma 7.2, if $u \in S^{l+2 m}(G)$, then $\{\mathbf{P} u, \mathbf{C} u\} \in \hat{\mathcal{S}}^{l}(G, \partial G)$ (the operator $\mathbf{C}=\mathbf{B}^{0}+\mathbf{B}^{1}$ is defined in Sec. 8).

Let us consider the operator

$$
\hat{\mathbf{L}}^{1}=\{\mathbf{P}, \mathbf{C}\}: S^{l+2 m}(G) \rightarrow \hat{\mathcal{S}}^{l}(G, \partial G)
$$

By Lemma 7.2, the operator $\hat{\mathbf{L}}^{1}$ is bounded.
Let $\psi$ and $\xi$ be functions defined before Lemma 8.1.
Theorem 10.1. Let condition 7.2 hold. Then for any sufficiently small $\varepsilon_{0}>0$, there exist bounded operators

$$
\hat{\mathbf{R}}_{\mathcal{K}}: \hat{\mathcal{S}}^{l}(G, \partial G) \rightarrow S^{l+2 m}(G), \quad \hat{\mathbf{M}}_{\mathcal{K}}, \hat{\mathbf{S}}_{\mathcal{K}}: \hat{\mathcal{S}}^{l}(G, \partial G) \rightarrow \hat{\mathcal{S}}^{l}(G, \partial G)
$$

such that

$$
\begin{equation*}
\mathbf{L}^{1} \hat{\mathbf{R}}_{\mathcal{K}} f=\psi f+\hat{\mathbf{M}}_{\mathcal{K}} f+\hat{\mathbf{S}}_{\mathcal{K}} f \tag{10.2}
\end{equation*}
$$

$\left\|\hat{\mathbf{M}}_{\mathcal{K}}\right\| \leq c_{1} \varepsilon_{0},\left\|\hat{\mathbf{S}}_{\mathcal{K}}\right\| \leq c_{2}$, and the squared operator $\hat{\mathbf{S}}_{\mathcal{K}}$ is compact. Moreover, the operator $\hat{\mathbf{S}}_{\mathcal{K}}$ can be written in the form $\hat{\mathbf{S}}_{\mathcal{K}}=\hat{\mathbf{U}}_{\mathcal{K}}+\hat{\mathbf{F}}_{\mathcal{K}}$, where $\left\|\hat{\mathbf{U}}_{\mathcal{K}}\right\| \leq c_{3}$, and the operator $\hat{\mathbf{F}}_{\mathcal{K}}$ is compact; the constants $c_{1}, c_{2}$ and $c_{3}>0$ are independent of $\varepsilon_{0}$.

Proof. The idea of the proof is similar to the idea of the proof of Lemma 8.1. We explain how to construct the operator $\hat{\mathbf{R}}_{\mathcal{K}}$. We perform the change of variables $y \rightarrow y^{\prime}$ from Sec. 6.1 in a neighborhood of the set $\mathcal{K}$ and denote $y^{\prime}=y$. We denote the functions $\psi, \xi$, and $f$ written in the new coordinates by the same symbols. Then the operator $\hat{\mathbf{R}}_{\mathcal{K}}$ is defined by the formula $\hat{\mathbf{R}}_{\mathcal{K}} f=\xi U$, where $U \in \mathcal{W}^{l+2 m}\left(K^{d}\right)$ (for any $d>0$ ) is a solution of problem (6.16), (6.17) with the right-hand side $\psi f$ (see Lemma 7.6).

The proof of the following lemma is similar to the proof of Lemma 8.2 (but we must use Lemma 10.1 instead of Lemma 8.1).
Lemma 10.1. Let condition 7.2 hold. Then there exists a bounded operator $\hat{\mathbf{R}}_{1}: \hat{\mathcal{S}}^{l}(G, \partial G) \rightarrow S^{l+2 m}(G)$ and a compact operator $\hat{\mathbf{T}}_{1}: \hat{\mathcal{S}}^{l}(G, \partial G) \rightarrow \hat{\mathcal{S}}^{l}(G, \partial G)$ such that

$$
\begin{equation*}
\hat{\mathbf{L}}^{1} \hat{\mathbf{R}}_{1}=\hat{\mathbf{I}}+\hat{\mathbf{T}}_{1} \tag{10.3}
\end{equation*}
$$

where $\hat{\mathbf{I}}$ is the identity operator in the space $\hat{\mathcal{S}}^{l}(G, \partial G)$.
10.2. Construction of the right regularizer in the case where $\mathbf{B}_{i \mu}^{2} \neq 0$. Let $\xi$ and $\psi$ be the functions defined before Lemma 8.1, but now we assume that the number $\delta>0$ is arbitrary (in particular, it is independent of $\varepsilon_{0}$ ).

The proof of the following lemma is similar to the proof of Lemma 8.3.
Lemma 10.2. Let condition 7.2 hold. Then for any $\delta, 0<\delta<1$, there exist bounded operators

$$
\hat{\mathbf{R}}_{\mathcal{K}}^{\prime}: \hat{\mathcal{S}}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) \rightarrow S^{l+2 m}(G), \quad \hat{\mathbf{M}}_{\mathcal{K}}^{\prime}, \hat{\mathbf{T}}_{\mathcal{K}}^{\prime}: \hat{\mathcal{S}}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) \rightarrow \hat{\mathcal{S}}^{l}(G, \partial G)
$$

such that

$$
\begin{equation*}
\mathbf{L}^{1} \hat{\mathbf{R}}_{\mathcal{K}}^{\prime} f^{\prime}=\psi\left\{0, f^{\prime}\right\}+\hat{\mathbf{M}}_{\mathcal{K}}^{\prime} f^{\prime}+\hat{\mathbf{T}}_{\mathcal{K}}^{\prime} f^{\prime} \tag{10.4}
\end{equation*}
$$

where $\left\|\hat{\mathbf{M}}_{\mathcal{K}}^{\prime}\right\| \leq c \delta$, the operator $\hat{\mathbf{T}}_{\mathcal{K}}^{\prime}$ is compact, and $c>0$ is independent of $\delta$.
For any function $f^{\prime} \in \hat{\mathcal{S}}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)$, assume that

$$
\hat{\mathbf{R}}_{1}^{\prime} f^{\prime}=\hat{\mathbf{R}}_{\mathcal{K}}^{\prime} f^{\prime}+\sum_{j=1}^{J} \mathbf{R}_{0 j}^{\prime}\left(\psi_{j}^{\prime} f^{\prime}\right)
$$

where the functions $\psi_{j}^{\prime}$ and the operator $\mathbf{R}_{0 j}^{\prime}$ are the same as in Sec. 8.2.
Using Lemma 10.2, it easy to verify that

$$
\begin{equation*}
\hat{\mathbf{L}}^{1} \hat{\mathbf{R}}_{1}^{\prime} f^{\prime}=\left\{0, f^{\prime}\right\}+\hat{\mathbf{M}}_{1}^{\prime} f^{\prime}+\hat{\mathbf{T}}_{1}^{\prime} f^{\prime} \tag{10.5}
\end{equation*}
$$

where

$$
\hat{\mathbf{M}}_{1}^{\prime}, \hat{\mathbf{T}}_{1}^{\prime}: \hat{\mathcal{S}}^{l+2 m-\mathbf{m}-1 / 2}(\partial G) \rightarrow \hat{\mathcal{S}}^{l}(G, \partial G)
$$

are bounded operators such that $\left\|\hat{\mathbf{M}}_{1}^{\prime}\right\| \leq c \delta$, where $c>0$ is independent of $\delta$, and the operator $\hat{\mathbf{T}}_{1}^{\prime}$ is compact.

Using the operators $\hat{\mathbf{R}}_{1}$ and $\hat{\mathbf{R}}_{1}^{\prime}$, we construct a right regularizer for problem (6.7), (6.8) in the case where $\mathbf{B}_{i \mu}^{2} \neq 0$. For this, we need the following concordance condition.
Condition 10.1. For any function $u \in S^{l+2 m}(G)$, we have $\mathbf{B}^{2} u \in \hat{\mathcal{S}}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)$ and

$$
\left\|\mathbf{B}^{2} u\right\|_{\hat{\mathcal{S}}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)} \leq c\|u\|_{W^{l+2 m}(G)}
$$

Remark 10.1. According to (6.5), the operator $\mathbf{B}^{2}$ corresponds to nonlocal terms with supports lying outside the set $\mathcal{K}$. Hence, if condition 10.1 holds for the functions $u \in S^{l+2 m}(G)$, then it is also fulfilled for the functions $u \in W^{l+2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$.
Remark 10.2. Using the example from Subsec. 6.2 (see Chap. 2), we explain how to achieve the fulfillment of condition 10.1.

Consider problem (6.9), (6.10) and assume, in addition, that the transformations $\Omega_{i s}$ in this problem correspond to condition (6.3) (i.e., the restriction on the structure of transformations $\Omega_{i s}$ ). Then, by virtue of the continuity of $\Omega_{i s}$, we have $\Omega_{i s}\left(\mathcal{O}_{\delta}(g)\right) \subset \mathcal{O}_{\varepsilon_{0} / 2}(\mathcal{K})$ for any point $g \in \overline{\Gamma_{i}} \cap \mathcal{K}$ if $\delta>0$ is sufficiently small. Hence, for any function $u \in W^{l+2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$, we have

$$
\begin{equation*}
\mathbf{B}_{i \mu}^{2} u(y)=0 \quad \text { for } \quad y \in \mathcal{O}_{\delta}(\mathcal{K}), \tag{10.6}
\end{equation*}
$$

since $1-\zeta\left(\Omega_{i s}(y)\right)=0$ for $y \in \mathcal{O}_{\delta}(\mathcal{K})$. Obviously, condition 10.1 holds in this case.
Instead of condition (6.3), we can assume the following: if $\Omega_{i s}(g) \notin \mathcal{K}$ (where $g \in \overline{\Gamma_{i}} \cap \mathcal{K}$ ), then the coefficients of the operators $B_{i \mu s}(y, D)$ vanish at the points $\Omega_{i s}(g)$. This also guarantees that $\mathbf{B}^{2} u \in \hat{\mathcal{S}}^{l+2 m-\mathbf{m}-1 / 2}(\partial G)$ for any function $u \in W^{l+2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$.

By virtue of Lemma 7.2 and condition 10.1, we have

$$
\{\mathbf{P} u, \mathbf{B} u\} \in \hat{\mathcal{S}}^{l}(G, \partial G) \quad \forall u \in S^{l+2 m}(G)
$$

Hence the operator

$$
\hat{\mathbf{L}}_{S}=\{\mathbf{P}, \mathbf{B}\}: S^{l+2 m}(G) \rightarrow \hat{\mathcal{S}}^{l}(G, \partial G)
$$

is well defined and bounded.
Lemma 10.3. Let conditions 7.2 and 10.1 hold. Then there exist a bounded operator $\hat{\mathbf{R}}: \hat{\mathcal{S}}^{l}(G, \partial G) \rightarrow$ $S^{l+2 m}(G)$ and a compact operator $\hat{\mathbf{T}}: \hat{\mathcal{S}}^{l}(G, \partial G) \rightarrow \hat{\mathcal{S}}^{l}(G, \partial G)$ such that

$$
\begin{equation*}
\hat{\mathbf{L}}_{S} \hat{\mathbf{R}}=\hat{\mathbf{I}}+\hat{\mathbf{T}} \tag{10.7}
\end{equation*}
$$

Proof. Assume that $\Phi=\mathbf{B}^{2} \hat{\mathbf{R}}_{1} f$, where $f=\left\{f_{0}, f^{\prime}\right\} \in \hat{\mathcal{S}}^{l}(G, \partial G)$, and $\hat{\mathbf{R}}_{1}$ is a operator from formula (10.3). Then, according to condition 10.1, the functions $\Phi$ and $\mathbf{B}^{2} \hat{\mathbf{R}}_{1}^{\prime} \Phi$ belong to the domain of the operator $\hat{\mathbf{R}}_{1}^{\prime}$. Hence we can define a bounded operator $\hat{\mathbf{R}}_{\mathcal{S}}: \hat{\mathcal{S}}^{l}(G, \partial G) \rightarrow S^{l+2 m}(G)$ by the formula

$$
\hat{\mathbf{R}}_{\mathcal{S}} f=\hat{\mathbf{R}}_{1} f-\hat{\mathbf{R}}_{1}^{\prime} \Phi+\hat{\mathbf{R}}_{1}^{\prime} \mathbf{B}^{2} \hat{\mathbf{R}}_{1}^{\prime} \Phi .
$$

It is easy to verify (similarly to the proof of Lemma 8.4 by using inequalities(10.3) and (10.5)) that

$$
\hat{\mathbf{L}}_{S} \hat{\mathbf{R}}_{\mathcal{S}}=\hat{\mathbf{I}}+M+T,
$$

where $M, T: \hat{\mathcal{S}}^{l}(G, \partial G) \rightarrow \hat{\mathcal{S}}^{l}(G, \partial G)$ are bounded operators such that $\|M\| \leq c \delta(c>0$ is independent of $\delta$ ) and the operator $T$ is compact.

The operator $\hat{\mathbf{I}}+M: \hat{\mathcal{S}}^{l}(G, \partial G) \rightarrow \hat{\mathcal{S}}^{l}(G, \partial G)$ is invertible if $\delta \leq 1 /(2 c)$. Hence, introducing the notation $\hat{\mathbf{R}}=\hat{\mathbf{R}}_{\mathcal{S}}(\hat{\mathbf{I}}+M)^{-1}$ and $\mathbf{T}=T(\hat{\mathbf{I}}+M)^{-1}$, we obtain (10.7).
10.3. Fredholm solvability of nonlocal problems. Since the subspace $S^{l+2 m}(G)$ has a finite dimension in $W^{l+2 m}(G)$, then there exists a finite-dimensional subspace $\mathcal{R}^{l}(G, \partial G)$ of the space $\mathcal{W}^{l}(G, \partial G)$ such that

$$
\{\mathbf{P} u, \mathbf{B} u\} \in \hat{\mathcal{S}}^{l}(G, \partial G)+\mathcal{R}^{l}(G, \partial G) \quad \forall u \in W^{l+2 m}(G) .
$$

Hence we can define a bounded operator

$$
\hat{\mathbf{L}}=\{\mathbf{P}, \mathbf{B}\}: W^{l+2 m}(G) \rightarrow \hat{\mathcal{S}}^{l}(G, \partial G) \dot{+} \mathcal{R}^{l}(G, \partial G) .
$$

Theorem 10.2. Let conditions 7.2 and 10.1 hold. Then $\hat{\mathbf{L}}$ is a Fredholm operator.
Proof. It follows from Lemmas 8.5 and 10.3 and [56, Theorem 15.2] that $\hat{\mathbf{L}}_{S}: S^{l+2 m}(G) \rightarrow \hat{\mathcal{S}}^{l}(G, \partial G)$ is a Fredholm operator. Since the domain $W^{l+2 m}(G)$ of the operator $\hat{\mathbf{L}}$ is an extension of the domain $S^{l+2 m}(G)$ of the operator $\hat{\mathbf{L}}_{S}$ to a finite-dimensional subspace and $\hat{\mathbf{L}}$ coincides with $\hat{\mathbf{L}}_{S}$ on $S^{l+2 m}(G)$, we see that $\hat{\mathbf{L}}$ is also a Fredholm operator.

## 11. Nonlocal Problems with Homogeneous Nonlocal Conditions

In this section, we study operators corresponding to problem (6.7), (6.8) with homogeneous boundary conditions. Using the results of Sec. 10, we show that if the line $\operatorname{Im} \lambda=1-l-2 m$ consists of only a regular eigenvalue, then this operator, unlike $\mathbf{L}$, is a Fredholm operator, if certain algebraic relations between the operators $\mathbf{P}, \mathbf{B}^{0}$, and $\mathbf{B}^{1}$ at points of the set $\mathcal{K}$ are fulfilled.
11.1. The absence of eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ on the line $\operatorname{Im} \lambda=1-l-2 m$ or the presence of a irregular eigenvalue. Let us introduce the space

$$
W_{B}^{l+2 m}(G)=\left\{u \in W^{l+2 m}(G): \mathbf{B} u=0\right\}
$$

Obviously, $W_{B}^{l+2 m}(G)$ is a closed subspace in $W^{l+2 m}(G)$. Consider a bounded operator $\mathbf{L}_{B}$ : $W_{B}^{l+2 m}(G) \rightarrow W^{l}(G)$ defined by the formula

$$
\mathbf{L}_{B} u=\mathbf{P} u, \quad u \in W_{B}^{l+2 m}(G) .
$$

To study problem (6.7), (6.8) with homogeneous nonlocal conditions, we need the following condition for the operators $B_{i \mu s}(y, D)$ (see e.g., [57, Chap. 2, Sec. 1]).

Condition 11.1. For all $i=1, \ldots, N$, the system of operators $\left\{B_{i \mu 0}(y, D)\right\}_{\mu=1}^{m}$ is normal on $\overline{\Gamma_{i}}$ and orders of the operators $B_{i \mu s}(y, D), s=0, \ldots, S_{i}$, do not exceed $2 m-1$.

In this subsection, we will prove the following result.
Theorem 11.1. Let condition 7.1 hold. Then $\mathbf{L}_{B}$ is a Fredholm operator.
Let the line $\operatorname{Im} \lambda=1-l-2 m$ contain an irregular eigenvalue $\lambda_{0}$ of the operator $\tilde{\mathcal{L}}(\lambda)$ and let condition 11.1 hold. Then the image of the operator $\mathbf{L}_{B}$ is not closed (and hence $\mathbf{L}_{B}$ is not a Fredholm operator).

The following lemma allows one to reduce nonlocal problems with inhomogeneous boundary conditions to problems with homogeneous boundary conditions.

Lemma 11.1. Let condition 11.1 hold. Then for functions

$$
f_{j \sigma \mu} \in H_{a}^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)
$$

such that $\operatorname{supp} f_{j \sigma \mu} \subset \mathcal{O}_{\varepsilon^{\prime}}(0)\left(\varepsilon^{\prime}>0\right.$ is fixed), there exists a function $V \in \mathcal{H}_{a}^{l+2 m}(K)$ such that $\operatorname{supp} V \subset \mathcal{O}_{2 \varepsilon^{\prime}}(0)$ and

$$
\begin{gather*}
\mathbf{B}_{j \sigma \mu}(y, D) V=f_{j \sigma \mu},  \tag{11.1}\\
\|V\|_{\mathcal{H}_{a}^{l+2 m}(K)} \leq c_{\varepsilon^{\prime}}\left\|\left\{f_{j \sigma \mu}\right\}\right\|_{\mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)} \tag{11.2}
\end{gather*}
$$

where $c_{\varepsilon^{\prime}}>0$ is independent of $f_{j \sigma \mu}$.
Proof. 1. Similarly to the proof of [58, Lemma 3.1], we construct functions $V_{j \sigma} \in H_{a}^{l+2 m}\left(K_{j}\right)$ such that

$$
\begin{gather*}
\left.B_{j \sigma \mu j 0}(y, D) V_{j \sigma}\right|_{\gamma_{j \sigma}}=f_{j \sigma \mu},  \tag{11.3}\\
\left\|V_{j \sigma}\right\|_{H_{a}^{l+2 m}\left(K_{j}\right)} \leq k_{2}\left\|\left\{f_{j \sigma \mu}\right\}\right\|_{\mathcal{H}_{a}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)} . \tag{11.4}
\end{gather*}
$$

Since supp $f_{j \sigma \mu} \subset \mathcal{O}_{\varepsilon^{\prime}}(0)$, we can assume that $\operatorname{supp} V_{j \sigma} \subset \mathcal{O}_{2 \varepsilon^{\prime}}(0)$.
2. Let us denote

$$
\delta=\min \left|(-1)^{\sigma} \omega_{j}+\omega_{j \sigma k s} \pm \omega_{k}\right| / 2, \quad j, k=1, \ldots, N, \quad \sigma=1,2, s=1, \ldots, S_{j \sigma k}
$$

and introduce functions $\zeta_{j \sigma} \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ such that $\zeta_{j \sigma}(\omega)=1$ for $\left|(-1)^{\sigma} \omega_{j}-\omega\right|<\delta / 2$ and $\zeta_{j \sigma}(\omega)=0$ for $\left|(-1)^{\sigma} \omega_{j}-\omega\right|>\delta$. Since the functions $\zeta_{j \sigma}$ are the multipliers in the space $H_{a}^{l+2 m}\left(K_{j}\right)$, it follows from (11.3) and (11.4) that the function $V=\left(\zeta_{11} V_{11}+\zeta_{12} V_{12}, \ldots, \zeta_{N 1} V_{N 1}+\zeta_{N 2} V_{N 2}\right)$ satisfies conditions (11.1) and (11.2).

Remark 11.1. Similar reasonings are not valid in Sobolev spaces since the functions $\zeta_{j \sigma}$ are not multipliers in $W^{l+2 m}\left(K_{j}\right)$. Moreover, we can construct functions $f_{j \sigma \mu}$ from the space $W^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)$ $(j=1, \ldots, N, \sigma=1,2, \mu=1, \ldots, m)$, for which there is no a function $V \in \mathcal{W}^{l+2 m}(K)$ satisfying conditions (11.1). Therefore, a problem with homogeneous nonlocal conditions is not equivalent to a problem with inhomogeneous boundary conditions (e.g., a situation is possible where the first problem is a Fredholm problem but the second is not).

As earlier, assume that $d=d(\varrho)=2 d_{2} \varrho$, where $d_{2}$ is defined in Eq. (6.15). To study the image of the operator $\mathbf{L}_{B}$, we use the following result (cf. Lemma 8.6).

Lemma 11.2. Let condition 11.1 hold and let the image of the operator $\mathbf{L}_{B}$ be closed. Then for sufficiently small $\varrho>0$ and all $U \in \mathcal{S}^{l+2 m}\left(K^{d}\right)$ such that

$$
\begin{equation*}
\left.\mathcal{B}_{j \sigma \mu}(D) U\right|_{\gamma_{j \sigma}^{2 e}}=0, \quad j=1, \ldots, N, \quad \sigma=1,2, \quad \mu=1, \ldots, m, \tag{11.5}
\end{equation*}
$$

the following estimate holds ${ }^{5}$ :

$$
\begin{equation*}
\|U\|_{\mathcal{W}^{l+2 m}\left(K^{e}\right)} \leq c\left(\left\|\left\{\mathcal{P}_{j}(D) U_{j}\right\}\right\|_{\mathcal{W}^{l}\left(K_{j}^{d}\right)}+\|U\|_{\mathcal{H}_{0}^{l+2 m-1}\left(K^{d}\right)}\right) . \tag{11.6}
\end{equation*}
$$

Proof. 1. Since the image of the operator $\mathbf{L}_{B}$ is closed, by Lemma 8.5, by the compactness of the embedding $W^{l+2 m}(G) \subset W^{l+2 m-1}(G)$, and [56, Theorem 7.1], we have

$$
\begin{equation*}
\|u\|_{W^{l+2 m}(G)} \leq c\left(\|\mathbf{P}(y, D) u\|_{W^{l}(G)}+\|u\|_{W^{l+2 m-1}(G)}\right) \tag{11.7}
\end{equation*}
$$

for all $u \in W_{B}^{l+2 m}(G)$. Let us substitute in (11.7) a function $u \in W_{B}^{l+2 m}(G)$ such that $\operatorname{supp} u \in$ $\mathcal{O}_{2 \varrho_{1}}(\mathcal{K}), 2 \varrho_{1}<\min \left\{\varepsilon_{0}, \varkappa_{1}\right\}$ into (11.7). By Eq. (6.5), for such functions we have $\mathbf{B}^{2} u=0$. Hence, using [57, Ch. 2, Lemma 3.2], we see that for sufficiently small $\varrho_{1}$, the estimate

$$
\begin{equation*}
\|U\|_{\mathcal{W}^{l+2 m}(K)} \leq k_{1}\left(\left\|\left\{\mathcal{P}_{j}(D) U_{j}\right\}\right\|_{\mathcal{W}^{l}(K)}+\|U\|_{\mathcal{W}^{l+2 m-1}(K)}\right) \tag{11.8}
\end{equation*}
$$

holds for all $U \in \mathcal{W}^{l+2 m}(K)$ such that $\operatorname{supp} U \subset \mathcal{O}_{2 \varrho_{1}}(0)$ and

$$
\begin{equation*}
\mathbf{B}_{j \sigma \mu}(y, D) U=0, \quad j=1, \ldots, N, \quad \sigma=1,2, \quad \mu=1, \ldots, m \tag{11.9}
\end{equation*}
$$

2. Let us show that if $\varrho_{2}<\varrho_{1} d_{1}$ is sufficiently small, then estimate (11.8) is valid for all $U \in$ $\mathcal{S}^{l+2 m}(K)$ such that $\operatorname{supp} U \subset \mathcal{O}_{2 \varrho 2}(0)$ and

$$
\begin{equation*}
\mathcal{B}_{j \sigma \mu}(D) U=0, \quad j=1, \ldots, N, \quad \sigma=1,2, \quad \mu=1, \ldots, m \tag{11.10}
\end{equation*}
$$

Assume that $\Phi_{j \sigma \mu}=\left.\mathbf{B}_{j \sigma \mu}(y, D) U\right|_{\gamma_{j \sigma}}$. Obviously,

$$
\begin{equation*}
\operatorname{supp} \Phi_{j \sigma \mu} \subset \mathcal{O}_{\varrho_{2} / d_{1}}(0) \subset \mathcal{O}_{\varrho_{1}}(0) \tag{11.11}
\end{equation*}
$$

We fix $a, 0<a<1$, and prove that

$$
\begin{equation*}
\left\|\left\{\Phi_{j \sigma \mu}\right\}\right\|_{\mathcal{H}_{0}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)} \leq k_{2} \varrho_{2}^{1-a}\|U\|_{\mathcal{W}^{l+2 m}(K)} \tag{11.12}
\end{equation*}
$$

By Eq. (11.10) and the compactness of the trace operator in weight spaces, it suffices to estimate summands of the form

$$
\left(a_{\alpha}(y)-a_{\alpha}(0)\right) D^{\alpha} U_{j}, \quad|\alpha|=m_{j \sigma \mu}, \quad a_{\beta}(y) D^{\beta} U_{j}, \quad|\beta| \leq m_{j \sigma \mu}-1
$$

where $a_{\alpha}$ and $a_{\beta}$ are infinitely differentiable functions. Using the restriction to the support of the functions $U_{j}$ and Lemmas 5.5 and 5.2 , we obtain that

$$
\begin{aligned}
& \left\|\left(a_{\alpha}(y)-a_{\alpha}(0)\right) D^{\alpha} U_{j}\right\|_{H_{0}^{l+2 m-m_{j \sigma \mu}}{ }_{\left(K_{j}\right)}} \\
& \qquad k_{3} \varrho_{2}^{1-a}\left\|\left(a_{\alpha}(y)-a_{\alpha}(0)\right) D^{\alpha} U_{j}\right\|_{H_{a-1}^{l+2 m-m_{j \sigma \mu}}\left(K_{j}\right)} \\
& \quad \leq k_{4} \varrho_{2}^{1-a}\left\|D^{\alpha} U_{j}\right\|_{H_{a}^{l+2 m-m_{j \sigma \mu}}\left(K_{j}\right)} \leq k_{5} \varrho_{2}^{1-a}\left\|U_{j}\right\|_{W^{l+2 m}\left(K_{j}\right)} .
\end{aligned}
$$

Similarly, using Lemma 5.2, we have

$$
\left\|a_{\beta}(y) D^{\beta} U_{j}\right\|_{H_{0}^{l+2 m-m_{j \sigma \mu}}\left(K_{j}\right)} \leq k_{6} \varrho_{2}^{1-a}\left\|U_{j}\right\|_{H_{a-1}^{l+2 m-1}\left(K_{j}\right)} \leq k_{7} \varrho_{2}^{1-a}\left\|U_{j}\right\|_{W^{l+2 m}\left(K_{j}\right)}
$$

Thus, estimate (11.12) is proved.
Now, by virtue of Eq. (11.11) and Lemma 11.1, there exists a function

$$
V=\left(V_{1}, \ldots, V_{N}\right) \in \mathcal{H}_{0}^{l+2 m}(K)
$$

such that $\operatorname{supp} V \subset \mathcal{O}_{2 \varrho_{1}}(0)$ and

$$
\begin{gather*}
\left.\mathbf{B}_{j \sigma \mu}(y, D) V\right|_{\gamma_{j \sigma}}=\Phi_{j \sigma \mu}  \tag{11.13}\\
\|V\|_{\mathcal{H}_{0}^{l+2 m}(K)} \leq c_{\varrho_{1}}\left\|\left\{\Phi_{j \sigma \mu}\right\}\right\|_{\mathcal{H}_{0}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)} \tag{11.14}
\end{gather*}
$$

where $c_{\varrho_{1}}$ is independent of $\varrho_{2}$.

[^4]Using Eq. (11.8) to estimate $U-V$ and inequalities (11.14) and (11.12), we have

$$
\begin{aligned}
\|U\|_{\mathcal{W}^{l+2 m}(K)} \leq\|U-V\|_{\mathcal{W}^{l+2 m}(K)} & +\|V\|_{\mathcal{W}^{l+2 m}(K)} \\
\leq & k_{8}\left(\left\|\left\{\mathcal{P}_{j}(D) U_{j}\right\}\right\|_{\mathcal{W}^{l}(K)}+\|U\|_{\mathcal{W}^{l+2 m-1}(K)}+\varrho_{2}^{1-a}\|U\|_{\mathcal{W}^{l+2 m}(K)}\right)
\end{aligned}
$$

Now, choosing sufficiently small $\varrho_{2}$, we obtain the estimate (11.8), which is valid for all $U \in \mathcal{W}^{l+2 m}(K)$ such that $\operatorname{supp} U \subset \mathcal{O}_{2 \varrho_{2}}(0)$, and relations (7.30) and (11.10) hold.
3. Now we omit the restriction $\operatorname{supp} U \subset \mathcal{O}_{2 \varrho_{2}}(0)$ and prove the estimate (11.6) for $\varrho<\varrho_{2} d_{1}$ and any $U \in \mathcal{W}^{l+2 m}\left(K^{d}\right)$ satisfying Eqs. (7.30) and (11.5).

Introduce a function $\psi \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ such that $\psi(y)=1$ for $|y| \leq \varrho, \operatorname{supp} \psi \subset \mathcal{O}_{2 \varrho}(0)$ and $\psi$ is independent of the polar angle $\omega$.

Assume that $\Psi_{j \sigma \mu}=\left.\mathcal{B}_{j \sigma \mu}(D)(\psi U)\right|_{\gamma_{j \sigma}}$. Obviously,

$$
\begin{equation*}
\operatorname{supp} \Psi_{j \sigma \mu} \subset \mathcal{O}_{\varrho / d_{1}}(0) \subset \mathcal{O}_{\varrho_{2}}(0) \tag{11.15}
\end{equation*}
$$

Let us show that

$$
\begin{equation*}
\left\|\Psi_{j \sigma \mu}\right\|_{H_{0}^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)} \leq k_{9} \sum_{k=1}^{N}\left(\left\|\mathcal{P}_{k}(D) U_{k}\right\|_{W^{l}\left(K_{k}^{d}\right)}+\left\|U_{k}\right\|_{H_{0}^{l+2 m-1}\left(K_{j}^{d}\right)}\right) . \tag{11.16}
\end{equation*}
$$

Taking into account Eq. (11.5), we represent the function $\Psi_{j \sigma \mu}$ in the form

$$
\begin{equation*}
\Psi_{j \sigma \mu}=\sum_{k, s} \Psi_{j \sigma \mu k s}+\sum_{(k, s) \neq(j, 0)} J_{j \sigma \mu k s}, \tag{11.17}
\end{equation*}
$$

where

$$
\begin{aligned}
\Psi_{j \sigma \mu k s} & =\left.\left(\left[B_{j \sigma \mu k s}(D), \psi\right] U_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}} \\
J_{j \sigma \mu k s} & =\left.\left(\psi\left(\mathcal{G}_{j \sigma k s} y\right)-\psi(y)\right)\left(B_{j \sigma \mu k s}(D) U_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}},
\end{aligned}
$$

where $[\cdot, \cdot]$ denotes the commutator.
Since the expression for $\Psi_{j \sigma \mu k s}$ contains derivatives of the functions $U_{k}$ of order not greater than $m_{j \sigma \mu}-1$, we have that

$$
\begin{equation*}
\left\|\Psi_{j \sigma \mu k s}\right\|_{H_{0}^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)} \leq k_{10}\|U\|_{\mathcal{H}_{0}^{l+2 m-1}\left(K^{d}\right)} . \tag{11.18}
\end{equation*}
$$

Now we repeat the reasonings of item 1 of the proof of Lemma 8.7 and obtain the inequality

$$
\begin{align*}
& \left\|J_{j \sigma \mu k s}\right\|_{H_{0}^{l+2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}\right)} \\
& \quad \leq k_{11}\left(\left\|\mathcal{P}_{k}(D) U_{k}\right\|_{W^{l}\left(\left\{d_{1} \varrho / 2<|y|<2 d_{2} \varrho\right\}\right)}+\left\|U_{k}\right\|_{W^{l+2 m-1}\left(\left\{d_{1} \varrho / 2<|y|<2 d_{2} \varrho\right\}\right)}\right) . \tag{11.19}
\end{align*}
$$

Equation (11.16) follows from Eqs. (11.17), (11.18). and (11.19).
4. By virtue of Eq. (11.15) and Lemma 11.1 (applied to the operators $\mathcal{B}_{j \sigma \mu}(D)$ ), there exists a function $V=\left(V_{1}, \ldots, V_{N}\right) \in \mathcal{H}_{0}^{l+2 m}(K)$ such that $\operatorname{supp} V \subset \mathcal{O}_{2 \varrho_{2}}(0)$ and

$$
\begin{gather*}
\mathcal{B}_{j \sigma \mu}(D) V=\Psi_{j \sigma \mu},  \tag{11.20}\\
\|V\|_{\mathcal{H}_{0}^{l+2 m}(K)} \leq k_{12}\left\|\left\{\Psi_{j \sigma \mu}\right\}\right\|_{\mathcal{H}_{0}^{l+2 m-\mathbf{m}-1 / 2}(\gamma)} . \tag{11.21}
\end{gather*}
$$

Using (11.8) to estimate $\psi U-V$, the Leibnitz inequality, and inequalities (11.21) and (11.16), we obtain

$$
\begin{aligned}
&\|U\|_{\mathcal{W}^{l+2 m}\left(K^{e}\right)} \leq\|\psi U\|_{\mathcal{W}^{l+2 m}(K)} \leq\|\psi U-V\|_{\mathcal{W}^{l+2 m}(K)}+\|V\|_{\mathcal{W}^{l+2 m}(K)} \\
& \leq k_{11}\left(\left\|\left\{\mathcal{P}_{j}(D) U_{j}\right\}\right\|_{\mathcal{W}^{l}\left(K^{d}\right)}+\|U\|_{\mathcal{H}_{0}^{l+2 m-1}\left(K^{d}\right)}\right)
\end{aligned}
$$

The lemma is proved.

Lemma 11.2 allows one to prove that if the $\operatorname{line} \operatorname{Im} \lambda=1-l-2 m$ contains an irregular eigenvalue, then $\mathbf{L}_{B}$, as $\mathbf{L}$, is not a Fredholm operator.
Lemma 11.3. Let the line $\operatorname{Im} \lambda=1-l-2 m$ contain an irregular eigenvalue $\lambda_{0}$ of the operator $\tilde{\mathcal{L}}(\lambda)$ and let condition 11.1 hold. Then the image of the operator $\mathbf{L}_{B}$ is not closed.

Proof. 1. Assume that the image of the operator $\mathbf{L}_{B}$ is closed. Denote an eigenvector and adjoint vectors corresponding to the eigenvalue $\lambda_{0}$ (see [23]) by $\varphi^{(0)}(\omega), \ldots, \varphi^{(\varkappa-1)}(\omega)$. The vectors $\varphi^{(k)}(\omega)$ belong to $\mathcal{W}^{l+2 m}(-\bar{\omega}, \bar{\omega})$ and satisfy the relations

$$
\begin{equation*}
\mathcal{P}_{j}(D) V_{j}^{k}=0, \quad \mathcal{B}_{j \sigma \mu}(D) V^{k}=0 \tag{11.22}
\end{equation*}
$$

where

$$
V^{k}=r^{i \lambda_{0}} \sum_{s=0}^{k} \frac{1}{s!}(i \ln r)^{k} \varphi^{(k-s)}(\omega), \quad k=0, \ldots, \varkappa-1 .
$$

Since $\lambda_{0}$ in an irregular eigenvalue, we see that the function $V^{k}(y)$ is not a vector-valued polynomial for some $k \geq 0$. For simplicity, we assume that $V^{0}=r^{i \lambda_{0}} \varphi^{(0)}(\omega)$ is not a vector-valued polynomial (the case where $k>0$ is similar).

Let $\varrho$ and $d=d(\varrho)$ be the same constants as in Lemma 11.2. Let us consider the sequence

$$
U^{\delta}=r^{\delta} V^{0} /\left\|r^{\delta} V^{0}\right\|_{\mathcal{W}^{l+2 m}\left(K^{e}\right)} .
$$

The denominator of the fraction is finite for any $\delta>0$, but

$$
\left\|r^{\delta} V^{0}\right\|_{\mathcal{W}^{l+2 m}\left(K^{e}\right)} \rightarrow \infty \quad \text { as } \quad \delta \rightarrow 0
$$

since $V^{0}$ is not a vector-valued polynomial. Nevertheless,

$$
\left\|r^{\delta} V^{0}\right\|_{\mathcal{H}_{0}^{l+2 m-1}\left(K^{d}\right)} \leq c
$$

where $c>0$ is independent of $\delta \geq 0$; hence,

$$
\begin{equation*}
\left\|U^{\delta}\right\|_{\mathcal{H}_{0}^{l+2 m-1}\left(K^{d}\right)} \rightarrow 0 \quad \text { as } \quad \delta \rightarrow 0 . \tag{11.23}
\end{equation*}
$$

Using (11.22), we can verify (similarly to the proof of Lemma 8.7) that

$$
\begin{gather*}
\left\|\left\{\mathcal{P}_{j}(D) U_{j}^{\delta}\right\}\right\|_{\mathcal{W}^{l}\left(K^{d}\right)} \rightarrow 0 \quad \text { as } \quad \delta \rightarrow 0,  \tag{11.24}\\
\left\|\left\{\mathcal{B}_{j \sigma \mu}(D) U^{\delta}\right\}\right\|_{\mathcal{H}_{0}^{l+2 m-\mathbf{m}-1 / 2}\left(\gamma^{3 \varrho}\right)} \rightarrow 0 \quad \text { as } \quad \delta \rightarrow 0 . \tag{11.25}
\end{gather*}
$$

2. Introduce a functions $\psi \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ such that $\psi(y)=1$ for $y \in \mathcal{O}_{2 \varrho}(0)$ and $\operatorname{supp} \psi \subset \mathcal{O}_{3 \varrho}(0)$.

Applying Lemma 11.1 to the operators $\mathcal{B}_{j \sigma \mu}(D)$ and the functions $f_{j \sigma \mu}=\left.\psi \mathcal{B}_{j \sigma \mu}(D) U^{\delta}\right|_{\gamma_{j \sigma}}$ (note that $\left.\operatorname{supp} f_{j \sigma \mu} \subset \mathcal{O}_{3 \varrho}(0)\right)$, we construct a function $W^{\delta} \in \mathcal{H}_{0}^{l+2 m}(K)(\delta>0)$ such that supp $W^{\delta} \subset \mathcal{O}_{6 \varrho}(0)$ and

$$
\begin{gather*}
\left.\mathcal{B}_{j \sigma \mu}(D) W^{\delta}\right|_{\gamma_{j \sigma}^{2 \varrho}}=\left.\mathcal{B}_{j \sigma \mu}(D) U^{\delta}\right|_{\gamma_{j \sigma}^{2 \rho}},  \tag{11.26}\\
\left\|W^{\delta}\right\|_{\mathcal{H}_{0}^{l+2 m}\left(K^{6 \varrho}\right)} \leq k_{1} \sum_{j, \sigma, \mu}\left\|\left\{\mathcal{B}_{j \sigma \mu}(D) U^{\delta}\right\}\right\|_{\mathcal{H}_{0}^{l+2 m-\mathrm{m}-1 / 2}\left(\gamma^{3 \varrho}\right)} . \tag{11.27}
\end{gather*}
$$

Moreover, the function $U^{\delta}-W^{\delta}$ satisfies relations (7.30); hence, we can apply Lemma 11.2 to the function $U^{\delta}-W^{\delta}$. Then estimate (11.6), the boundedness of the embedding operator $H_{0}^{l+2 m}\left(K_{j}^{6 \varrho}\right) \subset$ $W^{l+2 m}\left(K_{j}^{6 \varrho}\right)$, and inequality (11.27) imply the following inequality:

$$
\begin{align*}
& \left\|U^{\delta}\right\|_{\mathcal{W}^{l+2 m}\left(K^{\varrho}\right)} \leq\left\|U^{\delta}-W^{\delta}\right\|_{\mathcal{W}^{l+2 m}\left(K^{g}\right)}+\left\|W^{\delta}\right\|_{\mathcal{W}^{l+2 m}\left(K^{\varrho}\right)} \\
& \quad \leq k_{2}\left(\left\|\left\{\mathcal{P}_{j}(D) U_{j}^{\delta}\right\}\right\|_{\mathcal{W}^{l}\left(K^{d}\right)}+\left\|\left\{\mathcal{B}_{j \sigma \mu}(D) U^{\delta}\right\}\right\|_{\mathcal{H}_{0}^{l+2 m-\mathrm{m}-1 / 2}\left(\gamma^{3 \varrho}\right)}+\left\|U^{\delta}\right\|_{\mathcal{H}_{0}^{l+2 m-1}\left(K^{d}\right)}\right) . \tag{11.28}
\end{align*}
$$

Nevertheless, relations (11.23)-(11.25) contradict estimate (11.28) since $\left\|U^{\delta}\right\|_{\mathcal{W}^{l+2 m}\left(K^{g}\right)}=1$.

Proof of Theorem 11.1. The first item of Theorem 11.1 follows from Theorem 8.1; the second item follows from Lemma 11.3.
11.2. A regular eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$ on the line $\operatorname{Im} \lambda=1-l-2 m$. It remains to consider the case where the line $\operatorname{Im} \lambda=1-l-2 m$ contains only a regular eigenvalue. Let condition 7.2 hold. Prove that in this case, $\mathbf{L}_{B}$ is a Fredholm operator for fixed $l \geq 1$ if the following condition holds.

Condition 11.2. If $l \geq 1$, system (7.25) contains all the operators $D^{\xi} \mathcal{P}_{j}(D),|\xi|=l-1, j=1, \ldots, N$.
Theorem 11.2. Let conditions 7.2 and 10.1 hold. Then
(1) $\mathbf{L}_{B}: W_{B}^{2 m}(G) \rightarrow L_{2}(G)$ is a Fredholm operator;
(2) if $l \geq 1$ and condition 11.2 holds, then $\mathbf{L}_{B}: W_{B}^{l+2 m}(G) \rightarrow W^{l}(G)$ is a Fredholm operator;
(3) if $l \geq 1$ and condition 11.2 is violated but condition 11.1 holds, then the image of the operator $\mathbf{L}_{B}: W_{B}^{l+2 m}(G) \rightarrow W^{l}(G)$ is not closed (and hence the operator $\mathbf{L}_{B}$ is not a Fredholm operator).
Proof. 1. By Lemma 8.5, the kernel of the operator $\mathbf{L}_{B}$ is finite-dimensional. Let us study the image $\mathcal{R}\left(\mathbf{L}_{B}\right)$ of the operator $\mathbf{L}_{B}$.
2. First, we assume that $l \geq 1$ and condition 11.2 holds. We show that the set

$$
\begin{equation*}
\left\{f_{0} \in W^{l}(G):\left\{f_{0}, 0\right\} \in \hat{\mathcal{S}}^{l}(G, \partial G)\right\} \tag{11.29}
\end{equation*}
$$

is a closed finite-dimensional subspace in $W^{l}(G)$. Indeed, let $\hat{\psi}$ be a function from the definition of the space $\hat{\mathcal{S}}^{l}(G, \partial G)$ (see Sec. 10.1). Then the vector $\left\{f_{j}, 0\right\}$ of right-hand sides of problem (6.12), (6.13) corresponds to the vector of right-hand sides $\left\{\hat{\psi} f_{0}, 0\right\}$ of problem (6.7), (6.8). Obviously, $\mathcal{T}_{j \sigma \mu}\left\{f_{j}, 0\right\}=0$. Moreover, by virtue of condition 11.2, relations (7.27) are absent. Thus, by virtue of (10.1), the norm of the function $\left\{f_{0}, 0\right\} \in \hat{\mathcal{S}}^{l}(G, \partial G)$ in $\hat{\mathcal{S}}^{l}(G, \partial G)$ is equivalent to the function $f_{0}$ in $W^{l}(G)$ and set (11.29) is a subset in $W^{l}(G)$ consisting of functions satisfying condition (5.4). In other words, set (11.29) coincides with the space $S^{l}(G)$.

Now, since

$$
\hat{\mathcal{S}}^{l}(G, \partial G) \subset \hat{\mathcal{S}}^{l}(G, \partial G) \dot{\mathcal{R}^{l}}(G, \partial G),
$$

the set

$$
\begin{equation*}
\left\{f_{0} \in W^{l}(G):\left\{f_{0}, 0\right\} \in \hat{\mathcal{S}}^{l}(G, \partial G) \dot{+} \mathcal{R}^{l}(G, \partial G)\right\} \tag{11.30}
\end{equation*}
$$

(containing set (11.29)) is also a closed subspace with finite codimension in $W^{l}(G)$. On the other hand, $f_{0} \in \mathcal{R}\left(\mathbf{L}_{B}\right)$ if and only if $\left\{f_{0}, 0\right\} \in \mathcal{R}(\hat{\mathbf{L}})$, where $\hat{\mathbf{L}}$ is operator defined in Sec. 10.3. This and the fact that $\hat{\mathbf{L}}$ is a Fredholm operator imply that the image of the operator $\mathbf{L}_{B}$ is closed and has finite dimension.
3. Now assume that $l \geq 1$ and condition 11.2 is violated. Let us prove (using the results of Sec. 7.3, Chap. 2) that the image $\mathcal{R}\left(\mathbf{L}_{B}\right)$ of the operator $\mathbf{L}_{B}$ is not closed. Assume the contrary: let the image $\mathcal{R}\left(\mathbf{L}_{B}\right)$ be closed.

Since condition 11.2 is violated, we see that the set (7.27) is not empty; this means that for some $j$ and $\xi$, norm (7.28) contains the corresponding term $\left\|\mathcal{T}_{j \xi} f\right\|_{H_{0}^{1}\left(\mathbb{R}^{2}\right)}$. Similarly to the proof of Lemma 7.3, this implies that there exists a sequence $f^{\delta}=\left\{f_{j}^{\delta}, 0\right\} \in \hat{\mathcal{S}}^{l}(K, \gamma), \delta>0$, such that supp $f^{\delta} \subset \mathcal{O}_{\varepsilon}(0)$ and $f^{\delta}$ converges in $\mathcal{W}^{l}(K, \gamma)$ to $f^{0} \notin \hat{\mathcal{S}}^{l}(K, \gamma)$ for $\delta \rightarrow 0$.

By virtue of Lemma 7.6, for any function $f^{\delta}$ we can find a function $U^{\delta} \in \mathcal{W}^{l+2 m}\left(K^{d}\right)$ such that

$$
\begin{gather*}
\mathcal{P}_{j}(D) U_{j}^{\delta}=f_{j}^{\delta}, \quad \mathcal{B}_{j \sigma \mu}(D) U^{\delta}=0  \tag{11.31}\\
\left\|U^{\delta}\right\|_{\mathcal{H}_{0}^{l+2 m-1}\left(K^{d}\right)} \leq c\left\|f^{\delta}\right\|_{\mathcal{W}^{l}(K, \gamma)} \tag{11.32}
\end{gather*}
$$

( $c>0$ is independent of $\delta$ ) and $U^{\delta}$ corresponds to relations (7.30). By virtue of the second relation in (11.31) and relations (7.30), we can apply Lemma 11.2 to the function $U^{\delta}$. Using estimate (11.6),
the convergence of the sequence $f^{\delta}$ to $f^{0} \notin \hat{\mathcal{S}}^{l}(K, \gamma)$, and inequality (11.32), we obtain a contradiction (cf. the proof of Lemma 8.7).
4. In the case where $l=0$, the set of conditions (7.27) is empty since these conditions arise only if $l \geq 1$. Similarly to item 2 of the proof, we obtain the conclusion of the theorem.

## 12. Examples

We present two examples that illustrate the results of Chaps. 2 and 3 (detailed proofs can be found in [33]). In these examples, the set $\mathcal{K}$ consists of several orbits and hence we must use obvious generalizations of the theorems from previous sections to this case.

### 12.1. Example 1.

12.1.1. Problem with homogeneous nonlocal conditions. Let $\partial G \backslash \mathcal{K}=\bigcup_{i=1}^{2} \Gamma_{i}$, where $\Gamma_{i}$ are open in the topology of the boundary curves of class $C^{\infty}, \mathcal{K}=\overline{\Gamma_{1}} \cap \overline{\Gamma_{2}}=\left\{g_{1}, g_{2}\right\}, g_{1}$ and $g_{2}$ are endpoints of the curves $\overline{\Gamma_{1}}$ and $\overline{\Gamma_{2}}$. Assume that the domain $G$ coincides with a plane angle of spread $\pi$ in neighborhoods of the points $g_{1}$ and $g_{2}$.

Consider the nonlocal problem

$$
\begin{align*}
\Delta u & =f_{0}(y), \quad y \in G  \tag{12.1}\\
\left.u\right|_{\Gamma_{i}}+\left.b_{i} u\left(\Omega_{i}(y)\right)\right|_{\Gamma_{i}} & =f_{i}(y) \quad y \in \Gamma_{i}, \quad i=1,2, \tag{12.2}
\end{align*}
$$

where $b_{1}, b_{2} \in \mathbb{R}, \Omega_{i}$ is an infinitely smooth transformation mapping a neighborhood $\mathcal{O}_{i}$ of the curve $\Gamma_{i}$ onto $\Omega\left(\mathcal{O}_{i}\right)$ such that $\Omega\left(\Gamma_{i}\right) \subset G, \Omega_{i}\left(g_{j}\right)=g_{j}, j=1,2$, and the transformation $\Omega_{i}$ is a rotation by the angle $\pi / 2$ inwards the domain $G$ near the points $g_{1}$ and $g_{2}$ (see Fig. 12.1).


Fig. 12.1. The domain $G$ with the boundary $\partial G=\overline{\Gamma_{1}} \cup \overline{\Gamma_{2}}$.
Consider the operator $\mathbf{L}: W^{l+2}(G) \rightarrow \mathcal{W}^{l}(G, \partial G)$ acting by the formula

$$
\mathbf{L} u=\left(\Delta u,\left.u\right|_{\Gamma_{1}}+b_{1} u,\left.u\right|_{\Gamma_{2}}+b_{2} u\right)
$$

and corresponding to problem (12.1), (12.2). Using Theorem 8.1, we obtain the following result.
Let $l$ be even. Then $\mathbf{L}: W^{l+2}(G) \rightarrow \mathcal{W}^{l}(G, \partial G)$ is a Fredholm operator if and only if $b_{1}+b_{2} \neq 0$.
Let $l$ be odd. Then $\mathbf{L}: W^{l+2}(G) \rightarrow \mathcal{W}^{l}(G, \partial G)$ is not a Fredholm operator for any $b_{1}, b_{2} \in \mathbb{R}$.
12.1.2. Problem with homogeneous nonlocal conditions. Let us denote

$$
W_{B}^{l+2}(G)=\left\{u \in W^{l+2}(G):\left.u\right|_{\Gamma_{i}}+\left.b_{i} u\left(\Omega_{i}(y)\right)\right|_{\Gamma_{i}}=0, i=1,2\right\}
$$

and introduce the operator $\mathbf{L}_{B}: W_{B}^{l+2}(G) \rightarrow W^{l}(G)$ by the formula

$$
\mathbf{L}_{B} u=\Delta u, \quad u \in W_{B}^{l+2}(G) .
$$

Using Theorems 8.1 and 11.2, we obtain the following result.
Let $l$ be even. Then $\mathbf{L}_{B}: W_{B}^{l+2}(G) \rightarrow W^{l}(G)$ is a Fredholm operator for any $b_{1}, b_{2} \in \mathbb{R}$.
Let $l$ be odd and $l=4 k+1, k=0,1,2, \ldots$ Then $\mathbf{L}_{B}: W_{B}^{l+2}(G) \rightarrow W^{l}(G)$ is a Fredholm operator if and only if $b_{1}=b_{2}<1$.

Let $l$ be odd and $l=4 k+3, k=0,1,2, \ldots$. Then $\mathbf{L}_{B}: W_{B}^{l+2}(G) \rightarrow W^{l}(G)$ is a Fredholm operator if and only if $b_{1}=b_{2}>-1$.

### 12.2. Example 2.

12.2.1. Problem with inhomogeneous nonlocal conditions. Let the boundary $\partial G \in C^{\infty}$ of the domain $G$ coincide with the boundary of the square $(0,4 / 3) \times(0,4 / 3)$ outside circles $\mathcal{O}_{1 / 8}((4 i / 3,4 j / 3)), i, j=$ 0,1 . Introduce the notation $\Gamma_{1}=\left\{y \in \partial G: y_{1}<1 / 3, y_{2}<1 / 3\right\}, \Gamma_{2}=\left\{y \in \partial G: y_{1}>1, y_{2}>1\right\}$. Let $\Gamma_{3}$ and $\Gamma_{4}$ be the connected components of the set $\partial G \backslash\left(\overline{\Gamma_{1}} \cup \overline{\Gamma_{2}}\right)$. In this case, we have $\mathcal{K}=\left\{g_{1}, \ldots, g_{4}\right\}$, where $g_{1}=(1 / 3,0), g_{2}=(0,1 / 3), g_{3}=(4 / 3,1)$, and $g_{4}=(1,4 / 3)$ (see Fig. 12.2).


Fig. 12.2. The domain $G$ with the smooth boundary $\partial G=\bigcup_{i=1}^{4} \overline{\Gamma_{i}}$.
Let us consider the nonlocal problem

$$
\begin{align*}
& \Delta u=f_{0}(y), \quad y \in G,  \tag{12.3}\\
& \left.u(y)\right|_{\Gamma_{i}}+\left.b_{i} u\left(y+h_{i}\right)\right|_{\Gamma_{i}}=f_{i}(y), \quad y \in \Gamma_{i}, \quad i=1,2, \\
& \left.u(y)\right|_{\Gamma_{j}}=f_{j}(y), \quad y \in \Gamma_{j}, \quad j=3,4, \tag{12.4}
\end{align*}
$$

where $h_{1}=(1,1), h_{2}=(-1,-1)$, and $b_{1}, b_{2} \in \mathbb{R}$. Obviously, the set $\mathcal{K}$ consists of two orbits Orb ${ }_{1}$ and $\mathrm{Orb}_{2}$, where the orbit $\mathrm{Orb}_{1}$ contains the points $g_{1}$ and $g_{3}=g_{1}+h_{1}$ and the orbit Orb ${ }_{2}$ contains the points $g_{2}$ and $g_{4}=g_{2}+h_{2}$.

Consider the operator $\mathbf{L}: W^{l+2}(G) \rightarrow \mathcal{W}^{l}(G, \partial G)$ acting by the formula

$$
\mathbf{L} u=\left(\Delta u,\left.u(y)\right|_{\Gamma_{1}}+\left.b_{1} u\left(y+h_{1}\right)\right|_{\Gamma_{1}},\left.u(y)\right|_{\Gamma_{2}}+\left.b_{i} u\left(y+h_{2}\right)\right|_{\Gamma_{2}},\left.u(y)\right|_{\Gamma_{3}},\left.u(y)\right|_{\Gamma_{4}}\right)
$$

and corresponding to problem (12.3), (12.4). Using Theorems 8.1, we obtain the following result.
Let l be even. Then $\mathbf{L}: W^{l+2}(G) \rightarrow \mathcal{W}^{l}(G, \partial G)$ is a Fredholm operator if and only if $b_{1} b_{2}>0$.
Let $l$ be odd. Then $\mathbf{L}: W^{l+2}(G) \rightarrow \mathcal{W}^{l}(G, \partial G)$ is not a Fredholm operator for any $b_{1}, b_{2} \in \mathbb{R}$.
12.2.2. Problem with homogeneous nonlocal conditions. Let us denote

$$
W_{B}^{l+2}(G)=\left\{u \in W^{l+2}(G):\left.u\right|_{\Gamma_{i}}+\left.b_{i} u\left(y+h_{i}\right)\right|_{\Gamma_{i}}=0, i=1,2 ;\left.u\right|_{\Gamma_{j}}=0, j=3,4\right\}
$$

and introduce the operator $\mathbf{L}_{B}: W_{B}^{l+2}(G) \rightarrow W^{l}(G)$ by the formula

$$
\mathbf{L}_{B} u=\Delta u, \quad u \in W_{B}^{l+2}(G) .
$$

Applying Theorems 8.1 and 11.2, we obtain the following result.
Let l be even. Then $\mathbf{L}_{B}: W_{B}^{l+2}(G) \rightarrow W^{l}(G)$ is a Fredholm operator if and only if either $b_{1} b_{2}>0$, or $b_{1}=b_{2}=0$.

Let $l$ be odd. Then $\mathbf{L}_{B}: W_{B}^{l+2}(G) \rightarrow W^{l}(G)$ is a Fredholm operator if and only if $b_{1}=b_{2}=0$.

## Chapter 4

## GENERALIZED SOLUTIONS OF NONLOCAL ELLIPTIC PROBLEMS

## 13. Generalized Solutions of Nonlocal Problems

13.1. Generalized solutions. As in previous chapters, we assume that conditions 6.1-6.4 hold (with $l=0$ in the last condition). We also assume that the orders $m_{i \mu}$ of the operators $B_{i \mu s}(y, D)$ satisfy the inequalities

$$
m_{i \mu} \leq 2 m-1
$$

Here and in the next chapter, we will study so-called generalized solutions (see Definitions 13.2 and 13.3) of nonlocal boundary-value problem (6.7), (6.8):

$$
\begin{align*}
\mathbf{P}(y, D) u & =f_{0}(y), \quad y \in G,  \tag{13.1}\\
\mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u & =f_{i \mu}(y) \quad y \in \Gamma_{i}, \quad i=1, \ldots, N, \quad \mu=1, \ldots, m . \tag{13.2}
\end{align*}
$$

Let us introduce the notion of the generalized solution of problem (13.1), (13.2). First, like in [72], we state the corresponding definition for a "local" problem, i.e., in the case where $\mathbf{B}_{i \mu}^{1}=0$ and $\mathbf{B}_{i \mu}^{2}=0$.

Further, we assume that the number $\ell$ is fixed such that

$$
0 \leq \ell \leq 2 m-1
$$

Since $C^{\infty}\left(\overline{\Gamma_{i}}\right) \subset H_{2 m}^{2 m-k+1 / 2}\left(\Gamma_{i}\right)$, by virtue of Lemma 5.8, we have $C^{\infty}\left(\overline{\Gamma_{i}}\right) \subset H_{\ell}^{\ell-k+1 / 2}, k=1, \ldots, 2 m$. Hence the norm

$$
\begin{equation*}
\|\mathbf{u}\|_{\mathbf{W}^{\ell}(G)}=\left(\|\mathbf{u}\|_{W^{\ell}(G)}^{2}+\sum_{i=1}^{N} \sum_{k=1}^{2 m}\left\|D_{\nu_{i}}^{k-1} \mathbf{u}\right\|_{H_{\ell}^{\ell-k+1 / 2}\left(\Gamma_{i}\right)}^{2}\right)^{1 / 2} \tag{13.3}
\end{equation*}
$$

is finite for any function $\mathbf{u} \in C^{\infty}(\bar{G})$, where $\nu_{i}$ is an outward normal to the part $\Gamma_{i}$ of the boundary and

$$
D_{\nu_{i}}^{k-1} \mathbf{u}=\left.(-i)^{k-1} \frac{\partial^{k-1} \mathbf{u}}{\partial \nu_{i}^{k-1}}\right|_{\Gamma_{i}}
$$

Denote by $\mathbf{W}^{\ell}(G)$ the completion of the set $C^{\infty}(\bar{G})$ with respect to the norm ${ }^{6}$ (13.3).
It follows from (13.3) that the closure $\mathbf{S}$ of the operator

$$
\mathbf{u} \mapsto\left\{\left.\mathbf{u}\right|_{G}, D_{\nu_{i}}^{k-1} \mathbf{u}\right\}, \quad \mathbf{u} \in C^{\infty}(\bar{G})
$$

states an isometric correspondence between $\mathbf{W}^{\ell}(G)$ and a subspace of the direct product

$$
W^{\ell}(G) \times \prod_{i=1}^{N} \prod_{k=1}^{2 m} H_{\ell}^{\ell-k+1 / 2}\left(\Gamma_{i}\right)
$$

We identify the element $\mathbf{u} \in \mathbf{W}^{\ell}(G)$ with the element $\mathbf{S u}=\left\{u, u_{i k}\right\}$ and write

$$
\mathbf{u}=\left\{u, u_{i k}\right\} \in \mathbf{W}^{\ell}(G)
$$

Note that if $\ell \geq 1,\left\{\mathbf{u}^{n}\right\} \subset C^{\infty}(\bar{G})$, and $\mathbf{u}^{n} \rightarrow \mathbf{u}$ in $\mathbf{W}^{\ell}(G)$ as $n \rightarrow \infty$, then $\mathbf{u}^{n} \rightarrow u$ in $W^{\ell}(G)$. Hence, $D_{\nu_{i}}^{k-1} \mathbf{u}^{n} \rightarrow D_{\nu_{i}}^{k-1} u$ in $W^{\ell-k+1 / 2}\left(\Gamma_{i}\right) \subset H_{\ell}^{\ell-k+1 / 2}\left(\Gamma_{i}\right)$ for $k=1, \ldots, \ell$. Thus, the components $u_{i k}$ are uniquely defined by the component $u$ for $k=1, \ldots, \ell$ :

$$
u_{i k}=D_{\nu_{i}}^{k-1} u, \quad i=1, \ldots, N, \quad k=1, \ldots, \ell .
$$

However, generally speaking, the components $u_{i k}, k=\ell+1, \ldots, 2 m$, are not defined by the component $u$ (cf. [72, Sec. 2.2]). Therefore, the spaces $\mathbf{W}^{\ell}(G)$ and $W^{\ell}(G)$ are substantially different.

Let us also note that if $\ell_{1}$ and $\ell_{2}$ are integers, $0 \leq \ell_{1}, \ell_{2} \leq 2 m-1$, and $\ell_{1}<\ell_{2}$, then the space $\mathbf{W}^{\ell_{2}}(G)$ is a closed subspace of the space $\mathbf{W}^{\ell_{1}}(G)$ and

$$
\|\mathbf{u}\|_{\mathbf{W}^{\ell_{1}}(G)} \leq c\|\mathbf{u}\|_{\mathbf{W}^{\ell_{2}(G)}} \quad \forall \mathbf{u} \in \mathbf{W}^{\ell_{2}}(G)
$$

where $c>0$ is independent of $u$. This follows from Lemma 5.8.
Let us consider the family of functions $\left\{\varphi_{\delta}\right\}_{\delta>0} \subset C^{\infty}(\bar{G})$ such that $\varphi_{\delta}(y)=1$ for $y \in G \backslash \mathcal{O}_{\delta}(\mathcal{K})$ and $\varphi_{\delta}(y)=0 \operatorname{fpr} y \in \mathcal{O}_{\delta / 2}(\mathcal{K})$ and $\left|D^{\alpha} \varphi_{\delta}\right| \leq c_{\alpha} \delta^{-|\alpha|}$, where $c_{\alpha}>0$ is independent of $\delta$. It easy to see that

$$
\begin{array}{rllll}
\left\|\varphi_{\delta} u-u\right\|_{H_{a}^{l}(G)} & \rightarrow 0 & \text { as } \delta \rightarrow 0, & & u \in H_{a}^{l}(G), \\
& & l \geq 0, & a \in \mathbb{R}  \tag{13.4}\\
\left\|\varphi_{\delta} \psi-\psi\right\|_{H_{a}^{l-1 / 2}\left(\Gamma_{i}\right)} \rightarrow 0 & \text { as } \delta \rightarrow 0, & & \psi \in H_{a}^{l-1 / 2}\left(\Gamma_{i}\right), & \\
l \geq 1, & & a \in \mathbb{R} .
\end{array}
$$

Assume that for sufficiently small $\delta>0$ (in particular, for $\delta>0$, when $G \cap \mathcal{O}_{\delta}(g), g \in \mathcal{K}$ coincides with an angle), the function $\varphi_{\delta}(y)$ depends only on $\operatorname{dist}(y, \mathcal{K})$ if $y \in \mathcal{O}_{\delta}(\mathcal{K})$. This property implies that

$$
\begin{equation*}
\left.D_{\nu_{i}}^{l} \varphi_{\delta}\right|_{\Gamma_{i}}=0, \quad i=1, \ldots, N, \quad l \geq 1 \tag{13.5}
\end{equation*}
$$

Lemma 13.1. Let $\mathbf{u}=\left\{u, u_{i k}\right\} \in \mathbf{W}^{\ell}(G)$. Then $\varphi_{\delta} \mathbf{u}=\left\{\varphi_{\delta} u, \varphi_{\delta} u_{i k}\right\}$, i.e., the operator $\mathbf{S}$ commutes with the operator of multiplication by $\varphi_{\delta}$.
Proof. Let $\left\{\mathbf{u}^{n}\right\} \subset C^{\infty}(\bar{G})$ and $\mathbf{u}^{n} \rightarrow \mathbf{u}$ in $\mathbf{W}^{\ell}(G)$. On the one hand, this means that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \varphi_{\delta} \mathbf{u}^{n}=\varphi_{\delta} \mathbf{u} \quad \text { in } \quad \mathbf{W}^{\ell}(G) \tag{13.6}
\end{equation*}
$$

On the other hand, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \varphi_{\delta} \mathbf{u}^{n}=\varphi_{\delta} u \quad \text { in } W^{\ell}(G) \tag{13.7}
\end{equation*}
$$

[^5]and, by virtue of Eq. (13.5),
\[

$$
\begin{equation*}
\lim _{n \rightarrow \infty} D_{\nu_{i}}^{k-1}\left(\varphi_{\delta} \mathbf{u}^{n}\right)=\lim _{n \rightarrow \infty} \varphi_{\delta} D_{\nu_{i}}^{k-1} \mathbf{u}^{n}=\varphi_{\delta} u_{i k} \quad \text { in } \quad H_{\ell}^{\ell-k+1 / 2}\left(\Gamma_{i}\right) \tag{13.8}
\end{equation*}
$$

\]

It follows from (13.7) and (13.8) that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \varphi_{\delta} \mathbf{u}^{n}=\left\{\varphi_{\delta} u, \varphi_{\delta} u_{i k}\right\} \quad \text { in } \mathbf{W}^{\ell}(G) . \tag{13.9}
\end{equation*}
$$

Combining Eqs. (13.6) and (13.9), we complete the proof.
It follows from [72, Lemma 2.3.1] that $^{7}$

$$
\begin{array}{ll}
\left\|\varphi_{\delta} \mathbf{P}(y, D) \mathbf{u}\right\|_{W^{\ell-2 m}(G)} \leq c_{1 \delta}\|\mathbf{u}\|_{\mathbf{W}^{\ell}(G)} & \forall \mathbf{u} \in C^{\infty}(\bar{G}) \\
\left\|\varphi_{\delta} \mathbf{B}_{i \mu}^{0} \mathbf{u}\right\|_{W^{\ell-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)} \leq c_{2 \delta}\|\mathbf{u}\|_{\mathbf{W}^{\ell}(G)} & \forall \mathbf{u} \in C^{\infty}(\bar{G})
\end{array}
$$

where $c_{1 \delta}$ and $c_{2 \delta}>0$ are independent of $\mathbf{u}$. Hence, for all $\delta>0$ the closure $\mathbf{L}_{\delta}^{0}=\left\{\varphi_{\delta} \mathbf{P}(y, D), \varphi_{\delta} \mathbf{B}_{i \mu}^{0}\right\}$ of the operator

$$
\mathbf{u} \mapsto\left\{\varphi_{\delta} \mathbf{P}(y, D) \mathbf{u}, \varphi_{\delta} \mathbf{B}_{i \mu}^{0} \mathbf{u}\right\}, \quad \mathbf{u} \in C^{\infty}(\bar{G})
$$

is a bounded operator that maps the whole space $\mathbf{W}^{\ell}(G)$ to the space $W^{\ell-2 m}(G) \times \mathcal{W}^{\ell-\mathbf{m}-1 / 2}(\partial G)$, where

$$
\mathcal{W}^{\ell-\mathbf{m}-1 / 2}(\partial G)=\prod_{i=1}^{N} \prod_{\mu=1}^{m} W^{\ell-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right) .
$$

Introduce the notation

$$
\begin{aligned}
W^{\ell}(G, \mathcal{K}) & =\left\{g_{0}: \varphi_{\delta} g_{0} \in W^{\ell}(G) \forall \delta>0\right\}, \\
\mathcal{W}^{\ell-\mathbf{m}-1 / 2}(\partial G, \mathcal{K}) & =\left\{\left\{g_{i \mu}\right\}: \varphi_{\delta} g_{i \mu} \in W^{\ell-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right) \forall \delta>0\right\} .
\end{aligned}
$$

Definition 13.1. Let the number $\ell$ be fixed $(0 \leq \ell \leq 2 m-1)$. The function $\mathbf{u} \in \mathbf{W}^{\ell}(G)$ is called a strong generalized solutions of the local problem

$$
\begin{align*}
\mathbf{P}(y, D) \mathbf{u} & =g_{0}(y),  \tag{13.10}\\
\mathbf{B}_{i \mu}^{0} \mathbf{u} & =g_{i \mu}(y)  \tag{13.11}\\
& y \in \Gamma_{i}, \quad i=1, \ldots, N, \quad \mu=1, \ldots, m
\end{align*}
$$

with the right-hand side $\left\{g_{0}, g_{i \mu}\right\} \in W^{\ell-2 m}(G, \mathcal{K}) \times \mathcal{W}^{\ell-\mathbf{m}-1 / 2}(\partial G, \mathcal{K})$, if

$$
\mathbf{L}_{\delta}^{0} \mathbf{u}=\varphi_{\delta}\left\{g_{0}, g_{i \mu}\right\} \quad \forall \delta>0
$$

Assume that $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{W}^{0}(G, \partial G)$ and define a strong generalized solution $\mathbf{u}=\left\{u, u_{i k}\right\} \in \mathbf{W}^{\ell}(G)$ of problem (13.1), (13.2). First, we assume that the function $\mathbf{u}$ satisfies the relation

$$
\begin{equation*}
\varphi_{\delta} \mathbf{P}(y, D) \mathbf{u}=\varphi_{\delta} f_{0} \quad \text { in } \quad W^{\ell-2 m}(G) \quad \forall \delta>0 \tag{13.12}
\end{equation*}
$$

It follows from Eq. (13.12) and [72, Theorem 7.2.2] (theorem on the local increasing of smoothness) that $u \in W_{\text {loc }}^{2 m}(G)$ and

$$
\mathbf{P}(y, D) u=f_{0}(y), \quad \text { a.e. } y \in G .
$$

Moreover,

$$
\begin{equation*}
\mathbf{B}_{i \mu}^{1} \mathbf{u} \stackrel{\text { def }}{=} \mathbf{B}_{i \mu}^{1} u \in W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i} \backslash \overline{\mathcal{O}_{\delta}(\mathcal{K})}\right) \quad \forall \delta>0 \tag{13.13}
\end{equation*}
$$

and, by virtue of Eq. (6.6) (for $l=0$ ),

$$
\begin{equation*}
\mathbf{B}_{i \mu}^{2} \mathbf{u} \stackrel{\text { def }}{=} \mathbf{B}_{i \mu}^{2} u \in W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i} \backslash \overline{\mathcal{O}_{\varkappa_{2}}(\mathcal{K})}\right) . \tag{13.14}
\end{equation*}
$$

It follows from Eqs. (13.13) and (13.14) that

$$
\begin{equation*}
f_{i \mu}-\mathbf{B}_{i \mu}^{1} \mathbf{u}-\mathbf{B}_{i \mu}^{2} \mathbf{u} \in W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i} \backslash \overline{\mathcal{O}_{\varkappa_{2}}(\mathcal{K})}\right) . \tag{13.15}
\end{equation*}
$$

[^6]Assume that

$$
\mathbf{L}_{\varkappa_{2}}^{0} \mathbf{u}=\varphi_{\varkappa_{2}}\left\{f_{0}, f_{i \mu}-\mathbf{B}_{i \mu}^{1} \mathbf{u}-\mathbf{B}_{i \mu}^{2} \mathbf{u}\right\}
$$

It follows from this equation, Eq. (13.15), and [72, Theorem 7.2.1] (theorem on the local increasing of smoothness) that $u \in W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$. Hence, using Eq. (6.5) (for $l=0$ ), we have

$$
\begin{equation*}
\mathbf{B}_{i \mu}^{2} \mathbf{u} \stackrel{\text { def }}{=} \mathbf{B}_{i \mu}^{2} u \in W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right) \tag{13.16}
\end{equation*}
$$

Equations (13.13) and (13.16) yield the embedding

$$
\begin{equation*}
f_{i \mu}-\mathbf{B}_{i \mu}^{1} \mathbf{u}-\mathbf{B}_{i \mu}^{2} \mathbf{u} \in W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i} \backslash \overline{\mathcal{O}_{\delta}(\mathcal{K})}\right) \quad \forall \delta>0 \tag{13.17}
\end{equation*}
$$

Now we can state the following definition.
Definition 13.2. Let the number $\ell$ be fixed, $0 \leq \ell \leq 2 m-1$. The function $\mathbf{u} \in \mathbf{W}^{\ell}(G)$ is called a strong generalized solution of nonlocal problem (13.1), (13.2) with the right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in$ $\mathcal{W}^{0}(G, \partial G)$ if this function is a strong generalized solution of the local problem (13.10), (13.11) with the right-hand side $\left\{f_{0}, f_{i \mu}-\mathbf{B}_{i \mu}^{1} \mathbf{u}-\mathbf{B}_{i \mu}^{2} \mathbf{u}\right\}$.

It follows from Definition 13.2, Eq. (13.17), and [72, Theorem 7.2.1] (theorem on the local increasing of smoothness) that if $\mathbf{u}=\left\{u, u_{i k}\right\}$ is a strong generalized solution of problem (13.1), (13.2) with the right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{W}^{0}(G, \partial G)$, then $u \in W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\delta}(\mathcal{K})}\right)$ for all $\delta>0$,

$$
\begin{equation*}
\left(\varphi_{\delta} \mathbf{u}\right)_{i k}=D_{\nu_{i}}^{k-1}\left(\varphi_{\delta} u\right) \quad \forall \delta>0 \quad\left(\text { in } H_{\ell}^{\ell-k+1 / 2}\left(\Gamma_{i}\right), k=1, \ldots, 2 m\right) \tag{13.18}
\end{equation*}
$$

and

$$
\begin{gather*}
\mathbf{P}(y, D) u=f_{0}(y) \quad \text { a.e. } y \in G  \tag{13.19}\\
\mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u=f_{i \mu}(y) \quad \text { in } W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i} \backslash \overline{\mathcal{O}_{\delta}(\mathcal{K})}\right) \quad \forall \delta>0  \tag{13.20}\\
i=1, \ldots, N, \quad \mu=1, \ldots, m
\end{gather*}
$$

Thus, we see that the component $u$ of the strong generalized solution $\mathbf{u}$ is a generalized solution in the following case.

Definition 13.3. Let the number $\ell$ be fixed, $0 \leq \ell \leq 2 m-1$. A function $u$ is called a generalized solution of problem (13.1), (13.2) with the right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{W}^{0}(G, \partial G)$ if

$$
\begin{equation*}
u \in W^{\ell}(G) \cap W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\delta}(\mathcal{K})}\right) \quad \forall \delta>0 \tag{13.21}
\end{equation*}
$$

and $u$ satisfies Eqs. (13.19) and (13.20).
Moreover, for any $\delta>0$ and an integer $l \geq 0$, we can prove the existence of a number $\delta_{1}, 0<\delta_{1}<\delta$, such that for any generalized solution $u$ (in the sense of Definition 13.3) of problem (13.1), (13.2), the following estimate holds:

$$
\begin{equation*}
\|u\|_{W^{l+2 m}\left(G \backslash \overline{\mathcal{O}_{\delta}(\mathcal{K})}\right)} \leq c_{\delta}\left(\left\|f_{0}\right\|_{W^{l}\left(G \backslash \overline{\mathcal{O}_{\delta_{1}}(\mathcal{K})}\right)}+\left\|\left\{f_{i \mu}\right\}\right\|_{\mathcal{W}^{l+2 m-\mathbf{m}-1 / 2}\left(\partial G \backslash \overline{\left.\mathcal{O}_{\delta_{1}}(\mathcal{K})\right)}\right.}+\|u\|_{L_{2}\left(G \backslash \overline{\left.\mathcal{O}_{\delta_{1}}(\mathcal{K})\right)}\right)}\right) \tag{13.22}
\end{equation*}
$$

where $c_{\delta}>0$ is independent of the functions $u, f_{0}$, and $f_{i \mu}$.
Remark 13.1. We have shown that if $\mathbf{u}=\left\{u, u_{i \mu}\right\} \in \mathbf{W}^{\ell}(G)$ is a strong generalized solution of problem (13.1), (13.2), then its component $u$ is a generalized solution of the same problem. Let us show that the converse statement is also valid. Let $u$ be a generalized solution of problem $(13.1),(13.2)$ with the right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{W}^{0}(G, \partial G)$. Lemma 14.2 (see below) implies that $u \in W^{\ell}(G) \cap H_{2 m}^{2 m}(G)$. Hence, using Lemma 5.8, we obtain $D_{\nu_{i}}^{k-1} u \in H_{2 m}^{2 m-k+1 / 2}\left(\Gamma_{i}\right) \subset H_{\ell}^{\ell-k+1 / 2}\left(\Gamma_{i}\right)$, i.e.,

$$
\mathbf{u}=\left\{u, D_{\nu_{i}}^{k-1} u\right\} \in \mathbf{W}^{\ell}(G)
$$

Moreover, it is easy to verify that $\mathbf{u}$ is a strong generalized solution of problem (13.1), (13.2).

Let us show that if the function

$$
\mathbf{v}=\left\{u, v_{i k}\right\} \in \mathbf{W}^{\ell}(G)
$$

(with the same first component $u$ ) is a strong generalized solution of problem (13.1), (13.2), then

$$
\begin{equation*}
v_{i k}=D_{\nu_{i}}^{k-1} u \quad \text { in } \quad H_{\ell}^{\ell-k+1 / 2}\left(\Gamma_{i}\right), \quad k=1, \ldots, 2 m, \tag{13.23}
\end{equation*}
$$

i.e., the generalized solution uniquely defines the strong generalized solution. Indeed, since $\mathbf{v}$ is a strong generalized solution, we have (by Eq. (13.18))

$$
\begin{equation*}
\left(\varphi_{\delta} \mathbf{v}\right)_{i k}=D_{\nu_{i}}^{k-1}\left(\varphi_{\delta} u\right) \quad \forall \delta>0 \quad\left(\text { in } H_{\ell}^{\ell-k+1 / 2}\left(\Gamma_{i}\right), k=1, \ldots, 2 m\right) . \tag{13.24}
\end{equation*}
$$

From Eqs. (13.24) and (13.5) and Lemma 13.1 we obtain that

$$
\varphi_{\delta} v_{i k}=\left(\varphi_{\delta} \mathbf{v}\right)_{i k}=D_{\nu_{i}}^{k-1}\left(\varphi_{\delta} u\right)=\varphi_{\delta} D_{\nu_{i}}^{k-1} u \quad \text { in } H_{\ell}^{\ell-k+1 / 2}\left(\Gamma_{i}\right), \quad k=1, \ldots, \ell .
$$

Since $\delta$ is arbitrary, it follows from Eq. (13.4) that $v_{i k}=D_{\nu_{i}}^{k-1} u$ in $H_{\ell}^{\ell-k+1 / 2}\left(\Gamma_{i}\right), k=1, \ldots, \ell$.
Let $k=\ell+1, \ldots, 2 m$. Let us consider an arbitrary function $\psi \in C_{0}^{\infty}\left(\Gamma_{i}\right)$ and choose $\varphi_{\delta}$ such that $\varphi_{\delta}(y)=1$ for $y \in \operatorname{supp} \psi$. Using Lemma 13.1 and Eqs. (13.24) and (13.5), we have

$$
\left\langle v_{i k}-D_{\nu_{i}}^{k-1} u, \psi\right\rangle=\left\langle\varphi_{\delta} v_{i k}-\varphi_{\delta} D_{\nu_{i}}^{k-1} u, \psi\right\rangle=\left\langle\left(\varphi_{\delta} \mathbf{v}\right)_{i k}-D_{\nu_{i}}^{k-1}\left(\varphi_{\delta} u\right), \psi\right\rangle=0 .
$$

Since the set $C_{0}^{\infty}\left(\Gamma_{i}\right)$ is dense in $H_{-\ell}^{-(\ell-k+1 / 2)}\left(\Gamma_{i}\right)$, we see that $v_{i k}=D_{\nu_{i}}^{k-1} u$ in the set $H_{\ell}^{\ell-k+1 / 2}\left(\Gamma_{i}\right)$, $k=\ell+1, \ldots, 2 m$. Thus, Eq. (13.23) is proved and $\mathbf{u}=\mathbf{v}$.

Remark 13.1 states a bijective correspondence between generalized solutions and strong generalized solutions of the same nonlocal problem. In what follows, we will consider generalized solutions and hence it is convenient to use Definition 13.3.

In this chapter, we study problem (13.1), (13.2) with the following boundary conditions:

$$
\begin{align*}
\mathbf{P}(y, D) u=f_{0}(y), & y \in G  \tag{13.25}\\
\mathbf{B}_{i \mu} u \equiv \mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u=0, & y \in \Gamma_{i}, \quad i=1, \ldots, N, \quad \mu=1, \ldots, m, \tag{13.26}
\end{align*}
$$

where $f_{0} \in L_{2}(G)$.
Let us consider an unbounded operator $\mathbf{P}: \mathrm{D}(\mathbf{P}) \subset L_{2}(G) \rightarrow L_{2}(G)$ acting by the formula

$$
\mathbf{P} u=\mathbf{P}(y, D) u
$$

$$
u \in \mathrm{D}(\mathbf{P})=\left\{u \in L_{2}(G): u \text { satisfies Eq. (13.21), } \mathbf{B}_{i \mu} u=0, \mathbf{P}(y, D) u \in L_{2}(G)\right\}
$$

By Definition 13.3, the operator $\mathbf{P}$ corresponds to problem (13.25), (13.26).
Let us formulate the main result; it will be proved in Sec. 14.
Theorem 13.1. The operator $\mathbf{P}$ is a Fredholm operator.
Note that, unlike the case of a bounded operator $\mathbf{L}_{B}$ (see Sec. 11) corresponding to problem (13.25), (13.26), the Fredholm property of an unbounded operator $\mathbf{P}$ depends neither on spectral properties of the operator $\tilde{\mathcal{L}}(\lambda)$ nor on algebraic relations among operators $\mathbf{P}(y, D), \mathbf{B}_{i \mu}^{0}$, and $\mathbf{B}_{i \mu}^{1}$ at points of the set $\mathcal{K}$.
13.2. Nonlocal problems near the set $\mathcal{K}$. As earlier, we pay special attention to the behavior of solutions in a neighborhood of the set $\mathcal{K}$ of conjugation point of nonlocal conditions. Let us write out corresponding model problems.

Let us denote by $u_{j}(y)$ the function $u(y)$ for $y \in \mathcal{O}_{\varepsilon_{1}}\left(g_{j}\right)$. If $g_{j} \in \overline{\Gamma_{i}}, y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right)$, and $\Omega_{i s}(y) \in$ $\mathcal{O}_{\varepsilon_{1}}\left(g_{k}\right)$, then we denote $u\left(\Omega_{i s}(y)\right)$ by $u_{k}\left(\Omega_{i s}(y)\right)$. Then nonlocal problem (13.25), (13.26) in a $\varepsilon$ neighborhood of the set (orbit) $\mathcal{K}$ has the form

$$
\begin{gathered}
\mathbf{P}(y, D) u_{j}=f_{0}(y), \quad y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap G, \\
\left.B_{i \mu 0}(y, D) u_{j}(y)\right|_{\mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap \Gamma_{i}}+\left.\sum_{s=1}^{S_{i}}\left(B_{i \mu s}(y, D)\left(\zeta u_{k}\right)\right)\left(\Omega_{i s}(y)\right)\right|_{\mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap \Gamma_{i}}=f_{i \mu}(y), \\
y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap \Gamma_{i}, \quad i \in\left\{1 \leq i \leq N: g_{j} \in \overline{\Gamma_{i}}\right\}, \quad j=1, \ldots, N, \quad \mu=1, \ldots, m,
\end{gathered}
$$

where $f_{i \mu}=-\mathbf{B}_{i \mu}^{2} u$.
Let $y \mapsto y^{\prime}\left(g_{j}\right)$ be transformations of coordinates described in Sec. 6.1. Introduce the functions

$$
U_{j}\left(y^{\prime}\right)=u_{j}\left(y\left(y^{\prime}\right)\right), \quad f_{j}\left(y^{\prime}\right)=f_{0}\left(y\left(y^{\prime}\right)\right), \quad y^{\prime} \in K_{j}^{\varepsilon} ; \quad f_{j \sigma \mu}\left(y^{\prime}\right)=f_{i \mu}\left(y\left(y^{\prime}\right)\right), \quad y^{\prime} \in \gamma_{j \sigma}^{\varepsilon}
$$

where $\sigma=1(\sigma=2)$ if a transformation $y \mapsto y^{\prime}\left(g_{j}\right)$ maps $\mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap \Gamma_{i}$ to a side $\gamma_{j 1}$ (respectively, $\gamma_{j 2}$ ) of the angle $K_{j}$. Let us re-denote $y^{\prime}$ by $y$. Then, by condition 6.3 , problem (13.25), (13.26) has the form (cf. (6.12), (6.13))

$$
\begin{align*}
\mathbf{P}_{j}(y, D) U_{j}=f_{j}(y), & y \in K_{j}^{\varepsilon}  \tag{13.27}\\
\left.\mathbf{B}_{j \sigma \mu}\left(y, D_{y}\right) U \equiv \sum_{k, s}\left(B_{j \sigma \mu k s}(y, D) U_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}}=f_{j \sigma \mu}(y), & y \in \gamma_{j \sigma}^{\varepsilon} \tag{13.28}
\end{align*}
$$

Note that if $\mathbf{B}_{i \mu}^{2}=0$, then the right-hand side of problem (13.27), (13.28) coincides with the right-hand side of problem (6.12), (6.13).

## 14. Fredholm Solvability of Nonlocal Problems

14.1. Finite dimension of the kernel. Here, we prove that the kernel of the operator $\mathbf{P}$ has finite dimension. For this, we study the smoothness of generalized solutions of problem (13.25), (13.26) near the set $\mathcal{K}$.

Let $u$ be a generalized solution of problem (13.25), (13.26), and $U_{j}\left(y^{\prime}\right)=u_{j}\left(y\left(y^{\prime}\right)\right), j=1, \ldots, N$, be functions corresponding to the set (orbit) $\mathcal{K}$ and satisfying problem (13.27), (13.28) with the right-hand side $\left\{f_{j}, f_{j \sigma \mu}\right\}$.

By Eq. (13.21), we have

$$
\begin{equation*}
U_{j} \in W^{2 m}\left(K_{j}^{d_{2} \varepsilon} \backslash \overline{\mathcal{O}_{\delta}(0)}\right) \quad \forall \delta>0 \tag{14.1}
\end{equation*}
$$

where $d_{2}$ is defined in Eq. (6.15). It follows from the embedding $U_{j} \in W^{\ell}\left(K_{j}^{d_{2} \varepsilon}\right)$ and Lemma 5.2 that

$$
\begin{equation*}
U_{j} \in H_{a-2 m}^{0}\left(K_{j}^{d_{2} \varepsilon}\right) \tag{14.2}
\end{equation*}
$$

where $a>2 m-1$. Finally, $f_{j} \in L_{2}\left(K_{j}^{\varepsilon}\right)$ and, by virtue of Eq. (13.21) and embedding (6.5), where $l=0$, we have $f_{j \sigma \mu} \in W^{2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}^{\varepsilon}\right)$. Hence, by Lemma 5.2,

$$
\begin{equation*}
f_{j} \in H_{a}^{0}\left(K_{j}^{\varepsilon}\right), \quad f_{j \sigma \mu} \in H_{a}^{2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}^{\varepsilon}\right) \tag{14.3}
\end{equation*}
$$

where $a>2 m-1$.
Using the following two lemmas, we show that the embedding $U_{j} \in H_{a}^{2 m}\left(K_{j}^{\varepsilon / d_{2}^{3}}\right)$ is implied by Eqs. (14.1)-(14.3).

Let us denote $K_{j q}=K_{j} \cap\left\{\varepsilon d_{2}^{-3} d_{1}^{4-q} / 2<|y|<\varepsilon d_{2}^{-3} d_{2}^{4-q}\right\}$, where $q=0, \ldots, 4$.

Lemma 14.1. The following estimate holds for all $U \in \prod_{j} W^{2 m}\left(K_{j 0}\right)$ :

$$
\begin{align*}
\sum_{j}\left\|U_{j}\right\|_{W^{2 m}\left(K_{j 4}\right)} \leq c \sum_{j} & \left\{\left\|\mathbf{P}_{j}(y, D) U_{j}\right\|_{L_{2}\left(K_{j 1}\right)}\right. \\
& \left.+\sum_{\sigma, \mu}\left\|\left.\mathbf{B}_{j \sigma \mu}(y, D) U\right|_{\gamma_{j \sigma} \cap \overline{K_{j 1}}}\right\|_{W^{2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma} \cap \overline{K_{j 1}}\right)}+\left\|U_{j}\right\|_{L_{2}\left(K_{j 1}\right)}\right\}, \tag{14.4}
\end{align*}
$$

where $c>0$ is independent of $U$.
Proof. It follows from the general theory of elliptic operators that

$$
\begin{align*}
&\left\|U_{j}\right\|_{W^{2 m}\left(K_{j 4}\right)} \leq k_{1}\left(\left\|\mathbf{P}_{j}(y, D) U_{j}\right\|_{L_{2}\left(K_{j 3}\right)}\right. \\
&\left.+\sum_{\sigma, \mu}\left\|\left.B_{j \sigma \mu j 0}(y, D) U_{j}\right|_{\gamma_{j \sigma} \cap \overline{K_{j 3}}}\right\|_{W^{2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma} \cap \overline{K_{j 3}}\right)}+\left\|U_{j}\right\|_{L_{2}\left(K_{j 3}\right)}\right) . \tag{14.5}
\end{align*}
$$

Let $(k, s) \neq(j, 0)$. In this case, we see that $\mathcal{G}_{j \sigma k s}\left(\gamma_{j \sigma}\right) \cap \overline{K_{k 2}}$ is located inside the domain $K_{k 1}$. Therefore, we can use the continuity of the trace operator in Sobolev spaces and (similarly to Eq. (14.5)) we obtain the inequality

$$
\begin{align*}
&\left\|\left.B_{j \sigma \mu k s}(y, D) U_{k}\left(\mathcal{G}_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma} \cap \overline{K_{j 3}}}\right\|_{W^{2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma} \cap \overline{K_{j 3}}\right)} \\
& \leq k_{2}\left\|\left.B_{j \sigma \mu k s}(y, D) U_{k}\right|_{\mathcal{G}_{j \sigma k s}\left(\gamma_{j \sigma}\right) \cap \overline{K_{k 2}}}\right\|_{W^{2 m-m_{j \sigma \mu}-1 / 2}\left(\mathcal{G}_{j \sigma k s}\left(\gamma_{j \sigma}\right) \cap \overline{K_{k 2}}\right)} \\
& \leq k_{3}\left(\left\|\mathbf{P}_{j}(y, D) U_{k}\right\|_{L_{2}\left(K_{k 1}\right)}+\left\|U_{k}\right\|_{L_{2}\left(K_{k 1}\right)}\right) . \tag{14.6}
\end{align*}
$$

Estimates (14.5) and (14.6) yield Eq. (14.4).
Remark 14.1. Assume that the norms in $C^{0}\left(\overline{K_{j 1}}\right)$ of the coefficients $p_{j \alpha}$ of the operators $\mathbf{P}_{j}(y, D)$ and the norms in $C^{2 m-m_{j \sigma \mu}}\left(\overline{K_{j 0}}\right)$ of the coefficients $b_{j \sigma \mu k s \alpha}$ of the operators $B_{j \sigma \mu k s}(y, D)$ are bounded by a constant $C$. Let this constant $C$ bound norms in $C^{1}\left(\overline{K_{j 1}}\right)$ of the coefficients $p_{j \alpha},|\alpha|=2 m$, and the coefficients of higher derivatives in the operators $\mathbf{P}_{j}(y, D)$. Then the constant $c$ in inequality (14.4) depends only on $C$, the constant $A$ in (6.1), and the constant $D$ in (6.2).

Lemma 14.2. Fix arbitrary $a>0$. Assume that a function $U$ satisfies Eqs. (14.1) and (14.2) and is a solution of problem (13.27), (13.28) with the right-hand side $\left\{f_{j}, f_{j \sigma \mu}\right\}$, which satisfies Eq. (14.3). Then $U \in \prod_{j} H_{a}^{2 m}\left(K_{j}^{\varepsilon / d_{2}^{3}}\right)$ and

$$
\begin{equation*}
\sum_{j}\left\|U_{j}\right\|_{H_{a}^{2 m}\left(K_{j}^{\left.\varepsilon / d_{2}^{3}\right)}\right.} \leq c \sum_{j}\left\{\left\|f_{j}\right\|_{H_{a}^{0}\left(K_{j}^{\varepsilon}\right)}+\sum_{\sigma, \mu}\left\|f_{j \sigma \mu}\right\|_{H_{a}^{2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}^{\varepsilon}\right)}+\left\|U_{j}\right\|_{H_{a-2 m}^{0}\left(K_{j}^{\varepsilon}\right)}\right\}, \tag{14.7}
\end{equation*}
$$

where $c>0$ is independent of $U$.
Proof. Let us denote

$$
K_{j q}^{s}=K_{j} \cap\left\{\varepsilon d_{2}^{-3} d_{1}^{4-q} 2^{-s-1}<|y|<\varepsilon d_{2}^{-3} d_{2}^{4-q} 2^{-s}\right\}, \quad s=0,1,2, \ldots .
$$

Obviously,

$$
\begin{equation*}
\bigcup_{s=0}^{\infty} K_{j 1}^{s}=K_{j}^{\varepsilon}, \quad \bigcup_{s=0}^{\infty} K_{j 4}^{s}=K_{j}^{\varepsilon / d_{2}^{3}} \tag{14.8}
\end{equation*}
$$

Assume that $U_{j}^{s}\left(y^{\prime}\right)=U_{j}\left(2^{-s} y^{\prime}\right)$. We perform the change of variables $y=2^{-s} y^{\prime}$ in the equation

$$
\mathbf{P}_{j}(y, D) U_{j} \equiv \sum_{|\alpha| \leq 2 m} p_{j \alpha}(y) D_{y}^{\alpha} U_{j}(y)=f_{j}(y), \quad y \in K_{j 1}^{s}
$$

and in the nonlocal conditions

$$
\left.\sum_{k, s} \sum_{|\alpha| \leq m_{j \sigma \mu}} b_{j \sigma \mu k s \alpha}(x) D_{x}^{\alpha} U_{j}(x)\right|_{x=\mathcal{G}_{j \sigma k s} y}=f_{j \sigma \mu}(y), \quad y \in \gamma_{j \sigma} \cap \overline{K_{j 1}^{s}} .
$$

Multiplying both parts of the first equality by $2^{-s \cdot 2 m}$ and both parts of the second equality by $2^{-s \cdot m_{j \sigma \mu}}$, we have

$$
\begin{gather*}
\sum_{|\alpha| \leq 2 m} p_{j \alpha}^{s}\left(y^{\prime}\right) 2^{s(|\alpha|-2 m)} D_{y^{\prime}}^{\alpha} U_{j}^{s}\left(y^{\prime}\right)=2^{-s \cdot 2 m} f_{j}^{s}\left(y^{\prime}\right), \quad y^{\prime} \in K_{j 1}^{0},  \tag{14.9}\\
\left.\sum_{k, s} \sum_{|\alpha| \leq m_{j \sigma \mu}} b_{j \sigma \mu k s \alpha}^{s}\left(x^{\prime}\right) 2^{s\left(|\alpha|-m_{j \sigma \mu)}\right)} D_{x^{\prime}}^{\alpha} U_{j}^{s}\left(x^{\prime}\right)\right|_{x^{\prime}=\mathcal{G}_{j \sigma k s} y^{\prime}}=2^{-s \cdot m_{j \sigma \mu}} f_{j \sigma \mu}^{s}\left(y^{\prime}\right), \quad y^{\prime} \in \gamma_{j \sigma} \cap \overline{K_{j 1}^{0}}, \tag{14.10}
\end{gather*}
$$

where $p_{j \alpha}^{s}\left(y^{\prime}\right)=p_{j \alpha}\left(2^{-s} y^{\prime}\right), b_{j \sigma \mu k s \alpha}^{s}\left(x^{\prime}\right)=b_{j \sigma \mu k s \alpha}\left(2^{-s} x^{\prime}\right), f_{j}^{s}\left(y^{\prime}\right)=f_{j}\left(2^{-s} y^{\prime}\right)$, and $f_{j \sigma \mu}^{s}\left(y^{\prime}\right)=$ $f_{j \sigma \mu}\left(2^{-s} y^{\prime}\right)$. Applying Lemma 14.1 to problem (14.9), (14.10) we obtain the inequality

$$
\begin{align*}
& \sum_{j}\left\|U_{j}^{s}\right\|_{W^{2 m}\left(K_{j 4}^{0}\right)} \leq k_{1} \sum_{j}\left\{\left\|2^{-s \cdot 2 m} f_{j}^{s}\right\|_{L_{2}\left(K_{j 1}^{0}\right)}\right. \\
&\left.+\sum_{\sigma, \mu}\left\|2^{-s \cdot m_{j \sigma \mu}} f_{j \sigma \mu}^{s}\right\|_{W^{2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma} \cap \overline{K_{j 1}^{0}}\right)}+\left\|U_{j}^{s}\right\|_{L_{2}\left(K_{j 1}^{0}\right)}\right\} \tag{14.11}
\end{align*}
$$

where the constant $k_{1}$ is independent of $s$ by Remark 14.1.
Let us denote by $\Phi_{j \sigma \mu} \in H_{a}^{2 m-m_{j \sigma \mu}}\left(K_{j}\right)$ the function satisfying the following conditions: $\left.\Phi_{j \sigma \mu}\right|_{\gamma_{j \sigma}^{\varepsilon}}=$ $f_{j \sigma \mu}$ and

$$
\begin{equation*}
\left\|\Phi_{j \sigma \mu}\right\|_{H_{a}^{2 m-m_{j \sigma \mu}}\left(K_{j}^{\varepsilon}\right)} \leq 2\left\|f_{j \sigma \mu}\right\|_{H_{a}^{2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}^{\varepsilon}\right)} \tag{14.12}
\end{equation*}
$$

Then

$$
\left.\Phi_{j \sigma \mu}^{s}\right|_{\gamma_{j \sigma} \cap \overline{K_{j 1}^{0}}}=f_{j \sigma \mu}^{s}
$$

where $\Phi_{j \sigma \mu}^{s}\left(y^{\prime}\right)=\Phi_{j \sigma \mu}\left(2^{-s} y^{\prime}\right)$. Hence, Eq. (14.11) yields the inequality

$$
\begin{align*}
& \sum_{j}\left\|U_{j}^{s}\right\|_{W^{2 m}\left(K_{j 4}^{0}\right)} \leq k_{1} \sum_{j}\left\{\left\|2^{-s \cdot 2 m} f_{j}^{s}\right\|_{L_{2}\left(K_{j 1}^{0}\right)}\right. \\
&\left.+\sum_{\sigma, \mu}\left\|2^{-s \cdot m_{j \sigma \mu}} \Phi_{j \sigma \mu}^{s}\right\|_{W^{2 m-m_{j \sigma \mu}\left(K_{j 1}^{0}\right)}}+\left\|U_{j}^{s}\right\|_{L_{2}\left(K_{j 1}^{0}\right)}\right\} . \tag{14.13}
\end{align*}
$$

Conducting the inverse change of variables $y^{\prime}=2^{s} y$ in inequality (14.13), we obtain the inequality

$$
\begin{align*}
\sum_{j} \sum_{|\alpha| \leq 2 m}\left\|2^{-s|\alpha|} D_{y}^{\alpha} U_{j}\right\|_{L_{2}\left(K_{j 4}^{s}\right)} & \leq k_{1} \sum_{j}\left\{\left\|2^{-s \cdot 2 m} f_{j}\right\|_{L_{2}\left(K_{j 1}^{s}\right)}\right. \\
& \left.+\sum_{\sigma, \mu} \sum_{|\alpha| \leq 2 m-m_{j \sigma \mu}}\left\|2^{-s\left(|\alpha|+m_{j \sigma \mu}\right)} \Phi_{j \sigma \mu}\right\|_{L_{2}\left(K_{j 1}^{s}\right)}+\left\|U_{j}\right\|_{L_{2}\left(K_{j 1}^{s}\right)}\right\} . \tag{14.14}
\end{align*}
$$

Multiplying inequality (14.14) by $2^{-s(a-2 m)}$, summing over $s$, and taking into account Eqs. (14.12) and (14.8), we obtain (14.7).

Lemma 14.2 and Eq. (13.21) yield $u \in H_{a}^{2 m}(G), a>2 m-1$, where $u$ is an arbitrary generalized solution of problem (13.25), (13.26) with the right-hand side $f_{0} \in L_{2}(G)$.

It follows from Lemma 6.1 and [89, Theorem 3.2] that the set of solutions from $H_{a}^{2 m}(G)$ of problem (13.25), (13.26) with the right-hand side $f_{0}=0$ form a finite-dimensional subspace for almost all $a>2 m-1$. Thus, we proved the following result.
Lemma 14.3. The kernel of the operator $\mathbf{P}$ is finite-dimensional.
14.2. Closedness of the operator and its image. Finite dimension of the cokernel. To prove that the operator $\mathbf{P}$ is a Fredholm operator, we consider problem (13.25), (13.26) in spaces with weight $a$ such that $0<a \leq 1$. The following difficulty arises: if $u \in H_{a}^{2 m}(G)$, then, generally speaking, it is not necessary that $\mathbf{B}_{i \mu}^{2} u$ belongs to $H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$; therefore, the sum

$$
\mathbf{B}_{i \mu} u=\mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u
$$

may not belong to $H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$. We only can guarantee that $\mathbf{B}_{i \mu} u \in H_{a^{\prime}}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right), a^{\prime}>$ $2 m-1$ (this follows from the relation $\mathbf{B}_{i \mu} u \in W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ and Lemma 5.2). However, it is proved in Sec. 9 that

$$
\left\{\mathbf{P}(y, D) u, \mathbf{B}_{i \mu} u\right\} \in \mathcal{H}_{a}^{0}(G, \partial G) \dot{+} \mathcal{R}_{a}^{0}(G, \partial G) \quad \forall u \in H_{a}^{2 m}(G), \quad a>0
$$

where $\mathcal{R}_{a}^{0}(G, \partial G)$ is a finite-dimensional subspace embedded in $\{0\} \times \prod_{i, \mu} H_{a^{\prime}}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ for any $a^{\prime}>2 m-1$, i.e., the space $\mathcal{R}_{a}^{0}(G, \partial G)$ contains only functions of the form

$$
\left\{0, f_{i \mu}\right\}, \quad f_{i \mu} \in H_{a^{\prime}}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right), \quad f_{i \mu} \notin H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right) .
$$

We fix a number $a^{\prime}>2 m-1$. Then any function

$$
\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{H}_{a}^{0}(G, \partial G) \dot{+} \mathcal{R}_{a}^{0}(G, \partial G)
$$

can be uniquely represented in form

$$
\left\{f_{0}, f_{i \mu}\right\}=\left\{f_{0}, f_{i \mu}^{1}\right\}+\left\{0, f_{i \mu}^{2}\right\}
$$

where

$$
\left\{f_{0}, f_{i \mu}^{1}\right\} \in \mathcal{H}_{a}^{0}(G, \partial G), \quad\left\{0, f_{i \mu}^{2}\right\} \in \mathcal{R}_{a}^{0}(G, \partial G)
$$

and its norm is

$$
\left\|\left\{f_{0}, f_{i \mu}\right\}\right\|_{\mathcal{H}_{a}^{0}(G, \partial G)+\mathcal{R}_{a}^{0}(G, \partial G)}=\left(\left\|\left\{f_{0}, f_{i \mu}^{1}\right\}\right\|_{\mathcal{H}_{a}^{0}(G, \partial G)}^{2}+\sum_{i, \mu}\left\|f_{i \mu}^{2}\right\|_{H_{a^{\prime}}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)}^{2}\right)^{1 / 2}
$$

Moreover, by Theorem 9.1,

$$
\begin{equation*}
\mathbf{L}_{a}=\left\{\mathbf{P}(y, D), \mathbf{B}_{i \mu}\right\}: H_{a}^{2 m}(G) \rightarrow \mathcal{H}_{a}^{0}(G, \partial G)+\mathcal{R}_{a}^{0}(G, \partial G), \quad a>0, \tag{14.15}
\end{equation*}
$$

is a Fredholm operator for almost all $a>0$. Using the operator $\mathbf{L}_{a}$, we prove the following result.
Lemma 14.4. The operator $\mathbf{P}$ is closed, the image $\mathcal{R}(\mathbf{P})$ is closed, and $\operatorname{codim} \mathcal{R}(\mathbf{P})<\infty$.
Proof. 1. Let us consider an auxiliary unbounded operator

$$
\mathbf{P}_{a}: \mathrm{D}\left(\mathbf{P}_{a}\right) \subset L_{2}(G) \rightarrow L_{2}(G), \quad 0<a \leq 1 .
$$

This operator acts by the formula

$$
\mathbf{P}_{a} u=\mathbf{P}(y, D) u, \quad u \in \mathrm{D}\left(\mathbf{P}_{a}\right)=\left\{u \in H_{a}^{2 m}(G): \mathbf{B}_{i \mu} u=0, \mathbf{P}(y, D) u \in L_{2}(G)\right\} .
$$

We fix $a, 0<a \leq 1$, such that $\mathbf{L}_{a}$ is a Fredholm operator. We prove that $\mathbf{P}_{a}$ is also a Fredholm operator.

The Fredholm property of $\mathbf{L}_{a}$, the compactness of the embedding $H_{a}^{2 m}(G) \subset H_{a}^{0}(G)$ (see [53, Lemma 3.5]), and [56, Theorem 7.1] imply that

$$
\begin{equation*}
\|u\|_{H_{a}^{2 m}(G)} \leq k_{1}\left(\left\|\mathbf{L}_{a} u\right\|_{\mathcal{H}_{a}^{0}(G, \partial G)+\mathcal{R}_{a}^{0}(G, \partial G)}+\|u\|_{H_{a}^{0}(G)}\right) \tag{14.16}
\end{equation*}
$$

for all $u \in H_{a}^{2 m}(G)$. Now let $u \in \mathrm{D}\left(\mathbf{P}_{a}\right)$. Then

$$
\mathbf{L}_{a} u=\{\mathbf{P}(y, D) u, 0\}, \quad \mathbf{P}(y, D) u \in L_{2}(G) \subset H_{a}^{0}(G)
$$

and hence

$$
\left\|\mathbf{L}_{a} u\right\|_{\mathcal{H}_{a}^{0}(G, \partial G)+\mathcal{R}_{a}^{0}(G, \partial G)}=\|\mathbf{P}(y, D) u\|_{H_{a}^{0}(G)} .
$$

This relation, Eq. (14.16), and the continuity of the embedding $L_{2}(G) \subset H_{a}^{0}(G)$ for $a>0$ yield the inequality

$$
\begin{equation*}
\|u\|_{H_{a}^{2 m}(G)} \leq k_{2}\left(\|\mathbf{P}(y, D) u\|_{H_{a}^{0}(G)}+\|u\|_{H_{a}^{0}(G)}\right) \leq k_{3}\left(\|\mathbf{P}(y, D) u\|_{L_{2}(G)}+\|u\|_{L_{2}(G)}\right), \tag{14.17}
\end{equation*}
$$

where $u \in \mathrm{D}\left(\mathbf{P}_{a}\right)$. It follows from inequality (14.17) that the operator $\mathbf{P}_{a}$ is closed. Hence, using (14.17) and [56, Theorem 7.1] again, we obtain $\operatorname{dim} \operatorname{ker} \mathbf{P}_{a}<\infty\left(\right.$ clearly, $\left.\operatorname{ker} \mathbf{P}_{a}=\operatorname{ker} \mathbf{L}_{a}\right)$ and $\mathcal{R}\left(\mathbf{P}_{a}\right)$ is closed.

Consider an arbitrary function $f_{0} \in L_{2}(G)$. Obviously, $f_{0} \in H_{a}^{0}(G)$. By Corollary 9.1, there exist functionals $F_{1}, \ldots, F_{q_{0}}$ from the dual space $\mathcal{H}_{a}^{0}(G, \partial G)^{*}$ such that if $\left\langle\left\{f_{0}, 0\right\}, F_{q}\right\rangle=0, q=1, \ldots, q_{0}$, then problem (13.25), (13.26) has a solution $u \in H_{a}^{2 m}(G)$. Since

$$
\left|\left\langle\left\{f_{0}, 0\right\}, F_{q}\right\rangle\right| \leq k_{4}\left\|f_{0}\right\|_{H_{a}^{0}(G)} \leq k_{5}\left\|f_{0}\right\|_{L_{2}(G)},
$$

we see by the Riesz theorem on the general form of linear continuous functionals in Hilbert spaces that there exist functions $f_{1}, \ldots, f_{q_{0}} \in L_{2}(G)$ such that $\left\langle\left\{f_{0}, 0\right\}, F_{q}\right\rangle=\left(f_{0}, f_{q}\right)_{L_{2}(G)}, q=1, \ldots, q_{0}$. Hence $\operatorname{codim} \mathcal{R}\left(\mathbf{P}_{a}\right) \leq q_{0}$. Thus, $\mathbf{P}_{a}$ is a Fredholm operator.
2. Since $H_{a}^{2 m}(G) \subset H_{a-1}^{2 m-1}(G) \subset W^{\ell}(G)$ for $a \leq 1$ and $0 \leq \ell \leq 2 m-1$, we have the following relation:

$$
\begin{equation*}
\mathbf{P}_{a} \subset \mathbf{P} . \tag{14.18}
\end{equation*}
$$

It follows from Eq. (14.18) that the image $\mathcal{R}(\mathbf{P})$ is closed and $\operatorname{codim} \mathcal{R}(\mathbf{P}) \leq \operatorname{codim} \mathcal{R}\left(\mathbf{P}_{a}\right) \leq q_{0}$.
It remains to prove that $\mathbf{P}$ is closed ${ }^{8}$. Let us denote the basis in the space

$$
\mathcal{R}\left(\mathbf{P}_{a}\right)^{\perp}=\mathcal{R}(\mathbf{P}) \ominus \mathcal{R}\left(\mathbf{P}_{a}\right)
$$

by $h_{1}, \ldots, h_{k}$. Then there exist functions $v_{1}, \ldots, v_{k} \in \mathrm{D}(\mathbf{P})$ such that $\mathbf{P} v_{j}=h_{j}, j=1, \ldots, k$. Since $h_{j} \notin \mathcal{R}\left(\mathbf{P}_{a}\right)$, we have $v_{j} \notin \mathrm{D}\left(\mathbf{P}_{a}\right)$. It is also clear that the functions $v_{1}, \ldots, v_{k}$ are linearly independent since the functions $h_{1}, \ldots, h_{k}$ are linearly independent.

Consider a finite-dimensional space

$$
\mathcal{N}=\operatorname{Span}\left(v_{1}, \ldots, v_{k}, \operatorname{ker} \mathbf{P}\right) \ominus \operatorname{ker} \mathbf{P}_{a}
$$

It is easy to see that $\mathcal{N} \cap \mathrm{D}\left(\mathbf{P}_{a}\right)=\{0\}$. Indeed, if $u \in \mathcal{N} \cap \mathrm{D}\left(\mathbf{P}_{a}\right)$, then

$$
u=\sum_{i=1}^{k} \alpha_{i} v_{i}+v
$$

where $\alpha_{i}$ are some constants, $v \in \operatorname{ker} \mathbf{P}$. Then, taking into account Eq. (14.18), we have

$$
\sum_{i=1}^{k} \alpha_{i} h_{i}=\mathbf{P} u=\mathbf{P}_{a} u \in \mathcal{R}\left(\mathbf{P}_{a}\right) .
$$

Hence, $\alpha_{i}=0, i=1, \ldots, k$, and, consequently, $u=v$. Using Eq. (14.18) again, we see that $u=v \in \operatorname{ker} \mathbf{P}_{a}$. It follows from here and the definition of the space $\mathcal{N}$ that $u=0$.

Denote the graph of the operator $\mathbf{P}\left(\mathbf{P}_{a}\right)$ by $\operatorname{Gr} \mathbf{P}$ (respectively, $\left.\operatorname{Gr} \mathbf{P}_{a}\right)$. As is known, the operator $\mathbf{P}\left(\mathbf{P}_{a}\right)$ is closed if and only if its graph $\mathrm{Gr} \mathbf{P}\left(\mathrm{Gr} \mathbf{P}_{a}\right)$ is closed in $L_{2}(G) \times L_{2}(G)$.

Since $\operatorname{Gr} \mathbf{P}_{a}$ is closed (as the graph of a closed operator) and $\operatorname{Gr} \mathbf{P}_{a} \subset \operatorname{Gr} \mathbf{P}$ and the spaces $\mathcal{N}$ and $\mathcal{R}\left(\mathbf{P}_{a}\right)^{\perp}$ are finite-dimensional, we see that to prove that the operator $\mathbf{P}$ is closed, it suffices to show that

$$
\begin{equation*}
\operatorname{Gr} \mathbf{P} \subset \operatorname{Gr} \mathbf{P}_{a} \dot{+}\left(\mathcal{N} \times \mathcal{R}\left(\mathbf{P}_{a}\right)^{\perp}\right) \tag{14.19}
\end{equation*}
$$

[^7]Obviously, the sum in Eq. (14.19) is a direct product: if

$$
(u, f) \in \operatorname{Gr} \mathbf{P}_{a} \cap\left(\mathcal{N} \times \mathcal{R}\left(\mathbf{P}_{a}\right)^{\perp}\right)
$$

then $u \in \mathrm{D}\left(\mathbf{P}_{a}\right) \cap \mathcal{N}=\{0\}$ and hence $(u, f)=\left(u, \mathbf{P}_{a} u\right)=(0,0)$.
Further, let $(u, f) \in \mathrm{Gr} \mathbf{P}$, i.e., $u \in \mathrm{D} \mathbf{P}$ and $f=\mathbf{P} u$. Let us represent the function $f$ in the form

$$
f=f_{1}+f_{2},
$$

where $f_{1} \in \mathcal{R}\left(\mathbf{P}_{a}\right)$ and $f_{2} \in \mathcal{R}\left(\mathbf{P}_{a}\right)^{\perp}$. We choose an element $u_{1} \in \mathrm{D}\left(\mathbf{P}_{a}\right)$ such that $\mathbf{P}_{a} u_{1}=f_{1}$. Then $u_{2}=u-u_{1} \in \mathrm{D}(\mathbf{P})$ and $\mathbf{P} u_{2}=f_{2}$. Without loss of generality, we can assume that

$$
\begin{equation*}
u_{2} \perp \operatorname{ker} \mathbf{P}_{a} ; \tag{14.20}
\end{equation*}
$$

if this relation does not hold, then we consider the projection $u_{2 a}$ of the function $u_{2}$ to $\operatorname{ker} \mathbf{P}_{a}$ and replace $u_{1}$ by $u_{1}+u_{2 a}$ and $u_{2}$ by $u_{2}-u_{2 a}$. Obviously, $\left(u_{1}, f_{1}\right) \in \operatorname{Gr} \mathbf{P}_{a}$; taking into account Eq. (14.20), we have $\left(u_{2}, f_{2}\right) \in \mathcal{N} \times \mathcal{R}\left(\mathbf{P}_{a}\right)^{\perp}$. Thus, we have proved relation (14.19) and the lemma itself.

Theorem 13.1 follows from Lemmas 14.3 and 14.4.

## 15. Stability of the Index of a Differential Operator under Perturbations by Minor Terms

15.1. Passage to weight spaces. Let us introduce the operator

$$
\begin{equation*}
P^{\prime}(y, D)=\sum_{|\alpha| \leq 2 m-1} p_{\alpha}^{\prime}(y) D^{\alpha} \tag{15.1}
\end{equation*}
$$

corresponding to minor terms, where $p_{\alpha}^{\prime} \in C^{\infty}\left(\mathbb{R}^{2}\right)$. Consider the perturbed operator

$$
\mathbf{P}^{\prime}: \mathrm{D}\left(\mathbf{P}^{\prime}\right) \subset L_{2}(G) \rightarrow L_{2}(G)
$$

acting by the formula

$$
\begin{gathered}
\mathbf{P}^{\prime} u=\mathbf{P}(y, D) u+P^{\prime}(y, D) u \\
u \in \mathrm{D}\left(\mathbf{P}^{\prime}\right)=\left\{u \in L_{2}(G): u \text { satisfies Eq. }(13.21), \mathbf{B}_{i \mu} u=0, \mathbf{P}(y, D) u+P^{\prime}(y, D) u \in L_{2}(G)\right\} .
\end{gathered}
$$

According to Theorem 13.1, the unbounded operator $\mathbf{P}^{\prime}$ is a Fredholm operator (as well as $\mathbf{P}$ ). Let us formulate the main result of this section (see the proof in Sec. 15.2).
Theorem 15.1. ind $\mathbf{P}^{\prime}=\operatorname{ind} \mathbf{P}$.
Thus, minor terms in Eq. (13.25) do not affect the index of the unbounded operator $\mathbf{P}$. The difficulty is that, generally speaking, minor terms are neither compact perturbations nor $\mathbf{P}$-compact perturbations in the sense of Definition 1.2. If $\ell=2 m-1$, then the embedding $u \in \mathrm{D}(\mathbf{P})$ yields only $u \in W^{2 m-1}(G)$. This guarantees the $\mathbf{P}$-boundedness of the perturbation but not its $\mathbf{P}$-compactness. However, if $\ell<2 m-1$, then the embedding $u \in \mathrm{D}(\mathbf{P})$ does not yield the embedding $u \in W^{2 m-1}(G)$, and the perturbation is not even $\mathbf{P}$-bounded. Moreover, in this case, $\mathrm{D}\left(\mathbf{P}^{\prime}\right) \neq \mathrm{D}(\mathbf{P})$.

To overcome this difficulty, we introduce the operator $\mathbf{Q}: \mathrm{D}(\mathbf{Q}) \subset L_{2}(G) \rightarrow H_{a}^{0}(G)$ acting by the formula

$$
\begin{gather*}
\mathbf{Q} u=\mathbf{P}(y, D) u \\
u \in \mathrm{D}(\mathbf{Q})=\left\{u \in L_{2}(G): u \text { satisfies }(13.21), \mathbf{B}_{i \mu} u=0, \mathbf{P}(y, D) u \in H_{a}^{0}(G)\right\}, \tag{15.2}
\end{gather*}
$$

where $0 \leq 2 m-\ell-1<a<2 m-\ell$. We prove that ind $\mathbf{Q}=\operatorname{ind} \mathbf{P}$. On the other hand, we show that the operator $P^{\prime}(y, D)$ is a $\mathbf{Q}$-compact perturbation and hence it does not affect the index of the operators $\mathbf{Q}$ and $\mathbf{P}$.

Lemma 15.1. Let the line $\operatorname{Im} \lambda=a+1-2 m(2 m-\ell-1<a<2 m-\ell)$ not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then $\mathbf{Q}$ is a Fredholm operator and ind $\mathbf{Q}=\operatorname{ind} \mathbf{P}$.

Proof. 1. Consider the operator $\mathbf{L}_{a}$ defined by the formula (14.15) for $2 m-\ell-1<a<2 m-\ell$. According to Theorem 9.1, this operator is a Fredholm operator. Hence, by virtue of the compactness of the embedding $H_{a}^{2 m}(G) \subset L_{2}(G)$ (see [53, Lemma 3.5]), taking into account [56, Theorem 7.1], we have

$$
\begin{equation*}
\|u\|_{H_{a}^{2 m}(G)} \leq k_{1}\left(\left\|\mathbf{L}_{a} u\right\|_{\mathcal{H}_{a}^{0}(G, \partial G)+\mathcal{R}_{a}^{0}(G, \partial G)}+\|u\|_{L_{2}(G)}\right) . \tag{15.3}
\end{equation*}
$$

2. Let us introduce an unbounded operator $\dot{\mathbf{Q}}: \mathrm{D}(\dot{\mathbf{Q}}) \subset L_{2}(G) \rightarrow H_{a}^{0}(G)$ acting by the formula

$$
\begin{equation*}
\dot{\mathbf{Q}} u=\mathbf{P}(y, D) u, \quad u \in \mathrm{D}(\dot{\mathbf{Q}})=\left\{u \in H_{a}^{2 m}(G): \mathbf{B}_{i \mu} u=0\right\} . \tag{15.4}
\end{equation*}
$$

Since $H_{a}^{2 m}(G) \subset W^{\ell}(G)$ for $a<2 m-\ell$, we see that $\dot{\mathbf{Q}}$ is a reduction of the operator $\mathbf{Q}$, i.e., $\dot{\mathbf{Q}} \subset \mathbf{Q}$.
First, let us prove that $\dot{\mathbf{Q}}$ is a Fredholm operator. Let $u \in \mathrm{D}(\dot{\mathbf{Q}})$. Then $u \in \mathrm{D}\left(\mathbf{L}_{a}\right)=H_{a}^{2 m}(G)$, $\mathbf{P}(y, D) u \in H_{a}^{0}(G)$, and $\mathbf{B}_{i \mu} u=0$. Hence, Eq. (15.3) takes the form

$$
\begin{equation*}
\|u\|_{H_{a}^{2 m}(G)} \leq k_{1}\left(\|\dot{\mathbf{Q}} u\|_{H_{a}^{0}(G)}+\|u\|_{L_{2}(G)}\right) \quad \forall u \in \mathrm{D}(\dot{\mathbf{Q}}) \tag{15.5}
\end{equation*}
$$

It follows from (15.5) that the operator $\dot{\mathbf{Q}}$ is closed, $\operatorname{dim} \operatorname{ker} \dot{\mathbf{Q}}<\infty$, and $\mathcal{R}(\dot{\mathbf{Q}})=\overline{\mathcal{R}(\dot{\mathbf{Q}})}$ (to obtain the last two properties, one should apply [56, Theorem 7.1]).

Let us prove that $\operatorname{codim} \mathcal{R}(\dot{\mathbf{Q}})<\infty$. Since $\mathbf{L}$ is a Fredholm operator, there exist linearly independent functions $F_{1}, \ldots, F_{d} \in H_{a}^{0}(G)$ such that the function $f \in H_{a}^{0}(G)$ belongs to the image of the operator $\dot{\mathbf{Q}}$ if and only if $\left(f, F_{j}\right)_{H_{a}^{0}(G)}=0, j=1, \ldots, d$. Thus, $\dot{\mathbf{Q}}$ is a Fredholm operator.
3. Now we prove that $\mathbf{Q}$ is a Fredohlm operator. Since $\operatorname{ker} \mathbf{Q}=\operatorname{ker} \mathbf{P}$ and $\mathbf{P}$ is a Fredholm operator, we obtain

$$
\begin{equation*}
\operatorname{dim} \operatorname{ker} \mathbf{Q}=\operatorname{dim} \operatorname{ker} \mathbf{P}<\infty \tag{15.6}
\end{equation*}
$$

On the other hand, $\mathbf{Q}$ is an extension of the Fredholm operator $\dot{\mathbf{Q}} ;$ hence,

$$
\begin{equation*}
\mathcal{R}(\mathbf{Q})=\overline{\mathcal{R}(\mathbf{Q})}, \quad \operatorname{codim} \mathcal{R}(\mathbf{Q})<\infty \tag{15.7}
\end{equation*}
$$

Thus, $\mathbf{Q}$ is an extension of the Fredholm operator $\dot{\mathbf{Q}}$ and Eqs. (15.6) and (15.7) hold. Reasoning similar to that of item 2 of the proof of Lemma 14.4 shows that $\mathbf{Q}$ is a Fredholm operator.
4. By virtue of (15.6), we must prove that $\operatorname{codim} \mathcal{R}(\mathbf{Q})=\operatorname{codim} \mathcal{R}(\mathbf{P})$.

Let $\operatorname{codim} \mathcal{R}(\mathbf{Q})=d_{1}$, where $d_{1} \leq d$. Let us consider $f \in L_{2}(G)$. Then $f \in \mathcal{R}(\mathbf{P})$ if and only if $f \in \mathcal{R}(\mathbf{Q})$, owing to $L_{2}(G) \subset H_{a}^{0}(G)$. However, the embedding $f \in \mathcal{R}(\mathbf{Q})$ is equivalent to the expressions $\left(f, F_{j}\right)_{H_{a}^{0}(G)}=0, j=1, \ldots, d_{1}$, where $F_{1}, \ldots, F_{d_{1}} \in H_{a}^{0}(G)$ are linearly independent functions. Using the Schwartz inequality, the boundedness of the embedding $L_{2}(G) \subset H_{a}^{0}(G)$, and the Riesz theorem, we see that these expressions are equivalent to the following: $\left(f, f_{j}\right)_{L_{2}(G)}=0$, $j=1, \ldots, d_{1}$, where $f_{j} \in L_{2}(G)$. Moreover, the functions $f_{1}, \ldots, f_{d_{1}}$ are linearly independent. (In the opposite case, some linear combination of the functions $F_{1}, \ldots, F_{d_{1}}$ is orthogonal to any function from $L_{2}(G)$ in $H_{a}^{0}(G)$. This is impossible since $F_{1}, \ldots, F_{d_{1}}$ are linearly independent and $L_{2}(G)$ is dense in $H_{a}^{0}(G)$.) Thus, we have proved that $\operatorname{codim} \mathcal{R}(\mathbf{P})=d_{1}$.

Let us introduce the perturbed operator $\mathbf{Q}^{\prime}: \mathrm{D}\left(\mathbf{Q}^{\prime}\right) \subset L_{2}(G) \rightarrow H_{a}^{0}(G)$ acting by the formula

$$
\begin{gathered}
\mathbf{Q}^{\prime} u=\mathbf{P}(y, D) u+P^{\prime}(y, D) u \\
u \in \mathrm{D}\left(\mathbf{Q}^{\prime}\right)=\left\{u \in L_{2}(G): u \text { satisfies Eq. }(13.21), \mathbf{B}_{i \mu} u=0, \mathbf{P}(y, D) u+P^{\prime}(y, D) u \in H_{a}^{0}(G)\right\} .
\end{gathered}
$$

In the next subsection, we will show that ind $\mathbf{Q}^{\prime}=\operatorname{ind} \mathbf{Q}$ if there are no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ on the line $\operatorname{Im} \lambda=a+1-2 m$. Then, we will prove Theorem 15.1 by using the discreteness of the spectrum of the operator $\tilde{\mathcal{L}}(\lambda)$ and Lemma 15.1.

### 15.2. Compactness of minor terms in weight spaces.

Lemma 15.2. Let the line $\operatorname{Im} \lambda=a+1-2 m(2 m-\ell-1 \leq a \leq 2 m-\ell)$ not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then

$$
\|u\|_{W^{\ell}(G)} \leq c\left(\|\mathbf{Q} u\|_{H_{a}^{0}(G)}+\|u\|_{L_{2}(G)}\right) \quad \forall u \in \mathrm{D}(\mathbf{Q}) .
$$

Proof. Obviously, we must prove only the case where $\ell \geq 1$. Consider an unbounded operator $\hat{\mathbf{Q}}$ : $\mathrm{D}(\hat{\mathbf{Q}}) \subset W^{\ell}(G) \rightarrow H_{a}^{0}(G)$ acting by the formula $\hat{\mathbf{Q}} u=\mathbf{P}(y, D) u, u \in \mathrm{D}(\hat{\mathbf{Q}})=\mathrm{D}(\mathbf{Q})$. Since $\mathbf{Q}$ is a Fredholm operator, then $\hat{\mathbf{Q}}$ is also a Fredholm operator. Hence, the required estimate follows from the embedding $W^{\ell}(G) \subset L_{2}(G)(\ell \geq 1)$ and [56, Theorem 7.1].

Introduce a function $\psi_{j} \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ that is equal to 1 in a small neighborhood of the point $g_{j} \in \mathcal{K}$ and vanishes outside a larger neighborhood of the point $g_{j}$. The following lemma describes the behavior of the functions $u \in \mathrm{D}(\mathbf{Q})$ near the set $\mathcal{K}$.
Lemma 15.3. Let $2 m-\ell-1<a<2 m-\ell$ and a number a be sufficiently close to $2 m-\ell$. Then for any function $u \in \mathrm{D}(\mathbf{Q})$, we have

$$
\begin{equation*}
u(y)=\sum_{j=1}^{N} P_{j}(y)+v(y) \tag{15.8}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{j}(y)=\psi_{j}(y) \sum_{|\alpha| \leq \ell-1} p_{j \alpha}\left(y-g_{j}\right)^{\alpha}, \quad p_{j \alpha} \in \mathbb{C} \tag{15.9}
\end{equation*}
$$

and $v \in H_{2 m-\ell}^{2 m}(G)$ (if $\ell=0$, then we assume that $\left.P_{j}(y) \equiv 0\right)$; moreover,

$$
\begin{equation*}
\sum_{j, \alpha}\left|p_{j \alpha}\right|+\|v\|_{H_{2 m-\ell}^{2 m}(G)} \leq c\left(\|\mathbf{Q} u\|_{H_{a}^{0}(G)}+\|u\|_{L_{2}(G)}\right) . \tag{15.10}
\end{equation*}
$$

Proof. 1. It follows from inequality (13.22) that

$$
\begin{equation*}
\|u\|_{W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\delta}(\mathcal{K})}\right)} \leq k_{1 \delta}\left(\|\mathbf{Q} u\|_{H_{a}^{0}(G)}+\|u\|_{L_{2}(G)}\right) \quad \forall \delta>0 \tag{15.11}
\end{equation*}
$$

where $k_{1 \delta}$ is independent of $u$. Hence, it suffices to study the behavior of $u$ near the set $\mathcal{K}$.
2. Let $U_{j}\left(y^{\prime}\right)=u_{j}\left(y\left(y^{\prime}\right)\right), j=1, \ldots, N$, be functions corresponding to the set (orbit) $\mathcal{K}$ and satisfying Eqs. (13.27), (13.28) with the right-hand side $\left\{f_{j}, f_{j \sigma \mu}\right\}$.

By virtue of (15.11) and (6.5) (for $l=0$ ), we have $\left\{f_{j \sigma \mu}\right\} \in \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$. Since $\left\{f_{j}\right\} \in \mathcal{H}_{a}^{0}\left(K^{\varepsilon}\right)$, we have

$$
\begin{gather*}
\left\{f_{j}\right\} \in \mathcal{H}_{2 m}^{0}\left(K^{\varepsilon}\right), \quad\left\{f_{j \sigma \mu}\right\} \in \mathcal{H}_{2 m}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \\
\left\|\left\{f_{j}\right\}\right\|_{\mathcal{H}_{2 m}^{0}\left(K^{\varepsilon}\right)}+\left\|\left\{f_{j \sigma \mu}\right\}\right\|_{\mathcal{H}_{2 m}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)} \leq k_{2}\left(\|\mathbf{Q} u\|_{H_{a}^{0}(G)}+\|u\|_{L_{2}(G)}\right) \tag{15.12}
\end{gather*}
$$

Then $U \in \mathcal{W}^{\ell}\left(K^{\varepsilon_{1}}\right)$ and hence

$$
\begin{equation*}
U \in \mathcal{H}_{0}^{0}\left(K^{\varepsilon_{1}}\right) \tag{15.13}
\end{equation*}
$$

It follows from (15.11)-(15.13) and Lemma 14.2 that

$$
\begin{gather*}
U \in \mathcal{H}_{2 m}^{2 m}\left(K^{\varepsilon_{1}}\right) \\
\|U\|_{\mathcal{H}_{2 m}^{2 m}\left(K^{\varepsilon_{1}}\right)} \leq k_{3}\left(\|\mathbf{Q} u\|_{H_{a}^{0}(G)}+\|u\|_{L_{2}(G)}\right) . \tag{15.14}
\end{gather*}
$$

Assume that $\ell \geq 1$ (the case where $\ell=0$ is obvious). We prove that

$$
U=Q+\hat{U}
$$

where $\hat{U} \in \mathcal{H}_{2 m-\ell}^{2 m}\left(K^{\varepsilon}\right)$ and $Q=\left(Q_{1}, \ldots, Q_{N}\right)$ is a vector-valued polynomial of degree $\ell-1$ (if $\ell=0$, then there is no vector-valued polynomial $Q$ ). Inequality (15.10) follows from inequalities in (15.12) and (15.14) and the continuous dependence of the coefficients in asymptotic expansions below and the norm $\|\hat{U}\|_{\mathcal{H}_{2 m-\ell}^{2 m}\left(K^{\varepsilon_{1}}\right)}$ on the norms $\|U\|_{\mathcal{H}_{2 m}^{2 m}\left(K^{\varepsilon_{1}}\right)}$ and $\left\|\left\{f_{j}\right\}\right\|_{\mathcal{H}_{2 m}^{0}\left(K^{\varepsilon}\right)}$ and $\left\|\left\{f_{j \sigma \mu}\right\}\right\|_{\mathcal{H}_{2 m}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)}$.
3. Let $\delta=2 m-\ell-a$. Obviously, $0<\delta<1$. By Lemma 5.3, for every function $f_{j \sigma \mu} \in \mathcal{W}^{2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}^{\varepsilon}\right)$, there exists a polynomial $P_{j \sigma \mu}(r)$ of degree $2 m-m_{j \sigma \mu}-2$ (if $m_{j \sigma \mu}=2 m-1$, then $\left.P_{j \sigma \mu}(r) \equiv 0\right)$ such that

$$
\left\{f_{j \sigma \mu}-P_{j \sigma \mu}\right\} \in \mathcal{H}_{2 m-\ell-\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)
$$

(in this expression, the number $2 m-\ell-\delta$ can be substituted by any positive number). Using [26, Lemma 4.3], we construct a function

$$
\begin{equation*}
W^{1}=\sum_{s=0}^{\ell-1} \sum_{l=0}^{l_{1}} r^{s}(i \ln r)^{l} \varphi_{s l}^{1}(\omega) \in \mathcal{H}_{2 m}^{2 m}\left(K^{\varepsilon}\right), \tag{15.15}
\end{equation*}
$$

where $\varphi_{s l}^{1} \in \mathcal{W}^{2 m}(-\bar{\omega}, \bar{\omega})$, such that

$$
\left\{\mathbf{P}_{j}(y, D) W_{j}^{1}\right\} \in \mathcal{H}_{2 m-\ell-\delta}^{0}\left(K^{\varepsilon}\right), \quad\left\{\mathbf{B}_{j \sigma \mu}(y, D) W^{1}-P_{j \sigma \mu}\right\} \in \mathcal{H}_{2 m-\ell-\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)
$$

Hence,

$$
\left\{\mathbf{P}_{j}(y, D)\left(U_{j}-W_{j}^{1}\right)\right\} \in \mathcal{H}_{2 m-\ell-\delta}^{0}\left(K^{\varepsilon}\right), \quad\left\{\mathbf{B}_{j \sigma \mu}(y, D)\left(U_{j}-W^{1}\right)\right\} \in \mathcal{H}_{2 m-\ell-\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)
$$

It follows from (15.14) and (15.15) that $U-W^{1} \in \mathcal{H}_{2 m}^{2 m}\left(K^{\varepsilon}\right)$. By virtue of Lemma 6.1, we can choose a number $\delta, 0<\delta<1$, such that there are no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ in the strip $1-\ell-\delta \leq \operatorname{Im} \lambda<1-\ell$. Then, using [26, Theorem 2.2, Lemma 4.3], we obtain the following equality:

$$
U-W^{1}=W^{2}+\hat{U}
$$

where

$$
W^{2}=\sum_{n=1}^{n_{0}} \sum_{l=0}^{l_{2}} r^{i \mu_{n}}(i \ln r)^{l} \varphi_{n l}^{2}(\omega),
$$

$\left\{\mu_{1}, \ldots, \mu_{n_{0}}\right\}$ is the set of all eigenvalues that are located in the strip $1-\ell \leq \operatorname{Im} \lambda<1$ (actually, one should take eigenvalues from the strip $1-\ell-\delta \leq \operatorname{Im} \lambda<1$, but according to the choice of $\delta$, the strip $1-\ell-\delta \leq \operatorname{Im} \lambda<1-\ell$ does not contain any eigenvalue), $\varphi_{n l}^{2} \in \mathcal{W}^{2 m}(-\bar{\omega}, \bar{\omega})$ and $\hat{U} \in \mathcal{H}_{2 m-\ell-\delta}^{2 m}\left(K^{\varepsilon}\right) \subset$ $\mathcal{H}_{2 m-\ell}^{2 m}\left(K^{\varepsilon}\right)$.

Since $s \leq \ell-1$ (in the formula for $W^{1}$ ), $\operatorname{Re} i \mu_{n} \leq \ell-1$ (in the formula for $W^{2}$ ), and $W^{1}+W^{2}=U-\hat{U} \in \mathcal{W}^{\ell}\left(K^{\varepsilon}\right)$, we see, by virtue of [53, Lemma 4.20], that the function $W^{1}+W^{2}$ is a vector-valued polynomial of degree $\ell-1$.

The following result follows from Lemma 15.3.
Corollary 15.1. Let a number a satisfy the conditions of Lemma 15.3 and let $P^{\prime}(y, D)$ be a differential operator of order $2 m-1$ of the form (15.1). Then

$$
\begin{equation*}
\left\|P^{\prime}(y, D) u\right\|_{H_{2 m-\ell}^{1}(G)} \leq c\left(\|\mathbf{Q} u\|_{H_{a}^{0}(G)}+\|u\|_{L_{2}(G)}\right) \quad \forall u \in \mathrm{D}(\mathbf{Q}) \tag{15.16}
\end{equation*}
$$

Then we can prove that minor terms in Eq. (13.25) do not affect the index of the operator $\mathbf{Q}$.
Lemma 15.4. Let a number a satisfy the conditions of Lemma 15.1 and 15.3. Then the operators $\mathbf{Q}$ and $\mathbf{Q}^{\prime}$ are Fredholm operators and ind $\mathbf{Q}^{\prime}=\operatorname{ind} \mathbf{Q}$.
Proof. By Lemma 15.1, $\mathbf{Q}$ and $\mathbf{Q}^{\prime}$ are Fredholm operators.
Introduce the operator $P^{\prime}: \mathrm{D}\left(P^{\prime}\right) \subset L_{2}(G) \rightarrow H_{a}^{0}(G)$ acting by the formula $P^{\prime} u=P^{\prime}(y, D) u$, $u \in \mathrm{D}\left(P^{\prime}\right)=\mathrm{D}(\mathbf{Q})$. It follows from Corollary 15.1 and the compactness of the embedding $H_{2 m-\ell}^{1}(G) \subset$ $H_{a}^{0}(G)$ (see [53, Lemma 3.5]) that $\mathbf{Q}^{\prime}=\mathbf{Q}+P^{\prime}$ and $P^{\prime}$ is a $\mathbf{Q}$-compact operator. Hence, according to [49, Chap. 4, Theorem 5.26], we have ind $\mathbf{Q}^{\prime}=$ ind $\mathbf{Q}$.

Proof of Theorem 15.1. Lemma 6.1 implies that the spectrum of the operator $\tilde{\mathcal{L}}(\lambda)$ is discrete. Thus, there is a number $a$ satisfying the conditions of Lemma 15.3. Then, by virtue of Lemmas 15.1 and 15.4, $\operatorname{ind} \mathbf{P}^{\prime}=\operatorname{ind} \mathbf{Q}^{\prime}=\operatorname{ind} \mathbf{Q}=\operatorname{ind} \mathbf{P}$.

## 16. Stability of the Index under Perturbations of Nonlocal Conditions

16.1. Statement of the main result. In this section, we study the stability of the index of nonlocal operators under perturbations of nonlocal conditions. They are perturbed by operators of the same type as $\mathbf{B}_{i \mu}^{1}$ and $\mathbf{B}_{i \mu}^{2}$. This situation is more complicated than the situation considered in Sec. 15 since nonlocal perturbations explicitly change the domains of the corresponding unbounded operators. Hence, these perturbations cannot be considered as relatively compact. We offer a different approach based on the notion of a spread between closed operators.

Consider $m_{i \mu}$-ordered differential operators $C_{i \mu s}(y, D), i=1, \ldots, N, \mu=1, \ldots, m, s=1, \ldots, S_{i}^{\prime}$, of the same order as the operators $B_{i \mu s}$ and acting by the formula

$$
C_{i \mu s}(y, D) u=\sum_{|\alpha| \leq m_{i \mu}} c_{i \mu s \alpha}(y) D^{\alpha} u
$$

where $c_{i \mu s \alpha} \in C^{\infty}\left(\mathbb{R}^{2}\right)$. Introduce the operator $\mathbf{C}_{i \mu}^{1}$ by the formula

$$
\mathbf{C}_{i \mu}^{1} u=\sum_{s=1}^{S_{i}^{\prime}}\left(C_{i \mu s}(y, D)(\zeta u)\right)\left(\Omega_{i s}^{\prime}(y)\right), \quad y \in \Gamma_{i} \cap \mathcal{O}_{\varepsilon}(\mathcal{K}), \quad \mathbf{C}_{i \mu}^{1} u=0, \quad y \in \Gamma_{i} \backslash \mathcal{O}_{\varepsilon}(\mathcal{K}),
$$

where $\zeta$ and $\varepsilon$ are the same as in the definition of the operators $\mathbf{B}_{i \mu}^{1}$, and $\Omega_{i s}^{\prime}$ are the $C^{\infty}{ }_{-}$ diffeomorphisms with the same properties as $\Omega_{i s}$ (in particular, they satisfy Condition 6.3 , where $S_{i}$ and $\Omega_{i s}$ must be replaced by $S_{i}^{\prime}$ and $\Omega_{i s}^{\prime}$ ).

Consider the operators $\mathbf{C}_{i \mu}^{2}$ satisfying condition 6.4 with $l=0$; here $\mathbf{B}_{i \mu}^{2}$ must be replaced by $\mathbf{C}_{i \mu}^{2}$. Assume that

$$
\mathbf{C}_{i \mu}=\mathbf{C}_{i \mu}^{1}+\mathbf{C}_{i \mu}^{2} .
$$

In this section, we assume that the number $a$ satisfies the conditions of Lemma 15.3. In particular,

$$
2 m-\ell-1<a<2 m-\ell .
$$

We prove a theorem on the stability of the index under the following auxiliary conditions (see, e.g., [57, Chap. 2, Sec. 1]) that are assumed to be valid everywhere in this section (including the conditions of lemmas).
Condition 16.1. The system $\left\{\mathbf{B}_{i \mu}^{0}\right\}_{\mu=1}^{m}$ is normal on $\overline{\Gamma_{i}}, i=1, \ldots, N$.
Let $g_{i 1}, g_{i 2}, \tau_{i 1}, \tau_{i 2}, D_{\tau_{i 1}}^{\beta}$, and $D_{\tau_{i 2}}^{\beta}$ have the same sense as in Sec. 5.2.
Condition 16.2. If $\ell \geq m_{i \mu}-|\alpha|+1\left(|\alpha| \leq m_{i \mu}\right)$, then

$$
D^{\sigma} c_{i \mu s \alpha}\left(g_{i 1}\right)=D^{\sigma} c_{i \mu s \alpha}\left(g_{i 2}\right)=0, \quad|\sigma|=0, \ldots,(\ell-1)-\left(m_{i \mu}-|\alpha|\right) .
$$

Condition 16.3. If $\ell \geq m_{i \mu}+1$, then for any function $u \in W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$, the following relations hold:

$$
\left.D_{\tau_{i 1}}^{\beta}\left(\mathbf{C}_{i \mu}^{2} u\right)\right|_{y=g_{i 1}}=0,\left.\quad D_{\tau_{i 2}}^{\beta}\left(\mathbf{C}_{i \mu}^{2} u\right)\right|_{y=g_{i 2}}=0, \quad \beta=0, \ldots, \ell-1-m_{i \mu} .
$$

Remark 16.1. If we increase the index $\ell$, then conditions 16.2 and 16.3 become stricter. If, for example, we consider a nonlocal perturbation of the Dirichlet problem for the second-order equation (i.e., $m=1, m_{i \mu}=0$, and $\mu=1$ ) and find generalized solutions from $L_{2}(G)$ (i.e., $\ell=0$ ), then conditions 16.2 and 16.3 are satisfied automatically for any operators $\mathbf{C}_{i \mu}^{1}$ and $\mathbf{C}_{i \mu}^{2}$.

Lemma 16.1. Let conditions 16.2 and 16.3 hold. Then

$$
\begin{gather*}
\left\|\mathbf{C}_{i \mu}^{1} u\right\|_{H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)} \leq c_{1}\|u\|_{H_{a+\ell}^{2 m}(G)},  \tag{16.1}\\
\left\|\mathbf{C}_{i \mu}^{2} u\right\|_{H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)} \leq c_{2}\|u\|_{W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)} . \tag{16.2}
\end{gather*}
$$

Proof. 1. For any function $u \in H_{a+\ell}^{2 m}(G)$, we have

$$
\left.\left(D^{\alpha} u\right)\left(\Omega_{i s}^{\prime}(y)\right)\right|_{\Gamma_{i}} \in H_{a+\ell}^{2 m-|\alpha|-1 / 2}\left(\Gamma_{i}\right) \subset H_{a+\ell-\left(m_{i \mu}-|\alpha|\right)}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right) .
$$

Hence, by virtue of condition 16.2 and Lemma 5.6 we have

$$
\left.\left(c_{i \mu s \alpha} D^{\alpha} u\right)\left(\Omega_{i s}^{\prime}(y)\right)\right|_{\Gamma_{i}} \in H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right) .
$$

Estimate (16.1) follows from the boundedness of the mentioned embeddings and inequality (5.14).
2. Condition 6.4 (concerning $\mathbf{C}_{i \mu}^{2}$ ) implies that $\mathbf{C}_{i \mu}^{2} u \in W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ if the expression

$$
u \in W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)
$$

is valid. Now it follows from condition 16.3 and Lemma 5.4 that

$$
\mathbf{C}_{i \mu}^{2} u \in H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right) .
$$

Estimate (16.2) follows from Inequality (6.5) (that is applied to $\mathbf{C}_{i \mu}^{2}$ as $l=0$ ) and from (5.11).
Consider the operators $\mathbf{P}_{t}: \mathrm{D}\left(\mathbf{P}_{t}\right) \subset L_{2}(G) \rightarrow L_{2}(G), t \in \mathbb{C}$, acting by the formula

$$
\mathbf{P}_{t} u=\mathbf{P}(y, D) u
$$

$$
u \in \mathrm{D}\left(\mathbf{P}_{t}\right)=\left\{u \in L_{2}(G): u \text { satisfies (13.21), }\left(\mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+t \mathbf{C}_{i \mu}\right) u=0, \mathbf{P}(y, D) u \in L_{2}(G)\right\}
$$

Let us formulate the main result of this section (the proof will be given in Sec. 16.2).
Theorem 16.1. Let conditions $16.1-16.3$ hold. Then ind $\mathbf{P}_{t}=\operatorname{const}$ for all $t \in \mathbb{C}$.
16.2. Spread between nonlocal operators in weight spaces. As in Sec. 15, we first study the operators $\mathbf{Q}_{t}: \mathrm{D}\left(\mathbf{Q}_{t}\right) \subset L_{2}(G) \rightarrow H_{a}^{0}(G)$ acting by formula

$$
\begin{gathered}
\mathbf{Q}_{t} u=\mathbf{P}(y, D) u \\
u \in \mathrm{D}\left(\mathbf{Q}_{t}\right)=\left\{u \in L_{2}(G): u \text { satisfies }(13.21),\left(\mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+t \mathbf{C}_{i \mu}\right) u=0, \mathbf{P}(y, D) u \in H_{a}^{0}(G)\right\},
\end{gathered}
$$

where $t \in \mathbb{C}$. The operators $\mathbf{P}_{t}$ and $\mathbf{Q}_{t}$ correspond the following problem:

$$
\begin{align*}
\mathbf{P}(y, D) u & =f(y), \quad y \in G  \tag{16.3}\\
\left(\mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+t \mathbf{C}_{i \mu}\right) u=0, \quad y & \in \Gamma_{i}, \quad i=1, \ldots, N, \quad \mu=1, \ldots, m . \tag{16.4}
\end{align*}
$$

Remark 16.2. In the definition of the operator $\tilde{\mathcal{L}}(\lambda)$ (see Sec. 6.3), we considered the principal homogeneous parts of the operators $\mathbf{P}(y, D)$ and $B_{i \mu s}(y, D)$ at points of the set $\mathcal{K}$. By virtue of 16.2, the principal homogeneous parts of the operators $C_{i \mu s}(y, D)$ vanish at these points. Hence, for any $t$, the same operator $\tilde{\mathcal{L}}(\lambda)$ corresponds to problem (16.3), (16.4).

It follows from Remark 16.2 and Lemma 15.1 that $\mathbf{Q}_{t}$ is a Fredholm operator for chosen $a$. Hence, its graph $\operatorname{Gr} \mathbf{Q}_{t}$ is a closed subset in the Hilbert space $L_{2}(G) \times H_{a}^{0}(G)$; we equip this space with the following norm:

$$
\|(u, f)\|=\left(\|u\|_{L_{2}(G)}^{2}+\|f\|_{H_{a}^{0}(G)}^{2}\right)^{1 / 2} \quad \forall(u, f) \in L_{2}(G) \times H_{a}^{0}(G) .
$$

Assume

$$
\begin{equation*}
\delta\left(\mathbf{Q}_{t}, \mathbf{Q}_{t+s}\right)=\sup _{u \in \mathrm{D}\left(\mathbf{Q}_{t}\right):\left\|\left(u, \mathbf{Q}_{t} u\right)\right\|=1} \operatorname{dist}\left(\left(u, \mathbf{Q}_{t} u\right), \operatorname{Gr} \mathbf{Q}_{t+s}\right) . \tag{16.5}
\end{equation*}
$$

By Definition 1.3, the number

$$
\hat{\delta}\left(\mathbf{Q}_{t}, \mathbf{Q}_{t+s}\right)=\max \left\{\delta\left(\mathbf{Q}_{t}, \mathbf{Q}_{t+s}\right), \delta\left(\mathbf{Q}_{t+s}, \mathbf{Q}_{t}\right)\right\}
$$

is the spread between the operators $\mathbf{Q}_{t}$ and $\mathbf{Q}_{t+s}$.
The proof of the theorem on the stability of the index is based on [49, Chap. 4, Theorem 5.17] and the following result (it will be proved below).

Theorem 16.2. Let conditions 16.1-16.3 hold. Assume that the lines $\operatorname{Im} \lambda=a+1-2 m$ and $\operatorname{Im} \lambda=a+1+\ell-2 m$ do not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then

$$
\begin{equation*}
\hat{\delta}\left(\mathbf{Q}_{t}, \mathbf{Q}_{t+s}\right) \leq c_{t} s, \quad|s| \leq s_{t}, \tag{16.6}
\end{equation*}
$$

where $s_{t}>0$ is sufficiently small and $c_{t}>0$ is independent of $s$.
First, let us prove some auxiliary statements.
Lemma 16.2. Let the line $\operatorname{Im} \lambda=a+1+\ell-2 m$ not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then

$$
\begin{equation*}
\|u\|_{H_{a+\ell}^{2 m}(G)} \leq c_{t}\|(u, \mathbf{P}(y, D) u)\| \quad \forall u \in \mathrm{D}\left(\mathbf{Q}_{t+s}\right), \tag{16.7}
\end{equation*}
$$

where $c_{t}>0$ is independent of $s$ and $u$, under the condition that $|s|$ is sufficiently small.
Proof. 1. Consider the bounded operator

$$
\begin{equation*}
\mathbf{M}_{t}=\left\{\mathbf{P}(y, D), \mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+t \mathbf{C}_{i \mu}\right\}: H_{a+\ell}^{2 m}(G) \rightarrow \mathcal{H}_{a+\ell}^{0}(G, \partial G) . \tag{16.8}
\end{equation*}
$$

If $v \in H_{a+\ell}^{2 m}(G)$, then

$$
\begin{aligned}
\left(\mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+t \mathbf{C}_{i \mu}^{1}\right) v & \in H_{a+\ell}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right), \\
\mathbf{C}_{i \mu}^{2} v \in W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right) & \subset H_{a+\ell}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)
\end{aligned}
$$

(this follows condition 6.4 and item 1 of Lemma 5.3). Thus, the operator $\mathbf{M}_{t}$ is well defined.
By virtue of Theorem 9.1 and Remark 16.2, $\mathbf{M}_{t}$ is a Fredholm operator for any $t \in \mathbb{C}$. Hence, using [56, Theorem 7.1] and taking into account the compactness of the embedding $H_{a+\ell}^{2 m} \subset L_{2}(G)$ for $a<2 m-\ell$ (see [53, Lemma 3.5]), we obtain the following inequality:

$$
\begin{equation*}
\|u\|_{H_{a+\ell}^{2 m}(G)} \leq k_{1}\left(\left\|\mathbf{M}_{t} u\right\|_{\mathcal{H}_{a+\ell}^{0}(G, \partial G)}+\|u\|_{L_{2}(G)}\right) \quad \forall u \in H_{a+\ell}^{2 m}(G), \tag{16.9}
\end{equation*}
$$

where $k_{1}>0$ can depend on $t$, but is independent of $s$ and $u$.
2. Now we consider a function $u \in \mathrm{D}\left(\mathbf{Q}_{t+s}\right)$. By Lemma 15.3, $u \in H_{a+\ell}^{2 m}(G)$. By virtue of inequality (16.9), estimate (6.5) (with $l=0$ for $\mathbf{C}_{i \mu}^{2}$ ) and the boundedness of the embedding

$$
W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right) \subset H_{a+\ell}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)
$$

(see item 1 of Lemma 5.3), we have

$$
\|u\|_{H_{a+\ell}^{2 m}(G)} \leq k_{1}\left(\|\mathbf{P}(y, D) u\|_{H_{a+\ell}^{0}(G)}+\|u\|_{L_{2}(G)}\right)+k_{2}|s| \cdot\|u\|_{H_{a+\ell}^{2 m}(G)} \quad \forall u \in \mathrm{D}\left(\mathbf{Q}_{t+s}\right),
$$

where $k_{2}>0$ can depend on $t$ but is independent of $s$ and $u$. Choosing $|s| \leq 1 / 2 k_{2}$ and using the boundedness of the embedding $H_{a}^{0}(G) \subset H_{a+\ell}^{0}(G)$, we obtain (16.7).

The next result follows from Lemmas 16.1 and 16.2.
Corollary 16.1. Let the line $\operatorname{Im} \lambda=a+1+\ell-2 m$ not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. Then

$$
\begin{equation*}
\left\|\mathbf{C}_{i \mu} u\right\|_{H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)} \leq c_{t}\|(u, \mathbf{P}(y, D) u)\| \quad \forall u \in \mathrm{D}\left(\mathbf{Q}_{t+s}\right) \tag{16.10}
\end{equation*}
$$

where $c_{t}>0$ is independent of $s$ and $u$, under the condition that $|s|$ is sufficiently small.

The following lemma allows one to reduce problems with inhomogeneous nonlocal conditions to problems with homogeneous nonlocal conditions; here we use 16.1.

Lemma 16.3. Let $f_{i \mu} \in H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$. Then for any $t \in \mathbb{C}$ and $|s| \leq 1$, there exists a function $u \in H_{a}^{2 m}(G)$ such that

$$
\begin{gather*}
\left(\mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+(t+s) \mathbf{C}_{i \mu}\right) u=f_{i \mu}  \tag{16.11}\\
\|u\|_{H_{a}^{2 m}(G)} \leq c_{t} \sum_{i, \mu}\left\|f_{i \mu}\right\|_{H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)}, \tag{16.12}
\end{gather*}
$$

where $c_{t}>0$ is independent of $f_{i \mu}$ and $s$.
Proof. Using Lemma 11.1 and the method of the partition of unity, we construct a function $v \in H_{a}^{2 m}(G)$ such that

$$
\begin{gather*}
\operatorname{supp} v \subset \bar{G} \backslash \overline{G_{\rho}},  \tag{16.13}\\
\mathbf{B}_{i \mu}^{0} v=f_{i \mu}, \quad \mathbf{B}_{i \mu}^{1} v=0, \quad \mathbf{C}_{i \mu}^{1} v=0,  \tag{16.14}\\
\|v\|_{H_{a}^{2 m}(G)} \leq k_{1} \sum_{i, \mu}\left\|f_{i \mu}\right\|_{H_{a}^{2 m-m_{i \mu}}-1 / 2}\left(\Gamma_{i}\right) \tag{16.15}
\end{gather*}
$$

where $k_{1}>0$ is independent of $f_{i \mu}, t$, and $s$.
By virtue of (16.13) and (6.6) (with the operator $\mathbf{C}_{i \mu}^{2}$ instead of $\mathbf{B}_{i \mu}^{2}$ and $l=0$ ),

$$
\operatorname{supp} \mathbf{C}_{i \mu}^{2} v \subset \mathcal{O}_{\varkappa_{2}}(\mathcal{K})
$$

Moreover, by Lemma 16.1,

$$
\mathbf{C}_{i \mu}^{2} v \in H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right) .
$$

Hence, applying Lemma 11.1 and the method of partition of unity, we can construct a function $w \in H_{a}^{2 m}(G)$ such that

$$
\begin{gather*}
\operatorname{supp} w \subset \mathcal{O}_{\varkappa_{1}}(\mathcal{K})  \tag{16.16}\\
\mathbf{B}_{i \mu}^{0} w=-(t+s) \mathbf{C}_{i \mu}^{2} v, \quad \mathbf{B}_{i \mu}^{1} w=0, \quad \mathbf{C}_{i \mu}^{1} w=0  \tag{16.17}\\
\|w\|_{H_{a}^{2 m}(G)}^{2} \leq k_{1} \sum_{i, \mu}\left\|(t+s) \mathbf{C}_{i \mu}^{2} v\right\|_{H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)}
\end{gather*}
$$

Since $|s| \leq 1$, we can apply inequalities (16.2) and (16.15); from the last inequality we obtain

$$
\begin{align*}
&\|w\|_{H_{a}^{2 m}(G)} \leq k_{1} \sum_{i, \mu}(|t|+1)\left\|\mathbf{C}_{i \mu}^{2} v\right\|_{H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)} \\
& \leq k_{2}\|v\|_{H_{a}^{2 m}(G)} \leq k_{2} k_{1} \sum_{i, \mu}\left\|f_{i \mu}\right\|_{H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)}, \tag{16.18}
\end{align*}
$$

where $k_{2}>0$ can depend on $t$ but is independent of $f_{i \mu}$ and $s$.
By virtue of (16.16) and (16.2), $\mathbf{C}_{i \mu}^{2} w=0$. This and Eqs. (16.14) and (16.17) imply that $u=v+w$ satisfies relations (16.11). Inequality (16.12) follows from inequalities (16.15) and (16.18).
Remark 16.3. It is easy to see that if $\left(\mathbf{C}_{i \mu}^{2} v\right)(y)=0$ for $y \in \mathcal{O}_{\varkappa}(\mathcal{K})$ for some $\varkappa>0$ and any $v \in W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$, then Lemma 16.3 is valid for all $a \in \mathbb{R}$.
Proof of Theorem 16.2. 1. We prove inequalities that are similar to inequality (16.6), where $\hat{\delta}\left(\mathbf{Q}_{t}, \mathbf{Q}_{t+s}\right)$ must be replaced by $\delta\left(\mathbf{Q}_{t}, \mathbf{Q}_{t+s}\right)$ and $\delta\left(\mathbf{Q}_{t+s}, \mathbf{Q}_{t}\right)$. Let us prove the inequality

$$
\begin{equation*}
\delta\left(\mathbf{Q}_{t}, \mathbf{Q}_{t+s}\right) \leq c_{t}|s|, \quad|s| \leq s_{t} . \tag{16.19}
\end{equation*}
$$

The proof of the corresponding inequality for $\delta\left(\mathbf{Q}_{t+s}, \mathbf{Q}_{t}\right)$ is similar.

Fix an arbitrary number $t$ and consider the function $u \in \mathrm{D}\left(\mathbf{Q}_{t}\right)$. According to Definition (16.5), it suffices to find a function $v_{s} \in \mathrm{D}\left(\mathbf{Q}_{t+s}\right)$ (depending on $\left.u\right)$ such that

$$
\begin{equation*}
\left\|u-v_{s}\right\|_{L_{2}(G)}+\left\|\mathbf{P}(y, D) u-\mathbf{P}(y, D) v_{s}\right\|_{H_{a}^{0}(G)} \leq k_{1}|s| \cdot\|(u, \mathbf{P}(y, D) u)\|, \tag{16.20}
\end{equation*}
$$

where $|s|$ is sufficiently small, and $k_{1}, k_{2}, \ldots>0$ can depend on $t$ but are independent of $u$ and $s$.
Let us find $v_{s} \in \mathrm{D}\left(\mathbf{Q}_{t+s}\right)$ in the form

$$
\begin{equation*}
v_{s}=u+w_{s}, \tag{16.21}
\end{equation*}
$$

where $w_{s} \in H_{a}^{2 m}(G)$ is a solution of problem

$$
\begin{equation*}
\mathbf{P}(y, D) w_{s}=\sum_{j=1}^{J_{s}} \beta_{j}^{s} f_{j}^{s}, \quad\left(\mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+(t+s) \mathbf{C}_{i \mu}\right) w_{s}=-s \mathbf{C}_{i \mu} u \tag{16.22}
\end{equation*}
$$

let us define numbers $J_{s}$ and $\beta_{j}^{s}$ and functions $f_{j}^{s} \in H_{a}^{0}(G)$ such that a solution $w_{s} \in H_{a}^{2 m}(G)$ exists.
2. To solve problem (16.22), we note that, by virtue of the Corollary 16.1, we have $\mathbf{C}_{i \mu} u \in$ $H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$. Hence, applying Lemma 16.3, we can construct a function $W_{s} \in H_{a}^{2 m}(G)$ such that

$$
\begin{gather*}
\left(\mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+(t+s) \mathbf{C}_{i \mu}\right) W_{s}=-s \mathbf{C}_{i \mu} u,  \tag{16.23}\\
\left\|W_{s}\right\|_{H_{a}^{2 m}(G)} \leq k_{2}|s| \sum_{i, \mu}\left\|\mathbf{C}_{i \mu} u\right\|_{H_{a}^{2 m-m_{i \mu}-1 / 2}{ }_{\left(\Gamma_{i}\right)}} . \tag{16.24}
\end{gather*}
$$

From (16.24) and (16.10) we obtain the following inequality:

$$
\begin{equation*}
\left\|W_{s}\right\|_{H_{a}^{2 m}(G)} \leq k_{3}|s| \cdot\|(u, \mathbf{P}(y, D) u)\| . \tag{16.25}
\end{equation*}
$$

Obviously, Problem (16.22) is equivalent to the following problem:

$$
\begin{equation*}
\mathbf{P}(y, D) Y_{s}=-\mathbf{P}(y, D) W_{s}+\sum_{j=1}^{J_{s}} \beta_{j}^{s} f_{j}^{s}, \quad\left(\mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+(t+s) \mathbf{C}_{i \mu}\right) Y_{s}=0 \tag{16.26}
\end{equation*}
$$

where

$$
\begin{equation*}
Y_{s}=w_{s}-W_{s} \in H_{a}^{2 m}(G) . \tag{16.27}
\end{equation*}
$$

3. To solve problem (16.26), we consider the bounded operator

$$
\begin{equation*}
\mathbf{L}_{t}=\left\{\mathbf{P}(y, D), \mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+t \mathbf{C}_{i \mu}\right\}: H_{a}^{2 m}(G) \rightarrow \mathcal{H}_{a}^{0}(G, \partial G) . \tag{16.28}
\end{equation*}
$$

Note that, by Lemma 16.1, $\mathbf{C}_{i \mu}^{2} v \in H_{a}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ for any $v \in H_{a}^{2 m}(G)$; therefore, in the definition of the operator $\mathbf{L}_{t}$, we can write $\mathcal{H}_{a}^{0}(G, \partial G)$ instead of $\mathcal{H}_{a}^{0}(G, \partial G)+\mathcal{R}_{a}^{0}(G, \partial G)$. It follows from Theorem 9.1 and Remark 16.2 that $\mathbf{L}_{t}$ is a Fredholm operator for any $t \in \mathbb{C}$.

Let us decompose the space $H_{a}^{2 m}(G)$ into the orthogonal sum $H_{a}^{2 m}(G)=\operatorname{ker} \mathbf{L}_{t} \oplus E_{t}$, where $E_{t}$ is a closed subspace in $H_{a}^{2 m}(G)$. It is obvious that

$$
\begin{equation*}
\mathbf{L}_{t}^{\prime}=\left\{\mathbf{P}(y, D), \mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+t \mathbf{C}_{i \mu}\right\}: E_{t} \rightarrow \mathcal{H}_{a}^{0}(G, \partial G) \tag{16.29}
\end{equation*}
$$

is a Fredholm operator and its kernel is trivial. This implies that

$$
\begin{equation*}
\|u\|_{H_{a}^{2 m}(G)} \leq k_{4}\left\|\mathbf{L}_{t}^{\prime} u\right\|_{\mathcal{H}_{a}^{0}(G, \partial G)} \quad \forall u \in E_{t} . \tag{16.30}
\end{equation*}
$$

Let $J=\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{t}^{\prime}\right)$. Lemma 16.1 and [56, Sec. 16] imply that the operator

$$
\mathbf{L}_{t s}^{\prime}=\left\{\mathbf{P}(y, D), \mathbf{B}_{i \mu}^{0}+\mathbf{B}_{i \mu}^{1}+(t+s) \mathbf{C}_{i \mu}\right\}: E_{t} \rightarrow \mathcal{H}_{a}^{0}(G, \partial G)
$$

is also a Fredholm operator, its kernel is trivial, and $\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{t s}^{\prime}\right)=J$ under the condition $|s| \leq s_{t}$, where $s_{t}>0$ is sufficiently small. Moreover, using estimates (16.30), (16.1), and (16.2), we have for
all $u \in E_{t}$

$$
\begin{array}{r}
\|u\|_{H_{a}^{2 m}(G)} \leq k_{4}\left(\left\|\mathbf{L}_{t s}^{\prime} u\right\|_{\mathcal{H}_{a}^{0}(G, \partial G)}+s_{t} \sum_{i, \mu}\left\|\mathbf{C}_{i \mu} u\right\|_{\left.H_{a}^{2 m-m_{i \mu}-1 / 2}{ }_{\left(\Gamma_{i}\right)}\right)}\right. \\
\leq k_{5}\left(\left\|\mathbf{L}_{t s}^{\prime} u\right\|_{\mathcal{H}_{a}^{0}(G, \partial G)}+s_{t}\|u\|_{H_{a}^{2 m}(G)}\right) \tag{16.31}
\end{array}
$$

Choosing $s_{t} \leq 1 /\left(2 k_{6}\right)$, we obtain the inequality

$$
\begin{equation*}
\|u\|_{H_{a}^{2 m}(G)} \leq k_{6}\left\|\mathbf{L}_{t s}^{\prime} u\right\|_{\mathcal{H}_{a}^{0}(G, \partial G)} \quad \forall u \in E_{t} . \tag{16.32}
\end{equation*}
$$

Since $\mathbf{L}_{t s}^{\prime}$ is a Fredholm operator, we see that the set $\left\{f \in H_{a}^{0}(G):(f, 0) \in \mathcal{R}\left(\mathbf{L}_{t s}^{\prime}\right)\right\}$ is closed and has a finite codimension $J_{s}$ in $H_{a}^{0}(G)$. It is easy to see that $J_{s} \leq J$.

Let $f_{1}^{s}, \ldots, f_{J_{s}}^{s}$ be an orthonormal basis in the space

$$
H_{a}^{0}(G) \ominus\left\{f \in H_{a}^{0}(G):(f, 0) \in \mathcal{R}\left(\mathbf{L}_{t s}^{\prime}\right)\right\}
$$

Assume that $\beta_{j}^{s}=\left(\mathbf{P}(y, D) W_{s}, f_{j}^{s}\right)_{H_{a}^{0}(G)}$. Then problem (16.26) has a unique solution $Y_{s} \in E_{t}$; by virtue of (16.32) and (16.25), we have

$$
\begin{align*}
\left\|Y_{s}\right\|_{H_{a}^{2 m}(G)} \leq k_{6}\left(\left\|\mathbf{P}(y, D) W_{s}\right\|_{H_{a}^{0}(G)}+\right. & \left.\sum_{j=1}^{J_{s}}\left|\beta_{j}^{s}\right|\right) \\
& \leq k_{7}|s| \cdot\|(u, \mathbf{P}(y, D) u)\|+k_{6} J \max \left\{\beta_{1}^{s}, \ldots, \beta_{J_{s}}^{s}\right\} . \tag{16.33}
\end{align*}
$$

Applying the Schwartz inequality for estimating $\beta_{j}^{s}=\left(\mathbf{P}(y, D) W_{s}, f_{j}^{s}\right)_{H_{a}^{0}(G)}$, and using (16.25), we obtain

$$
\left|\beta_{j}^{s}\right| \leq\left\|\mathbf{P}(y, D) W_{s}\right\|_{H_{a}^{0}(G)} \leq k_{8}|s| \cdot\|(u, \mathbf{P}(y, D) u)\| .
$$

Hence the inequality follows from here and (16.33):

$$
\begin{equation*}
\left\|Y_{s}\right\|_{H_{a}^{2 m}(G)} \leq k_{9}|s| \cdot\|(u, \mathbf{P}(y, D) u)\| . \tag{16.34}
\end{equation*}
$$

4. Taking into account Eqs. (16.27), we obtain the inequalities

$$
\begin{gather*}
\left\|w_{s}\right\|_{L_{2}(G)} \leq k_{10}\left\|w_{s}\right\|_{H_{a}^{2 m}(G)} \leq k_{11}|s| \cdot\|(u, \mathbf{P}(y, D) u)\|,  \tag{16.35}\\
\left\|\mathbf{P}(y, D) w_{s}\right\|_{H_{a}^{0}(G)} \leq k_{12}\left\|w_{s}\right\|_{H_{a}^{2 m}(G)} \leq k_{12} k_{11}|s| \cdot\|(u, \mathbf{P}(y, D) u)\|, \tag{16.36}
\end{gather*}
$$

where $w_{s}=Y_{s}+W_{s}$ is a solution of problem (16.22) from Eqs. (16.25) and (16.34).
The boundedness of the embedding $H_{a}^{2 m}(G) \subset W^{m}(G)$ implies that the function $v_{s}$ defined in (16.21) belongs to $W^{m}(G)$; moreover, by virtue of the second relation in (16.22), we have $v_{s} \in \mathrm{D}\left(\mathbf{Q}_{t+s}\right)$. From (16.21), (16.35), and (16.36) we obtain the required inequality (16.20).
Proof of Theorem 16.1. Let us fix two arbitrary numbers $t_{1}$ and $t_{2} \in \mathbb{C}$. By virtue of lemma 15.1 and Remark 16.2, $\mathbf{Q}_{t}$ are Fredholm operators for all $t$ from an interval $I_{t_{1} t_{2}} \subset \mathbb{C}$ with endpoints $t_{1}$ and $t_{2}$. Covering every point of the interval $I_{t_{1} t_{2}}$ by a circle of a sufficiently small radius, choosing a finite subcover, and applying Theorem 16.2 and [49, Chap. 4, Theorem 5.17], we see that

$$
\operatorname{ind} \mathbf{Q}_{t_{1}}=\operatorname{ind} \mathbf{Q}_{t_{2}}
$$

This and Lemma 15.1 imply that

$$
\operatorname{ind} \mathbf{P}_{t_{1}}=\operatorname{ind} \mathbf{P}_{t_{2}}
$$

Theorem 16.1 is proved.

## 17. Instability of Index

17.1. Intersection of the support of nonlocal terms with conjugation points of boundary conditions.
17.1.1. Statement of the problem. Let $G$ be a bounded domain in $\mathbb{R}^{2}$ and let $\partial G \backslash\left\{g_{1}, g_{2}\right\}=\Gamma_{1} \cup \Gamma_{2}$, where $\Gamma_{i}$ are open (in $\partial G$ topology) smooth curves, $\overline{\Gamma_{1}} \cap \overline{\Gamma_{2}}=\left\{g_{1}, g_{2}\right\}, g_{1}$ and $g_{2}$ are the endpoints of the curves $\overline{\Gamma_{1}}$ and $\overline{\Gamma_{2}}$. Assume that the domain $G$ coincides with a plane angle of nonzero spread in some neighborhood of the points $g_{j}$. We also assume that the domain $G$ coincides with a plane angle of spread $2 \omega_{0}$ in a neighborhood $\mathcal{O}_{\varepsilon}\left(g_{j}\right)$, where $0<\omega_{0}<\pi$. Consider the following nonlocal problem in the domain $G$ :

$$
\begin{gather*}
\Delta u=f(y), \quad y \in G,  \tag{17.1}\\
\left.u\right|_{\Gamma_{1}}-\left.(1+t) u\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}}=0,\left.\quad u\right|_{\Gamma_{2}}-\left.(1-t) u\left(\Omega_{2}(y)\right)\right|_{\Gamma_{2}}=0, \tag{17.2}
\end{gather*}
$$

where $t \in \mathbb{C}$ is a parameter and $\Omega_{i}$ is a $C^{\infty}$-diffeomorphism defined in a neighborhood of the curve $\Gamma_{i}$. Assume that $\Omega_{i}\left(\Gamma_{i}\right) \subset G, \Omega_{i}\left(g_{1}\right)=g_{1}, \Omega_{i}\left(g_{2}\right)=g_{2}$, and the transform $\Omega_{i}$ in neighborhoods $\mathcal{O}_{\varepsilon}\left(g_{1}\right)$ and $\mathcal{O}_{\varepsilon}\left(g_{2}\right)$ of the points $g_{1}$ and $g_{2}$ is a rotation by an angle $\omega_{0}$ inwards the domain $G$ (see Fig. 17.1).


Fig. 17.1. Problem (17.1), (17.2)
Consider the unbounded operator $\mathbf{P}_{t}: \mathrm{D}\left(\mathbf{P}_{t}\right) \subset L_{2}(G) \rightarrow L_{2}(G)$ acting by the formula

$$
\begin{gathered}
\mathbf{P}_{t} u=\Delta u, \quad u \in \mathrm{D}\left(\mathbf{P}_{t}\right) \\
\mathrm{D}\left(\mathbf{P}_{t}\right)=\left\{u \in W^{1}(G) \cap W^{2}\left(G \backslash \overline{\mathcal{O}_{\delta}(\mathcal{K})}\right) \quad \forall \delta>0: \Delta u \in L_{2}(G) \quad \text { and } u \text { satisfies }(17.2)\right\} .
\end{gathered}
$$

By Theorem 13.1 (more precisely, by virtue of its generalization to the case where the set $\mathcal{K}$ consists of several orbits), $\mathbf{P}_{t}$ is a Fredholm operator for any $t \in \mathbb{C}$.

Let us prove the following result.
Theorem 17.1. There exists a number $t_{0}>0$ such that $\operatorname{ind} \mathbf{P}_{0}>\operatorname{ind} \mathbf{P}_{t}=$ const for $0<|t| \leq t_{0}$.
17.1.2. Proof of Theorem 17.1. Consider a model problem near the point $g_{1}$. For this, we take the coordinate system with origin at the point $g_{1}$ and the axis $O x_{1}$ coinciding with the bisector of the angle made by the boundary of the domain near $g_{1}$. The model nonlocal problem with a parameter $\lambda \in \mathbb{C}$ has the form (cf. (6.18))

$$
\begin{gather*}
\varphi^{\prime \prime}-\lambda^{2} \varphi=0, \quad|\varphi|<\omega_{0}  \tag{17.3}\\
\varphi\left(-\omega_{0}\right)-(1+t) \varphi(0)=0, \quad \varphi\left(\omega_{0}\right)-(1-t) \varphi(0)=0 \tag{17.4}
\end{gather*}
$$

where $\varphi(\omega)=\tilde{u}(\omega, \lambda)$ for a fixed $\lambda$. Obviously, the same model problem will correspond to the point $g_{2}$.
We see that the eigenvalues of problem (17.3), (17.4) are independent of $t$ and have the form

$$
\begin{equation*}
\lambda_{k}=\frac{\pi k}{\omega_{0}} i, \quad k=0, \pm 1, \pm 2, \ldots \tag{17.5}
\end{equation*}
$$

Now we are interested in the location of the eigenvalues relative to the strip $-1 \leq \operatorname{Im} \lambda \leq 0$. Since $0<\omega_{0}<\pi$, we see that there is a unique eigenvalue $\lambda_{0}=0$ in this strip; it corresponds to a unique
(up to a constant factor) eigenvector

$$
\begin{equation*}
\varphi_{0}(\omega)=-\frac{t}{\omega_{0}} \omega+1 \tag{17.6}
\end{equation*}
$$

and a unique (up to an eigenvector) adjoint vector ${ }^{9} \varphi_{1}(\omega)=0$.
Lemma 17.1. There exists a number $t_{0}>0$ such that

$$
\operatorname{codim} \mathcal{R}\left(\mathbf{P}_{0}\right) \leq \operatorname{codim} \mathcal{R}\left(\mathbf{P}_{t}\right)=\text { const }
$$

as $0<|t| \leq t_{0}$.
Proof. 1. Consider the operator $\mathbf{N}_{t}: H_{0}^{2}(G) \rightarrow \mathcal{H}_{0}^{0}(G, \partial G)$ acting by the formula

$$
\mathbf{N}_{t}=\left(\Delta u,\left.u\right|_{\Gamma_{1}}-\left.(1+t) u\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}},\left.u\right|_{\Gamma_{2}}-\left.(1-t) u\left(\Omega_{2}(y)\right)\right|_{\Gamma_{2}}\right) .
$$

Since there are no eigenvalues of problem (17.3), (17.4) on the line $\operatorname{Im} \lambda=-1$, according to [85, Theorem 3.4], $\mathbf{N}_{t}$ is a Fredholm operator for all $t$. Since, on the one hand, the operator $\left.u \mapsto u\left(\Omega_{j}(y)\right)\right|_{\Gamma_{j}}$ is bounded from $H_{0}^{2}(G)$ to $H_{0}^{3 / 2}\left(\Gamma_{j}\right)$ and, on the other hand, small perturbations of Fredholm operators do not change their indices (see [56, Sec 16]), we have ind $\mathbf{N}_{t}=$ const for all $t$ from a sufficiently small neighborhood of any fixed point $t^{\prime} \in \mathbb{C}$. Hence

$$
\begin{equation*}
\text { ind } \mathbf{N}_{t}=\text { const }, \quad t \in \mathbb{C} \tag{17.7}
\end{equation*}
$$

2. Let us prove that

$$
\begin{equation*}
\operatorname{codim} \mathcal{R}\left(\mathbf{N}_{t}\right)=\text { const, } \quad|t| \leq t_{0}, \tag{17.8}
\end{equation*}
$$

where $t_{0}>0$ is sufficiently small. By virtue of (17.7) it suffices to show that

$$
\begin{equation*}
\operatorname{dim} \operatorname{ker} \mathbf{N}_{t}=0, \quad|t| \leq t_{0} \tag{17.9}
\end{equation*}
$$

Let $t=0$ and $u \in \operatorname{ker} \mathbf{N}_{0}$. It follows from (13.22) that the function $u$ is infinitely differentiable outside an arbitrarily small neighborhood of the set $\left\{g_{1}, g_{2}\right\}$. On the other hand, $u \in H_{0}^{2}(G) \subset W_{2}^{2}(G)$; hence, by the Sobolev embedding theorem, $u \in C^{\infty}(G) \cap C(\bar{G})$ and

$$
\begin{equation*}
u\left(g_{1}\right)=u\left(g_{2}\right)=0 . \tag{17.10}
\end{equation*}
$$

Since for $t=0$ the coefficients of the problem are real, we can assume, without loss of generality, that the function $u(y)$ is real-valued. If the function $|u(y)|$ has a maximum inside the domain $G$, then, by the maximum principle, $u=$ const in $\bar{G}$; hence, by virtue of (17.10) we have that $u=0$. If $|u(y)|$ has a maximum at a part of the boundary $\Gamma_{i}$, then, according to nonlocal conditions (17.2), which have the following form:

$$
\left.u\right|_{\Gamma_{1}}=\left.u\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}},\left.\quad u\right|_{\Gamma_{2}}=\left.u\left(\Omega_{2}(y)\right)\right|_{\Gamma_{2}}
$$

for $t=0$, the function $|u(y)|$ also has a maximum inside the domain $G$; then $u=0$ by the above. Finally, if $|u(y)|$ has a maximum at the point $g_{1}$ or $g_{2}$, then, by virtue (17.10), we have $u=0$.

Thus, we have proved that $\operatorname{dim} \operatorname{ker} \mathbf{N}_{0}=0$. It follows from [56, Sec 16] that $\operatorname{dim} \operatorname{ker} \mathbf{N}_{t} \leq \operatorname{dim} \operatorname{ker} \mathbf{N}_{0}=0$ for sufficiently small $|t|$; Eq. (17.9) follows from here; hence, (17.8) is proved.
3. We prove that

$$
\begin{equation*}
\mathcal{R}\left(\mathbf{P}_{t}\right)=\left\{f \in L_{2}(G):(f, 0,0) \in \mathcal{R}\left(\mathbf{N}_{t}\right)\right\}, \quad 0 \neq t \in \mathbb{C} . \tag{17.11}
\end{equation*}
$$

Since any solution $u \in H_{0}^{2}(G)$ of problem (17.1), (17.2) with the right-hand side $f \in L_{2}(G)$ belongs to $W^{1}(G)$, we have

$$
\begin{equation*}
\mathcal{R}\left(\mathbf{P}_{t}\right) \supset\left\{f \in L_{2}(G):(f, 0,0) \in \mathcal{R}\left(\mathbf{N}_{t}\right)\right\}, \quad t \in \mathbb{C} . \tag{17.12}
\end{equation*}
$$

[^8]To prove the inverse embedding for $t \neq 0$, consider an arbitrary function $f \in \mathcal{R}\left(\mathbf{P}_{t}\right)$. Let $u \in W^{1}(G)$ be a solution of problem (17.1), (17.2) with the right-hand side $f$. Using Lemma 14.2 we obtain $u \in H_{a+1}^{2}(G)$ for any $a>0$. By virtue of (17.5), we can find a sufficiently small $a>0$ such that there is a unique eigenvalue $\lambda_{0}=0$ of problem (17.1), (17.2) in the strip $-1 \leq \operatorname{Im} \lambda<a$. It follows from [85, Theorem 3.3] (theorem on the asymptotic behavior of solutions of nonlocal problems) that

$$
\begin{equation*}
u(y)=c_{j} \varphi_{0}(\omega)+d_{j} \varphi_{0}(\omega) \ln r+v_{j}(y), \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right) \tag{17.13}
\end{equation*}
$$

where $(\omega, r)$ are the polar coordinates with pole at the point $g_{j}, \varphi_{0}(\omega)$ is given by Eq. (17.6), and $v_{j} \in H_{0}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right)\right)$. Note that

$$
u \in W_{2}^{1}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right)\right), \quad v_{j} \in W_{2}^{1}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right)\right),
$$

but

$$
\varphi_{0}(\omega) \notin W_{2}^{1}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right)\right), \quad \varphi_{0}(\omega) \ln r \notin W_{2}^{1}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right)\right)
$$

for $t \neq 0$. Hence, $c_{j}=d_{j}=0$ in Eq. (17.13); thus, $u \in H_{0}^{2}(G)$. Therefore, we have proved that $(f, 0,0) \in \mathcal{R}\left(\mathbf{N}_{t}\right)$, i.e.,

$$
\begin{equation*}
\mathcal{R}\left(\mathbf{P}_{t}\right) \subset\left\{f \in L_{2}(G):(f, 0,0) \in \mathcal{R}\left(\mathbf{N}_{t}\right)\right\}, \quad 0 \neq t \in \mathbb{C} \tag{17.14}
\end{equation*}
$$

Equation (17.11) follows from Eqs. (17.12) and (17.14).
4. Let us prove that

$$
\begin{equation*}
\operatorname{codim}\left\{f \in L_{2}(G):(f, 0,0) \in \mathcal{R}\left(\mathbf{N}_{t}\right)\right\}=\operatorname{codim} \mathcal{R}\left(\mathbf{N}_{t}\right), \quad t \in \mathbb{C} \tag{17.15}
\end{equation*}
$$

In Eq. (17.15), the codimension of the subspace

$$
\left\{f \in L_{2}(G):(f, 0,0) \in \mathcal{R}\left(\mathbf{N}_{t}\right)\right\}
$$

is calculated in the space $H_{0}^{0}(G)$ and the codimension of the subspace $\mathcal{R}\left(\mathbf{N}_{t}\right)$ is calculated in the space $H_{0}^{0}(G) \times H_{0}^{3 / 2}\left(\Gamma_{1}\right) \times H_{0}^{3 / 2}\left(\Gamma_{2}\right)$.

We fix $t \in \mathbb{C}$ and assume that

$$
J_{1}=\operatorname{codim}\left\{f \in L_{2}(G):(f, 0,0) \in \mathcal{R}\left(\mathbf{N}_{t}\right)\right\}, \quad J_{2}=\operatorname{codim} \mathcal{R}\left(\mathbf{N}_{t}\right)
$$

Let $f \in L_{2}(G)$ and $(f, 0,0) \in \mathcal{R}\left(\mathbf{N}_{t}\right)$. This is equivalent to the following relations:

$$
\left((f, 0,0), F_{j}\right)_{\mathcal{H}_{0}^{0}(G, \partial G)}=0, \quad j=1, \ldots, J_{2},
$$

where $F_{j}$ are functions that form a basis in the orthogonal complement to the subspace $\mathcal{R}\left(\mathbf{N}_{t}\right)$ of the space $\mathcal{H}_{0}^{0}(G, \partial G)$. By virtue of the Riesz theorem on the general form of linear continuous functionals in Hilbert spaces, these relations are equivalent to the following:

$$
\left(f, \hat{f}_{j}\right)_{L_{2}(G)}=0, \quad j=1, \ldots, J_{2},
$$

where $\hat{f}_{j}$ are some functions from the space $L_{2}(G)$. Thus,

$$
\begin{equation*}
J_{1} \leq J_{2} \tag{17.16}
\end{equation*}
$$

(the equality holds if and only if the functions $\hat{f}_{j}$ are linearly independent).
Let us prove the inverse embedding. Let $F=\left(f, f_{1}, f_{2}\right)$ be an arbitrary function from $\mathcal{R}\left(\mathbf{N}_{t}\right)$. Then there exists a function $u \in H_{0}^{2}(G)$ such that

$$
\begin{aligned}
\Delta u & =f(y), \quad y \in G \\
\left.u\right|_{\Gamma_{1}}-\left.(1+t) u\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}} & =f_{1},\left.\quad u\right|_{\Gamma_{2}}-\left.(1-t) u\left(\Omega_{2}(y)\right)\right|_{\Gamma_{2}}=f_{2} .
\end{aligned}
$$

Using [58, Lemma 3.1], we construct a function $v \in H_{0}^{2}(G)$ such that

$$
\begin{gather*}
\left.v\right|_{\Gamma_{1}}-\left.(1+t) v\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}}=f_{1},\left.\quad v\right|_{\Gamma_{2}}-\left.(1-t) v\left(\Omega_{2}(y)\right)\right|_{\Gamma_{2}}=f_{2}, \\
\|v\|_{H_{0}^{2}(G)} \leq k_{1}\left(\left\|f_{1}\right\|_{H_{0}^{3 / 2}\left(\Gamma_{1}\right)}+\left\|f_{2}\right\|_{H_{0}^{3 / 2}\left(\Gamma_{2}\right)}\right), \tag{17.17}
\end{gather*}
$$

where $k_{1}>0$ is independent of $f_{1}$ and $f_{2}$.

Obviously, the function $w=u-v \in H_{0}^{2}(G)$ is a solution of the problem

$$
\begin{gathered}
\Delta w=f(y)-\Delta v, \quad y \in G \\
\left.w\right|_{\Gamma_{1}}-\left.(1+t) w\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}}=0,\left.\quad w\right|_{\Gamma_{2}}-\left.(1-t) w\left(\Omega_{2}(y)\right)\right|_{\Gamma_{2}}=0 .
\end{gathered}
$$

Hence,

$$
f-\Delta v \in L_{2}(G), \quad(f-\Delta v, 0,0) \in \mathcal{R}\left(\mathbf{N}_{t}\right)
$$

This is equivalent to the relations

$$
\left(f-\Delta v, f_{j}^{\prime}\right)_{L_{2}(G)}=0, \quad j=1, \ldots, J_{1},
$$

where $f_{j}^{\prime} \in L_{2}(G)$ are functions that form a basis in the orthogonal complement of the subspace $\left\{f \in L_{2}(G):(f, 0,0) \in \mathcal{R}\left(\mathbf{N}_{t}\right)\right\}$ in the space $L_{2}(G)$. By virtue of the Riesz theorem on the general form of linear continuous functionals in Hilbert spaces and estimate (17.17), these relations are equivalent to

$$
\left(F, F_{j}^{\prime}\right)_{\mathcal{H}_{0}^{0}(G, \partial G)}=0, \quad j=1, \ldots, J_{1}
$$

where $F_{j}^{\prime}$ are some functions from the space $\mathcal{H}_{0}^{0}(G, \partial G)$. Thus,

$$
\begin{equation*}
J_{2} \leq J_{1} \tag{17.18}
\end{equation*}
$$

the equality holds if and only if functions $F_{j}^{\prime}$ are linearly independent.
Equation (17.15) follows from inequalities (17.16) and (17.18).
The following equality follows from relations (17.15) and (17.8):

$$
\begin{equation*}
\operatorname{codim}\left\{f \in L_{2}(G):(f, 0,0) \in \mathcal{R}\left(\mathbf{N}_{t}\right)\right\}=\text { const }, \quad|t| \leq t_{0} \tag{17.19}
\end{equation*}
$$

Combining (17.11), (17.12) for $t=0$, and (17.19), we complete the proof.
Lemma 17.2. Let a number $t_{0}>0$ be the same as in Lemma 17.1. Then

$$
\operatorname{dim} \operatorname{ker} \mathbf{P}_{0}>\operatorname{dim} \operatorname{ker} \mathbf{P}_{t}=0
$$

for $0<|t| \leq t_{0}$.
Proof. 1. Let $0<|t| \leq t_{0}$ and $u \in \operatorname{ker} \mathbf{P}_{t}$. Similarly to item 3 of the proof of Lemma 17.1, we can show that $u \in H_{0}^{2}(G)$; hence, $u \in \operatorname{ker} \mathbf{N}_{t}$. By virtue of Eq. (17.9), we have $u=0$; thus, $\operatorname{dim} \operatorname{ker} \mathbf{P}_{t}=0$.
2. Let $t=0$. Then $u=$ const belongs to $\operatorname{ker} \mathbf{P}_{0}$.

Proof of Theorem 17.1. Applying Lemmas 17.2 and 17.1, we obtain
ind $\mathbf{P}_{0}=\operatorname{dim} \operatorname{ker} \mathbf{P}_{0}-\operatorname{codim} \mathcal{R}\left(\mathbf{P}_{0}\right)>-\operatorname{codim} \mathcal{R}\left(\mathbf{P}_{0}\right) \geq-\operatorname{codim} \mathcal{R}\left(\mathbf{P}_{t}\right)=\operatorname{ind} \mathbf{P}_{t}, \quad 0<|t| \leq t_{0}$.
Theorem 17.1 is proved.
17.1.3. Nonlocal terms with supports in small neighborhoods of conjugation points. Now we show that the index of an operator can change even if the supports of nonlocal terms are concentrated in an arbitrary small neighborhood of a conjugation point of the boundary conditions.

Let $G, \Gamma_{i}$, and $g_{j}$ be the same as above. Consider the following nonlocal problem in the domain $G$ :

$$
\begin{align*}
& \Delta u=f(y), \quad y \in G  \tag{17.20}\\
&\left.u\right|_{\Gamma_{1}}-\left.(1+t) \xi(y) u\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}}=0,\left.\quad u\right|_{\Gamma_{2}}-\left.(1-t) \xi(y) u\left(\Omega_{2}(y)\right)\right|_{\Gamma_{2}}=0, \tag{17.21}
\end{align*}
$$

where $\xi \in C^{\infty}\left(\mathbb{R}^{2}\right)$, the support of the function $\xi$ is concentrated in an arbitrarily small neighborhood of the points $g_{1}$ and $g_{2}$, and $\xi(y)=1$ near these points (see Fig. 17.2).

Consider the unbounded operator $\mathbf{P}_{t}^{\prime}: \mathrm{D}\left(\mathbf{P}_{t}^{\prime}\right) \subset L_{2}(G) \rightarrow L_{2}(G)$ acting by the formula

$$
\begin{gathered}
\mathbf{P}_{t}^{\prime} u=\Delta u, \quad u \in \mathrm{D}\left(\mathbf{P}_{t}^{\prime}\right) \\
\mathrm{D}\left(\mathbf{P}_{t}^{\prime}\right)=\left\{u \in W^{1}(G) \cap W^{2}\left(G \backslash \overline{\mathcal{O}_{\delta}(\mathcal{K})}\right) \forall \delta>0: \Delta u \in L_{2}(G) \text { and } u \text { satisfies }(17.21)\right\} .
\end{gathered}
$$



Fig. 17.2. Problem (17.20), (17.21)

By Theorem 13.1 (more precisely, by virtue of its generalization to the case where the set $\mathcal{K}$ consists of several orbits), $\mathbf{P}_{t}$ is a Fredholm operator for any $t \in \mathbb{C}$.

Let us formulate the main result of this subsection.
Theorem 17.2. There exists a number $t_{0}>0$ such that $\operatorname{ind} \mathbf{P}_{0}^{\prime}>\operatorname{ind} \mathbf{P}_{t}^{\prime}=$ const for $0<|t| \leq t_{0}$.
Proof. Nonlocal conditions (17.2) differ from nonlocal conditions (17.21) by the operators

$$
\left.u \mapsto(1+t)(1-\xi(y)) u\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}},\left.\quad u \mapsto(1-t)(1-\xi(y)) u\left(\Omega_{2}(y)\right)\right|_{\Gamma_{2}} .
$$

Since the coefficients $(1 \pm t)(1-\xi(y))$ of the nonlocal terms vanish near the points $g_{1}$ and $g_{2}$, we have ind $\mathbf{P}_{t}^{\prime}=\operatorname{ind} \mathbf{P}_{t}$ for all $t \in \mathbb{C}$ (by virtue of Theorem 16.1), and the statement of the theorem follows from Theorem 17.1.
17.2. Case where there are no conjugation points in the support of nonlocal terms. In this subsection, we show that the index of an operator can change if and only if the support of nonlocal terms does not intersect with the set of conjugation points of the boundary conditions (and even lies inside the domain).
17.2.1. Statement of the problem. Let $G, \Gamma_{i}$, and $g_{j}$ be as above. Assume that

$$
0<\omega_{0}<\pi / 2
$$

and consider the following problem in the domain $G$ :

$$
\begin{gather*}
\Delta u=f(y), \quad y \in G  \tag{17.22}\\
\left.u\right|_{\Gamma_{1}}+\left.t u(\Omega(y))\right|_{\Gamma_{1}}=0,\left.\quad u\right|_{\Gamma_{2}}=0, \tag{17.23}
\end{gather*}
$$

where $t \in \mathbb{R}$ and $\Omega$ is a $C^{\infty}$-diffeomorphism defined in a neighborhood of the curve $\Gamma_{1}$. Assume that $\overline{\Omega\left(\Gamma_{1}\right)} \subset G$ (see Fig. 17.3).

Consider the unbounded operator $\mathbf{P}_{t}: \mathrm{D}\left(\mathbf{P}_{t}\right) \subset L_{2}(G) \rightarrow L_{2}(G)$ acting by the formula

$$
\begin{gathered}
\mathbf{P}_{t} u=\Delta u, \quad u \in \mathrm{D}\left(\mathbf{P}_{t}\right) \\
\mathrm{D}\left(\mathbf{P}_{t}\right)=\left\{u \in W^{1}(G) \cap W^{2}\left(G \backslash \overline{\mathcal{O}_{\delta}(\mathcal{K})}\right) \forall \delta>0: \Delta u \in L_{2}(G) \text { and } u \text { satisfies }(17.23)\right\} .
\end{gathered}
$$

By Theorem 13.1 (more precisely, by virtue of its generalization to the case where the set $\mathcal{K}$ consists of several orbits), $\mathbf{P}_{t}$ is a Fredholm operator for any $t \in \mathbb{C}$.

Let us formulate the main result of this subsection.
Theorem 17.3. There exists a number $t_{0}>0$ such that $0=\operatorname{ind} \mathbf{P}_{0}>\operatorname{ind} \mathbf{P}_{t}$ as $0<|t| \leq t_{0}$.


Fig. 17.3. Problem (17.22), (17.23)
17.2.2. Proof of Theorem 17.3. As is known, the local operator $\mathbf{P}_{0}$ is an isomorphism and hence

$$
\begin{equation*}
\operatorname{ind} \mathbf{P}_{0}=0 \tag{17.24}
\end{equation*}
$$

Let us study the operators $\mathbf{P}_{t}$. The same (local, since the support of nonlocal terms lies outside the set $\left.\left\{g_{1}, g_{2}\right\}\right)$ model problem with a parameter $\lambda \in \mathbb{C}$ corresponds to each point $g_{1}, g_{2}$ :

$$
\begin{gather*}
\varphi^{\prime \prime}-\lambda^{2} \varphi=0, \quad|\varphi|<\omega_{0}  \tag{17.25}\\
\varphi\left(-\omega_{0}\right)=\varphi\left(\omega_{0}\right)=0 . \tag{17.26}
\end{gather*}
$$

It can be directly verified that the eigenvalues of this problem have the form

$$
\begin{equation*}
\lambda_{k}=\frac{\pi k}{2 \omega_{0}} i, \quad k= \pm 1, \pm 2, \ldots \tag{17.27}
\end{equation*}
$$

Lemma 17.3. dim $\operatorname{ker} \mathbf{P}_{t}=0$ as $0<|t| \leq 1$.
Proof. Let $u \in \operatorname{ker} \mathbf{P}_{t}$. Since $0<\omega_{0}<\pi / 2$, it follows from Eq. (17.27) that there are no eigenvalues of problem (17.25), (17.26) in the bound $-1 \leq \operatorname{Im} \lambda<0$. In Sec. 18 (see Theorem 18.1), we show that in this case $u \in W_{2}^{2}(G)$. It follows from Eq. (13.22) that the function $u$ is infinitely differentiable outside an arbitrary neighborhood of the set $\left\{g_{1}, g_{2}\right\}$; by the Sobolev embedding theorem, $u \in C^{\infty}(G) \cap C(\bar{G})$.

Since $t \in \mathbb{R}$, we see that all coefficients of problem (17.22), (17.23) are real; therefore, without loss of generality, we assume that the function $u(y)$ is real-valued. If the function $|u(y)|$ has a maximum inside the domain $G$, then by the maximum principle, $u=$ const in $\bar{G}$; then, by virtue of the second condition in Eq. (17.23), we see that $u=0$. If $|u(y)|$ has a maximum on $\Gamma_{1}$, then it follows from the first condition in Eq. (17.23) and from the relation $|t| \leq 1$ that $|u(y)|$ has a maximum inside the domain $G$; by the above, we see that $u=0$. Finally, if $|u(y)|$ has a maximum on $\overline{\Gamma_{2}}$, by virtue of the continuity and the second condition in Eq. (17.23), we obtain $u=0$.
Lemma 17.4. There exists a number $t_{0}>0$ such that $\operatorname{codim} \mathcal{R}\left(\mathbf{P}_{t}\right)>0$ for $0<|t| \leq t_{0}$.
Proof. 1. Consider the operator $\mathbf{M}_{t}: H_{a+1}^{2}(G) \rightarrow \mathcal{H}_{a+1}^{0}(G \partial G), a>0$, acting by the formula

$$
\mathbf{M}_{t}=\left(\Delta u,\left.u\right|_{\Gamma_{1}}+\left.t u(\Omega(y))\right|_{\Gamma_{1}},\left.u\right|_{\Gamma_{2}}\right) .
$$

Since the embedding operator $W_{2}^{3 / 2}\left(\Gamma_{1}\right) \subset H_{a+1}^{3 / 2}\left(\Gamma_{1}\right)$ is bounded (by Lemma 5.3) and $\overline{\Omega\left(\Gamma_{1}\right)} \subset G$, we have

$$
\begin{equation*}
\|u(\Omega(y))\|_{H_{a+1}^{3 / 2}\left(\Gamma_{1}\right)} \leq k_{1}\left\|u\left(y^{\prime}\right)\right\|_{W_{2}^{3 / 2}\left(\Omega\left(\Gamma_{1}\right)\right)} \leq k_{2}\|u\|_{H_{a+1}^{2}(G)} . \tag{17.28}
\end{equation*}
$$

Hence, if $u \in H_{a+1}^{2}(G)$ and $a>0$, we see that $\mathbf{M}_{t} u \in \mathcal{H}_{a+1}^{0}(G, \partial G)$. Thus, the operator $\mathbf{M}_{t}$ is well defined.

According to [58, Theorem 10.5], the local operator $\mathbf{M}_{0}$ is an isomorphism if

$$
\begin{equation*}
0<a<\pi /\left(2 \omega_{0}\right) . \tag{17.29}
\end{equation*}
$$

Let us fix a number $a$ satisfying (17.29). Since $\mathbf{M}_{0}$ is an isomorphism and estimate (17.28) holds, we see that the operator $\mathbf{M}_{t}$ is also an isomorphism for $0 \leq|t| \leq t_{0}$, where $t_{0}=t_{0}(a)$ is sufficiently small.
2. Construct a function $u \in H_{a+1}^{2}(G)$ satisfying nonlocal conditions (17.23) such that

$$
u\left(\Omega\left(g_{1}\right)\right)=1
$$

For this, we consider a function $v \in C^{\infty}(G)$ such that $v(y)=1$ for $y \in \overline{\Omega\left(\Gamma_{1}\right)}$ and $\operatorname{supp} v \subset G$. In this case, $v(\Omega(y))=1$ for $y \in \Gamma_{1}$; hence $\left.v(\Omega(y))\right|_{\Gamma_{1}} \in H_{a+1}^{3 / 2}\left(\Gamma_{1}\right)$.

Now we consider a function $w \in H_{a+1}^{2}(G)$ such that

$$
\left.w\right|_{\Gamma_{1}}=-\left.t v(\Omega(y))\right|_{\Gamma_{1}},\left.\quad w\right|_{\Gamma_{2}}=0, \quad \operatorname{supp} w \cap \overline{\Omega\left(\Gamma_{1}\right)}=\varnothing
$$

(it exists by [58, Lemma 3.1]). It is easy to see that $u=v+w$ is a desired function (see Fig. 17.3).
3. We approximate the function $f=\Delta u \in H_{a+1}^{0}(G)$ by functions $f_{n} \in L_{2}(G), n=1,2, \ldots$, in the space $H_{a+1}^{0}(G)$ :

$$
\begin{equation*}
\left\|f_{n}-f\right\|_{H_{a+1}^{0}(G)} \rightarrow 0, \quad n \rightarrow \infty \tag{17.30}
\end{equation*}
$$

If $\operatorname{codim} \mathcal{R}\left(\mathbf{P}_{t}\right)=0$ for $0<|t| \leq t_{0}$, then for any function $f_{n} \in L_{2}(G)$, there exists a general solution $u_{n} \in W^{1}(G)$ of problem (17.22), (17.23) with the right-hand side $f_{n}$; by Lemma 17.3, this solution is unique. Moreover, by Lemma 14.2, we obtain that $u_{n} \in H_{a+1}^{2}(G)$.

The fact that $\mathbf{M}_{t}$ is an isomorphism and Eq. (17.30) implies that

$$
\left\|u_{n}-u\right\|_{H_{a+1}^{2}(G)} \leq k_{3}\left\|f_{n}-f\right\|_{H_{a+1}^{0}(G)} \rightarrow 0, \quad n \rightarrow \infty .
$$

Hence, by the Sobolev embedding theorem, we have

$$
\begin{equation*}
u_{n}\left(\Omega\left(g_{1}\right)\right) \rightarrow u\left(\Omega\left(g_{1}\right)\right)=1, \quad n \rightarrow \infty \tag{17.31}
\end{equation*}
$$

On the other hand, there are no eigenvalues of problem (17.25), (17.26) in the strip $-1 \leq \operatorname{Im} \lambda<0$. In Sec. 18 (see Theorem 18.1), we will show that in this case $u_{n} \in W_{2}^{2}(G)$. By the Sobolev embedding theorem, we obtain that $u_{n} \in C(\bar{G})$; by the second relation in Eq. (17.23), we see that $u_{n}\left(g_{1}\right)=0$. Then from the first relation in Eq. (17.23) we obtain that $u_{n}\left(\Omega\left(g_{1}\right)\right)=0$ (for $t \neq 0$ ). This contradicts Eq. (17.31). Thus, we have proved that $\operatorname{codim} \mathcal{R}\left(\mathbf{P}_{t}\right)>0$.

Theorem 17.3 follows from Eq. (17.24) and Lemmas 17.3 and 17.4.
Remark 17.1. Let $\mathbf{I}$ be a unique operator in $L_{2}(G)$ and $\lambda \in \mathbb{C}$. By Lemma 15.1, minor terms in an elliptic equation do not affect the index of the unbounded nonlocal operator $\mathbf{P}_{t}$. Thus,

$$
\operatorname{ind}\left(\mathbf{P}_{t}-\lambda \mathbf{I}\right)=\operatorname{ind} \mathbf{P}_{t}<0
$$

for $0<|t| \leq t_{0}$, where $t_{0}>0$ is sufficiently small. Therefore, if $0<|t| \leq t_{0}$, then the spectrum of the operator $\mathbf{P}_{t}$ coincides with the whole complex plane.

## Chapter 5

## SMOOTHNESS OF GENERALIZED SOLUTIONS OF NONLOCAL ELLIPTIC PROBLEMS

## 18. Preservation of Smoothness of Generalized Solutions

18.1. Statement of the problem. As in previous chapters, we assume that conditions 6.1-6.4 hold (condition 6.4 holds with $l=0$ ). As in Chap. 4 , we assume that the orders $m_{i \mu}$ of the differential operators $B_{i \mu s}(y, D)$ satisfy the inequalities

$$
m_{i \mu} \leq 2 m-1
$$

We study the smoothness of generalized solutions (see Definition 13.3) of a nonlocal boundary problem (6.7), (6.8):

$$
\begin{align*}
& \mathbf{P}(y, D) u=f_{0}(y), \quad y \in G,  \tag{18.1}\\
& \mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u=f_{i \mu}(y) \quad y \in \Gamma_{i}, \quad i=1, \ldots, N, \quad \mu=1, \ldots, m . \tag{18.2}
\end{align*}
$$

We say that the smoothness of generalized solutions is preserved if any generalized solution of problem (18.1), (18.2) (with any right-hand side $\left\{f_{0}, f_{i \mu}\right\}$ from some subset of the space $\mathcal{W}^{0}(G, \partial G)$; this subset can be defined by different ways in different cases) belongs to $W^{2 m}(G)$. If there exists a generalized solution of problem (18.1), (18.2) that does not belong to $W^{2 m}(G)$, then we say that the smoothness of generalized solutions is violated.

Let us consider a model problem that corresponds the set (orbit) $\mathcal{K}$.
Denote the function $u(y)$ for $y \in \mathcal{O}_{\varepsilon_{1}}\left(g_{j}\right)$ by $u_{j}(y)$. If $g_{j} \in \overline{\Gamma_{i}}, y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right)$, and $\Omega_{i s}(y) \in \mathcal{O}_{\varepsilon_{1}}\left(g_{k}\right)$, then we denote the function $u\left(\Omega_{i s}(y)\right)$ by $u_{k}\left(\Omega_{i s}(y)\right)$. In this notation, nonlocal problem (18.1), (18.2) in a $\varepsilon$-neighborhood of the set (orbit) $\mathcal{K}$ has the form

$$
\begin{gathered}
\mathbf{P}(y, D) u_{j}=f_{0}(y), \quad y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap G, \\
\left.B_{i \mu 0}(y, D) u_{j}(y)\right|_{\mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap \Gamma_{i}}+\left.\sum_{s=1}^{S_{i}}\left(B_{i \mu s}(y, D)\left(\zeta u_{k}\right)\right)\left(\Omega_{i s}(y)\right)\right|_{\mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap \Gamma_{i}}=\psi_{i \mu}(y), \\
y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap \Gamma_{i}, \quad i \in\left\{1 \leq i \leq N: g_{j} \in \overline{\Gamma_{i}}\right\}, \quad j=1, \ldots, N, \quad \mu=1, \ldots, m,
\end{gathered}
$$

where

$$
\psi_{i \mu}=f_{i \mu}-\mathbf{B}_{i \mu}^{2} u
$$

Let $y \mapsto y^{\prime}\left(g_{j}\right)$ be the change of variables described in Sec. 6.1 (see Chap. 2). Introduce the functions

$$
\begin{gather*}
U_{j}\left(y^{\prime}\right)=u\left(y\left(y^{\prime}\right)\right), \quad f_{j}\left(y^{\prime}\right)=f_{0}\left(y\left(y^{\prime}\right)\right), \quad y^{\prime} \in K_{j}^{\varepsilon} ; \\
f_{j \sigma \mu}\left(y^{\prime}\right)=f_{i \mu}\left(y\left(y^{\prime}\right)\right), \quad B_{j \sigma \mu}^{u}\left(y^{\prime}\right)=\left(\mathbf{B}_{i \mu}^{2} u\right)\left(y\left(y^{\prime}\right)\right),  \tag{18.3}\\
\psi_{j \sigma \mu}\left(y^{\prime}\right)=f_{j \sigma \mu}\left(y^{\prime}\right)-B_{j \sigma \mu}^{u}\left(y^{\prime}\right), \quad y^{\prime} \in \gamma_{j \sigma}^{\varepsilon},
\end{gather*}
$$

where $\sigma=1(\sigma=2)$ if the transformation $y \mapsto y^{\prime}\left(g_{j}\right)$ maps $\mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap \Gamma_{i}$ to the side $\gamma_{j 1}$ (respectively, $\gamma_{j 2}$ ) of the angle $K_{j}$. Let us denote $y^{\prime}$ by $y$ again. Then, by virtue of condition 6.3, problem (18.1), (18.2) has the following form (cf. (6.12), (6.13) and (13.27), (13.28)):

$$
\begin{gather*}
\mathbf{P}_{j}(y, D) U_{j}=f_{j}(y), \quad y \in K_{j}^{\varepsilon},  \tag{18.4}\\
\mathbf{B}_{j \sigma \mu}(y, D) U \equiv \sum_{k, s}\left(B_{j \sigma \mu k s}(y, D) U_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right)=\psi_{j \sigma \mu}(y), \quad y \in \gamma_{j \sigma}^{\varepsilon} . \tag{18.5}
\end{gather*}
$$

Note that the right-hand side of problem (18.4), (18.5) coincides with the right-hand side of problem (13.27), (13.28), if $f_{i \mu}=0$. If, moreover, $\mathbf{B}_{i \mu}^{2}=0$, then the right-hand side of problem (18.4), (18.5) coincides with the right-hand side of problem (6.12), (6.13).
18.2. Statement of the main result. Here we study the case where the following condition holds.

Condition 18.1. The line $\operatorname{Im} \lambda=1-2 m$ does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$.
Recall that the index $\ell$ in the definition of the generalized solution is fixed and satisfies the inequalities

$$
0 \leq \ell \leq 2 m-1
$$

Denote by $\Lambda$ the set of eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ that lie in the strip $1-2 m<\operatorname{Im} \lambda<1-\ell$ (this set can be empty). Let us also denote $i \Lambda=\{i \lambda: \lambda \in \Lambda\}$.
Condition 18.2. All eigenvalues from the set $\Lambda$ are regular.

Recall that the notion of a regular eigenvalue was introduced in Definition 7.1.
Condition 18.2 means that if $\ell=2 m-1$ (e.g., if $\ell=m=1$ ), then $\Lambda=\varnothing$ and if $\ell \leq 2 m-2$, then $i \Lambda \subset\{\ell, \ldots, 2 m-2\}$.

In the case where $\ell \leq 2 m-2$, we need additional conditions.
Let $W^{-2 m}\left(-\omega_{j}, \omega_{j}\right)$ be the space dual to $W^{2 m}\left(-\omega_{j}, \omega_{j}\right)$. Let us introduce the space

$$
\mathcal{W}^{-2 m}(-\bar{\omega}, \bar{\omega})=\prod_{j=1}^{N} W^{-2 m}\left(-\omega_{j}, \omega_{j}\right) .
$$

Consider the operator

$$
(\tilde{\mathcal{L}}(\lambda))^{*}: \mathcal{W}^{0}[-\bar{\omega}, \bar{\omega}] \rightarrow \mathcal{W}^{-2 m}(-\bar{\omega}, \bar{\omega})
$$

conjugated to the operator

$$
\tilde{\mathcal{L}}(\lambda): \mathcal{W}^{2 m}(-\bar{\omega}, \bar{\omega}) \rightarrow \mathcal{W}^{0}[-\bar{\omega}, \bar{\omega}] .
$$

The operator $(\tilde{\mathcal{L}}(\lambda))^{*}$ maps an element $\left\{\zeta_{j}, \chi_{j \sigma \mu}\right\} \in \mathcal{W}^{0}[-\bar{\omega}, \bar{\omega}]$ to $(\tilde{\mathcal{L}}(\lambda))^{*}\left\{\zeta_{j}, \chi_{j \sigma \mu}\right\}$ by the following rule:

$$
\left\langle\varphi,(\tilde{\mathcal{L}}(\lambda))^{*}\left\{\zeta_{j}, \chi_{j \sigma \mu}\right\}\right\rangle=\sum_{j}\left(\tilde{\mathcal{P}}_{j}\left(\omega, D_{\omega}, \lambda\right) \varphi_{j}, \zeta_{j}\right)_{L_{2}\left(-\omega_{j}, \omega_{j}\right)}+\sum_{j, \sigma, \mu} \tilde{\mathcal{B}}_{j \sigma \mu}\left(\omega, D_{\omega}, \lambda\right) \varphi \overline{\chi_{j \sigma \mu}}
$$

for all $\varphi \in \mathcal{W}^{2 m}(-\bar{\omega}, \bar{\omega})$, where $\langle\cdot, \cdot\rangle$ denotes a sesquilinear form on a corresponding pair of dual spaces.

For any number $s \in\{\ell, \ldots, 2 m-2\}$, we denote by $J_{s}$ the set of all indices $\left(j^{\prime}, \sigma^{\prime}, \mu^{\prime}\right)$, for which

$$
\begin{equation*}
s \leq m_{j^{\prime} \sigma^{\prime} \mu^{\prime}}-1 \tag{18.6}
\end{equation*}
$$

(i.e., indices corresponding to differential operators of sufficiently high order (namely, order $\geq s+1$ ) in the boundary conditions). Let us also denote the space consisting of vectors $\left\{c_{j \sigma \mu}\right\}\left(c_{j \sigma \mu} \in \mathbb{C}\right)$ satisfying the following relations:

$$
c_{j^{\prime} \sigma^{\prime} \mu^{\prime}}=0, \quad\left(j^{\prime}, \sigma^{\prime}, \mu^{\prime}\right) \in J_{s} .
$$

by $C_{s}$.
Condition 18.3. If $\ell \leq 2 m-2$, then for any $s \in i \Lambda$, the following conditions hold:
(1) $J_{s} \neq \varnothing$;
(2) $\left\langle\left\{0, c_{j \sigma \mu}\right\}, \psi\right\rangle=0$ for all $\left\{c_{j \sigma \mu}\right\} \in C_{s}$ and $\psi \in \operatorname{ker}(\tilde{\mathcal{L}}(-i s))^{*}$;
(3) let $\varphi_{c} \in \mathcal{W}^{2 m}(-\bar{\omega}, \bar{\omega})$ be a solution of the equation $\tilde{\mathcal{L}}(-i s) \varphi_{c}=\left\{0, c_{j \sigma \mu}\right\}$, where $\left\{c_{j \sigma \mu}\right\} \in C_{s}$ (this solution exists by item 2 and is defined with accuracy of an element $\left.\varphi_{0} \in \operatorname{ker} \tilde{\mathcal{L}}(-i s)\right)$. Then for any vector $\left\{c_{j \sigma \mu}\right\} \in C_{s}$, the function $r^{s} \varphi_{c}(\omega)$ is a homogeneous polynomial (of order s).

Remark 18.1. 1. Item 1 in condition 18.3 is necessary for the fulfillment of item 2. Indeed, consider an eigenvalue $\lambda_{s}=-i s \in \Lambda$ and assume that $J_{s}=\varnothing$. Then $C_{s}=\prod_{j, \sigma, \mu} \mathbb{C}$. Thus, if item 2 is valid, then the equation $\tilde{\mathcal{L}}\left(\lambda_{s}\right) \varphi_{c}=\left\{0, c_{j \sigma \mu}\right\}$ is solvable for any $c_{j \sigma \mu} \in \mathbb{C}$. It is easy to see that in this case the equation $\tilde{\mathcal{L}}\left(\lambda_{s}\right) \varphi=\left\{\tilde{f}_{j}, c_{j \sigma \mu}\right\}$ is also solvable for any $\left\{\tilde{f}_{j}, c_{j \sigma \mu}\right\} \in \mathcal{W}^{0}[-\bar{\omega}, \bar{\omega}]$. Hence, by Lemma 6.1, the operator $\tilde{\mathcal{L}}\left(\lambda_{s}\right)$ is an isomorphism; this is impossible since $\lambda_{s}$ is an eigenvalue;
2. Item 2 is a necessary and sufficient condition for the existence of solutions $\varphi_{c}$ for all $\left\{c_{j \sigma \mu}\right\} \in C_{s}$ from item 3 .

Remark 18.2. Assume that condition 18.2 holds. If item 3 of condition 18.3 holds for some solution $\varphi_{c}$, then it holds for any solution $\varphi_{c}+\varphi_{0}$, where $\varphi_{0} \in \operatorname{ker} \tilde{\mathcal{L}}(-i s)$ since $-i s$ is a regular eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$.

Condition 18.4. If $\ell \leq 2 m-2$, then the following statement holds for any $s \in\{\ell, \ldots, 2 m-2\} \backslash i \Lambda$. Let $\varphi_{c} \in \mathcal{W}^{2 m}(-\bar{\omega}, \bar{\omega})$ be a solution ${ }^{10}$ of the equation $\tilde{\mathcal{L}}(-i s) \varphi_{c}=\left\{0, c_{j \sigma \mu}\right\}$, where $\left\{c_{j \sigma \mu}\right\} \in C_{s}$. Then for any vector $\left\{c_{j \sigma \mu}\right\} \in C_{s}$, the function $r^{s} \varphi_{c}(\omega)$ is a homogeneous polynomial of (order $s$ ).
Remark 18.3. Assume that condition 18.2 holds.

1. If conditions 18.3 and 18.4 hold, then the problem

$$
\begin{equation*}
\mathcal{P}_{j}(D) V=0, \quad \mathcal{B}_{j \sigma \mu}(D) V=c_{j \sigma \mu} r^{s-m_{j \sigma \mu}} \tag{18.7}
\end{equation*}
$$

has a solution $V(y)$, which is a homogeneous polynomial of order $s$ for any vector $\left\{c_{j \sigma \mu}\right\} \in C_{s}$, $s=\ell, \ldots, 2 m-2$. Indeed, substituting the function $V=r^{s} \varphi_{c}(\omega)$ into Eq. (18.7), we obtain the equation $\tilde{\mathcal{L}}(-i s) \varphi_{s}=\left\{0, c_{j \sigma \mu}\right\}$. By conditions 18.3 and 18.4, this equation has a solution $\varphi_{c}$ such that the function $V=r^{s} \varphi_{c}(\omega)$ is a homogeneous polynomial of order $s$.
2. If either condition 18.3 or condition 18.4 is violated, then one can find a vector $\left\{c_{j \sigma \mu}\right\} \in C_{s}$ for which problem (18.7) has a solution of the form

$$
\begin{equation*}
V=r^{s} \varphi_{c}(\omega)+r^{s}(i \ln r) \sum_{n=1}^{J} c_{n} \varphi^{(n)}(\omega) \tag{18.8}
\end{equation*}
$$

where $s \in\{\ell, \ldots, 2 m-2\}, c_{n} \in \mathbb{C}, \varphi_{c}, \varphi^{(n)} \in \mathcal{W}^{2 m}(-\bar{\omega}, \bar{\omega})$ and $J=J(s)$; moreover, the function $V$ is not a polynomial with respect to variables $y_{1}$ and $y_{2}$.

Indeed, if condition 18.4 is violated, then the statement is obvious (with $c_{1}=\cdots=c_{J}=0$ ). Assume that condition 18.3 is violated. If items 1 and 2 of condition 18.4 hold and item 3 is violated, then the statement is obvious again (with $c_{1}=\cdots=c_{J}=0$ ). Assume that either item 1 or item 2 is violated. In both cases, item 2 is also violated (see Remark 18.1). In other words, there exist a regular eigenvalue $\lambda_{s}=-i s \in \Lambda$ and a numerical vector $\left\{c_{j \sigma \mu}\right\} \in C_{s}$ such that the element $\left\{0, c_{j \sigma \mu}\right\}$ is not orthogonal to the kernel $\operatorname{ker}\left(\tilde{\mathcal{L}}\left(\lambda_{s}\right)\right)^{*}$.

Let us denote a basis in $\operatorname{ker} \tilde{\mathcal{L}}\left(\lambda_{s}\right)$ by $\varphi^{(1)}, \ldots, \varphi^{(J)}(J \geq 1)$. Since $\lambda_{s}$ is a regular eigenvalue, we see that no eigenvector $\varphi^{(n)}$ has adjoint vectors. Let us substitute a function $V$ of the form (18.8) in Eqs. (18.7). As a result, we obtain

$$
\begin{equation*}
\tilde{\mathcal{L}}\left(\lambda_{s}\right) \varphi_{c}=\left\{0, c_{j \sigma \mu}\right\}-\left.\sum_{n=1}^{J} c_{n} \frac{d \tilde{\mathcal{L}}(\lambda)}{d \lambda}\right|_{\lambda=\lambda_{s}} \varphi^{(n)} \tag{18.9}
\end{equation*}
$$

Note that by Lemma 6.1

$$
\operatorname{dim} \operatorname{ker}\left(\tilde{\mathcal{L}}\left(\lambda_{s}\right)\right)^{*}=\operatorname{dim} \operatorname{ker} \tilde{\mathcal{L}}\left(\lambda_{s}\right)=J
$$

Let $\psi^{(1)}, \ldots, \psi^{(J)}$ be a basis in the space $\operatorname{ker}\left(\tilde{\mathcal{L}}\left(\lambda_{s}\right)\right)^{*}$. By [26, Lemma 3.2], the matrix

$$
\left\|\left\langle\left.\frac{d \tilde{\mathcal{L}}(\lambda)}{d \lambda}\right|_{\lambda=\lambda_{s}} \varphi^{(n)}, \psi^{(k)}\right\rangle\right\|_{n, k=1, \ldots, J}
$$

is nondegenerate. Hence we can choose constants $c_{n}$ such that the right-hand side of Eq. (18.9) will be orthogonal to the kernel $\operatorname{ker}\left(\tilde{\mathcal{L}}\left(\lambda_{s}\right)\right)^{*}$; hence, a solution $\varphi_{c}$ of Eqs. (18.9) exists. Moreover, since the element $\left\{0, c_{j \sigma \mu}\right\}$ is not orthogonal to the $\operatorname{kernel} \operatorname{ker}\left(\tilde{\mathcal{L}}\left(\lambda_{s}\right)\right)^{*}$, we see that the vector $\left(c_{1}, \ldots, c_{J}\right)$ is nonzero. Thus, the function $V$ of the form (18.8) is not a polynomial with respect to variables $y_{1}$ and $y_{2}$.

Let us formulate the main result of this section.
Theorem 18.1. Let conditions 18.1-18.4 hold and let $u$ be a generalized solution of problem (18.1), (18.2) with the right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{W}^{0}(G, \partial G)$. Then $u \in W^{2 m}(G)$.

[^9]Remark 18.4. By Theorem 18.1, any generalized solution of problem (18.1), (18.2) belongs to $W^{2 m}(G)$. The right-hand sides $f_{i \mu}$ in the nonlocal conditions belong to $W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ (this is natural). However, we do not impose any additional restrictions (as concordance conditions at points of the set $\mathcal{K}$ ) on the behavior of the functions $f_{i \mu}$ and the coefficients of nonlocal operators. In fact, the functions $f_{i \mu} \in W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ are not quite arbitrary. For example, if $m=1, m_{i 1}=0$, and $\mathbf{B}_{i 1}^{1}=0, \mathbf{B}_{i 1}^{2}=0$ (i.e., a "local" Dirichlet problem) and the solution $u$ belongs to $W^{2}(G)$, then, by the Sobolev embedding theorem, we have

$$
\begin{equation*}
f_{i 1}(g)=f_{j 1}(g), \quad g \in \overline{\Gamma_{i}} \cap \overline{\Gamma_{j}} \neq \varnothing . \tag{18.10}
\end{equation*}
$$

Theorem 18.1 shows that if conditions 18.1-18.4 hold, then the existence of a general solution guarantees the fulfillment of relations of the form (18.10). In Sec. 19, we prove that if condition 18.1 is violated, then for any generalized solution be smooth, we must impose special concordance conditions on the right-hand sides $f_{i \mu}$.
18.3. Statement of the main results. Let $U_{j}\left(y^{\prime}\right)=u_{j}\left(y\left(y^{\prime}\right)\right), j=1, \ldots, N$, be the functions corresponding to the set (orbit) $\mathcal{K}$ and satisfying relations (18.4) and (18.5) with the right-hand sides $\left\{f_{j}, \psi_{j \sigma \mu}\right\}$.

It follows from the proof of Lemma 15.3 that

$$
\begin{equation*}
U=Q+\hat{U} \tag{18.11}
\end{equation*}
$$

where $\hat{U} \in \mathcal{H}_{2 m-\ell}^{2 m}\left(K^{\varepsilon}\right)$, and $Q=\left(Q_{1}, \ldots, Q_{N}\right)$ is a vector-valued polynomial of order $\ell-1$ (if $\ell=0$, then there is no polynomial $Q$ ). Using this fact, we prove the following lemma.

Lemma 18.1. Let conditions 18.2-18.4 hold. Then

$$
\begin{equation*}
U=W+U^{\prime}, \tag{18.12}
\end{equation*}
$$

where $W=\left(W_{1}, \ldots, W_{N}\right)$ is a vector-valued polynomial of order $2 m-2, U^{\prime} \in \mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right)(\delta$ is such that $0<\delta<1$ and the strip $1-2 m<\operatorname{Im} \lambda \leq 1-2 m+\delta$ does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ ) and

$$
\begin{align*}
\left\{\mathbf{P}_{j}(y, D) U_{j}^{\prime}\right\} & \in \mathcal{H}_{0}^{0}\left(K^{\varepsilon}\right), \\
\left\{\mathbf{B}_{j \sigma \mu}(y, D) U^{\prime}\right\} & \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \cap \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) . \tag{18.13}
\end{align*}
$$

Proof. 1. The function $\hat{U}$ from Eq. (18.11) belongs to $\mathcal{H}_{2 m-\ell}^{2 m}\left(K^{\varepsilon}\right)$ and, by Eqs. (18.4), (18.5), and (18.11), is a solution of the following problem:

$$
\begin{align*}
\mathbf{P}_{j}(y, D) \hat{U}_{j} & =f_{j}-\mathbf{P}_{j}(y, D) Q_{j}, & & y \in K_{j}^{\varepsilon}, \\
\mathbf{B}_{j \sigma \mu}(y, D) \hat{U} & =\psi_{j \sigma \mu}-\mathbf{B}_{j \sigma \mu}(y, D) Q, & & y \in \gamma_{j \sigma}^{\varepsilon} . \tag{18.14}
\end{align*}
$$

Since $\left\{f_{j}\right\} \in \mathcal{W}^{0}\left(K^{\varepsilon}\right)$ and $Q$ is a vector-valued polynomial, we see that

$$
\begin{equation*}
\left\{f_{j}-\mathbf{P}_{j}(y, D) Q_{j}\right\} \in \mathcal{H}_{0}^{0}\left(K^{\varepsilon}\right) \tag{18.15}
\end{equation*}
$$

Further, $\psi_{j \sigma \mu}-\mathbf{B}_{j \sigma \mu}(y, D) Q \in W^{2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j}^{\varepsilon}\right)$. Hence, by Lemma 5.3, there exists a polynomial $P_{j \sigma \mu}(r)$ of order $2 m-m_{j \sigma \mu}-2$ (if $m_{j \sigma \mu}=2 m-1$, then $\left.P_{j \sigma \mu}(r) \equiv 0\right)$ such that

$$
\begin{equation*}
\left\{\psi_{j \sigma \mu}-\mathbf{B}_{j \sigma \mu}(y, D) Q-P_{j \sigma \mu}\right\} \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \cap \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \tag{18.16}
\end{equation*}
$$

for any $0<\delta<1$. Moreover, since

$$
\left\{\psi_{j \sigma \mu}-\mathbf{B}_{j \sigma \mu}(y, D) Q\right\}=\left\{\mathbf{B}_{j \sigma \mu}(y, D) \hat{U}\right\} \in \mathcal{H}_{2 m-\ell}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right),
$$

we see that any polynomial $P_{j \sigma \mu}(r)$ consists of monomials of order $\max \left(0, \ell-m_{j \sigma \mu}\right), \ldots, 2 m-m_{j \sigma \mu}-2$ (in particular, if $\ell=2 m-1$, then there is no polynomial $P_{j \sigma \mu}(r)$ ).
2. Let us write the polynomial $P_{j \sigma \mu}(r)$ in the form

$$
\begin{equation*}
P_{j \sigma \mu}(r)=c_{j \sigma \mu} r^{\ell-m_{j \sigma \mu}}+c_{j \sigma \mu}^{\prime} r^{\ell-m_{j \sigma \mu}+1}+\ldots, \tag{18.17}
\end{equation*}
$$

where, in particular, $c_{j \sigma \mu}=0$ for all $j, \sigma$, and $\mu$ such that $\ell \leq m_{j \sigma \mu}-1$ (cf. (18.6) as $s=\ell$ ). Hence $\left\{c_{j \sigma \mu}\right\} \in C_{\ell}$.

Consider the following auxiliary problem:

$$
\begin{equation*}
\mathcal{P}_{j}(D) W^{\ell}=0, \quad \mathcal{B}_{j \sigma \mu}(D) W^{\ell}=c_{j \sigma \mu} r^{\ell-m_{j \sigma \mu}} . \tag{18.18}
\end{equation*}
$$

By conditions 18.3 and 18.4 (see Remark 18.3), problem (18.18) has a solution $W^{\ell}(y)$, which is a homogeneous vector-valued polynomial of order $\ell$.

Using (18.17) and (18.18) and expanding the coefficients of operators $\mathbf{B}_{j \sigma \mu}(y, D)$ in the Taylor series, we obtain the embeddings

$$
\begin{align*}
\left\{\mathbf{P}_{j}(y, D) W_{j}^{\ell}\right\} & \in \mathcal{H}_{0}^{0}\left(K^{\varepsilon}\right), \\
\left\{\mathbf{B}_{j \sigma \mu}(y, D) W^{\ell}-P_{j \sigma \mu}+P_{j \sigma \mu}^{\prime}\right\} & \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \cap \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right), \tag{18.19}
\end{align*}
$$

where $P_{j \sigma \mu}^{\prime}(r)$ is a polynomial consisting of monomials of order $\max \left(0, \ell-m_{j \sigma \mu}+1\right), \ldots, 2 m-m_{j \sigma \mu}-2$.
It follows from (18.15), (18.16), and (18.19) that

$$
\begin{align*}
\left\{f_{j}-\mathbf{P}_{j}(y, D)\left(Q_{j}+W_{j}^{\ell}\right)\right\} & \in \mathcal{H}_{0}^{0}\left(K^{\varepsilon}\right) \\
\left\{\psi_{j \sigma \mu}-\mathbf{B}_{j \sigma \mu}(y, D)\left(Q+W^{\ell}\right)-P_{j \sigma \mu}^{\prime}\right\} & \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \cap \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) . \tag{18.20}
\end{align*}
$$

3. Repeating the procedure from item 2 of the proof finitely many times (every time we use conditions 18.3 and 18.4), we obtain the embeddings

$$
\begin{align*}
\left\{f_{j}-\mathbf{P}_{j}(y, D)\left(Q_{j}+W_{j}^{\ell}+\cdots+W_{j}^{2 m-2}\right)\right\} & \in \mathcal{H}_{0}^{0}\left(K^{\varepsilon}\right), \\
\left\{\psi_{j \sigma \mu}-\mathbf{B}_{j \sigma \mu}(y, D)\left(Q+W^{\ell}+\cdots+W^{2 m-2}\right)\right\} & \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \cap \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right), \tag{18.21}
\end{align*}
$$

where $W^{s}$ is a homogeneous vector-valued polynomial of order $s, s=\ell, \ldots, 2 m-2$. (Let us note that a homogeneous vector-valued polynomial of order $2 m-1$ belongs to $\mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right)$.) If $\ell=2 m-1$, then there are no polynomials $W^{s}$ in Eq. (18.21); in this case, the second relation in Eq. (18.21) follows from Eq. (18.16), where $P_{j \sigma \mu}=0$.

Equations (18.14) and (18.21) yield the embeddings

$$
\begin{align*}
\left\{\mathbf{P}_{j}(y, D)\left(\hat{U}_{j}-W_{j}^{\ell}-\cdots-W_{j}^{2 m-2}\right)\right\} & \in \mathcal{H}_{0}^{0}\left(K^{\varepsilon}\right), \\
\left\{\mathbf{B}_{j \sigma \mu}(y, D)\left(\hat{U}-W^{\ell}-\cdots-W^{2 m-2}\right)\right\} & \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \cap \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \tag{18.22}
\end{align*}
$$

4. Since the line $\operatorname{Im} \lambda=1-2 m+\delta$ does not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ and relations (18.22) hold, we see that it follows from [26, Theorem 2.2, Lemma 4.3] and conditions 18.2-18.4 that the function $\hat{U}+W^{\ell}+\cdots+W^{2 m-2}$ belongs to the space $\mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right)$ with accuracy up to a vector-valued polynomial. This vector-valued polynomial consists of vector-valued monomials of order $\min _{s \in i \Lambda} s, \ldots, 2 m-2$ (this vector-valued polynomial is absent if $\ell=2 m-1$ ). In other words, there exists a vector-valued polynomial $\hat{W}$ consisting of vector-valued monomials of orders $l, \ldots, 2 m-2$ such that

$$
\begin{align*}
\hat{U}+\hat{W} & \in \mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right), \\
\left\{\mathbf{P}_{j}(y, D)\left(\hat{U}_{j}+\hat{W}_{j}\right)\right\} & \in \mathcal{H}_{0}^{0}\left(K^{\varepsilon}\right),  \tag{18.23}\\
\left\{\mathbf{B}_{j \sigma \mu}(y, D)(\hat{U}+\hat{W})\right\} & \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \cap \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) .
\end{align*}
$$

Now the conclusion of the lemma follows from Eqs. (18.11) and (18.23).
Lemma 18.2. Let conditions 18.1-18.4 hold. Then $U \in \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$.

Proof. It follows from Eq. (18.13), Lemma 7.1, and Corollary 7.1 that there exists a function $V \in$ $\mathcal{H}_{\delta}^{2 m}(K) \cap \mathcal{W}^{2 m}(K)$ such that

$$
\begin{align*}
\left\{\mathbf{P}_{j}(y, D)\left(U_{j}^{\prime}-V_{j}\right)\right\} & \in \mathcal{H}_{0}^{0}\left(K^{\varepsilon}\right), \\
\left\{\mathbf{B}_{j \sigma \mu}(y, D)\left(U^{\prime}-V\right)\right\} & \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) . \tag{18.24}
\end{align*}
$$

By virtue of Eq. (18.24) and the fact that the strip $1-2 m \leq \operatorname{Im} \lambda \leq 1-2 m+\delta$ does not contain the eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$, we can apply [26, Theorem 2.2$]$ (theorem on the asymptotic behavior of solutions of nonlocal problems). Thus, we obtain the embedding $U^{\prime}-V \in \mathcal{H}_{0}^{2 m}\left(K^{\varepsilon}\right) \subset \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$. The conclusion of the lemma follows from here and Lemma 18.1.

Theorem 18.1 follows from (13.21) and Lemma 18.2.

## 19. "Bounded" Case. Concordance Condition

19.1. Behavior of generalized solution near conjugation points. Let $\Lambda$ be the same set of eigenvalues as in Sec. 18. In this section, we assume that the following condition holds (instead of Condition 18.1).
Condition 19.1. The line $\operatorname{Im} \lambda=1-2 m$ contains a unique eigenvalue $\lambda=i(1-2 m)$ of the operator $\tilde{\mathcal{L}}(\lambda)$, and this eigenvalue is regular.

The fundamental difference of results obtained in this section from results of Sec. 18 is in the behavior of generalized solutions near the set (orbit) $\mathcal{K}$. Lemma 18.1 is still valid if condition 19.1 holds. However, Lemma 18.2 becomes invalid since, if there is an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$ on the line $\operatorname{Im} \lambda=1-2 m$, then we cannot apply Lemma 7.1 and Corollary 7.1 from Sec. 7.1 (see Chap 2). Therefore, we will use results of Secs. 7.2 and 7.3 (see Chap. 2). Moreover, we impose special concordance conditions on the behavior of the functions $f_{i \mu}$ and the coefficients of the nonlocal operators near the set (orbit) $\mathcal{K}$.

Let $\tau_{j \sigma}$ and $D_{\tau_{j \sigma}}^{\beta}$ are the same as in Sec. 5.2 (see Chap. 2). Consider the operators

$$
D_{\tau_{j \sigma}}^{2 m-m_{j \sigma \mu}-1} \mathcal{B}_{j \sigma \mu}(D) U \equiv D_{\tau_{j \sigma}}^{2 m-m_{j \sigma \mu}-1}\left(\sum_{k, s}\left(B_{j \sigma \mu k s}(D) U_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right)\right)
$$

Using the chain rule for differentiation, we have

$$
\begin{equation*}
D_{\tau_{j \sigma}}^{2 m-m_{j \sigma \mu}-1} \mathcal{B}_{j \sigma \mu}(D) U \equiv \sum_{k, s}\left(\hat{B}_{j \sigma \mu k s}(D) U_{k}\right)\left(\mathcal{G}_{j \sigma k s} y\right) \tag{19.1}
\end{equation*}
$$

where $\hat{B}_{j \sigma \mu k s}(D)$ are homogeneous differential operators of order $2 m-1$ with constant coefficients. Formally replacing nonlocal operators in Eq. (19.1) by the corresponding local operators, we obtain

$$
\begin{equation*}
\hat{\mathcal{B}}_{j \sigma \mu}(D) U \equiv \sum_{k, s} \hat{B}_{j \sigma \mu k s}(D) U_{k}(y) \tag{19.2}
\end{equation*}
$$

which coincides with the operators in Eq. (7.2) for $l=0$.
It was shown in Sec. 7.2 (see Chap. 2) that if condition 19.1 holds, then the system of operators (19.2) is linearly independent. Denote the maximal linearly subsystem of system (19.2) by

$$
\begin{equation*}
\left\{\hat{\mathcal{B}}_{j^{\prime} \sigma^{\prime} \mu^{\prime}}(D)\right\} \tag{19.3}
\end{equation*}
$$

Then any operator $\hat{\mathcal{B}}_{j \sigma \mu}(D)$, which does not belong to system (19.3), can be represented in form

$$
\begin{equation*}
\hat{\mathcal{B}}_{j \sigma \mu}(D)=\sum_{j^{\prime}, \sigma^{\prime}, \mu^{\prime}} \beta_{j \sigma \mu}^{j^{\prime} \sigma^{\prime} \mu^{\prime}} \hat{\mathcal{B}}_{j^{\prime} \sigma^{\prime} \mu^{\prime}}(D) \tag{19.4}
\end{equation*}
$$

where $\beta_{j \sigma \mu}^{j^{\prime} \sigma^{\prime} \mu^{\prime}}$ are some constants.

Introduce the notion of a concordance condition. Let $\left\{Z_{j \sigma \mu}\right\} \in \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$ be a vector consisting of functions defined on intervals $\gamma_{j \sigma}^{\varepsilon}$. Consider the functions

$$
\left.Z_{j \sigma \mu}^{0}(r)=\left.Z_{j \sigma \mu}(y)\right|_{y=\left(r \cos \omega_{j}, r(-1)^{\sigma}\right.} \sin \omega_{j}\right) .
$$

Every function $Z_{j \sigma \mu}^{0}$ belongs to $W^{2 m-m_{j \sigma \mu}-1 / 2}(0, \varepsilon)$.
Definition 19.1. Let $\beta_{j \sigma \mu}^{j^{\prime} \sigma^{\prime} \mu^{\prime}}$ be constants from relations (19.4). If the relations

$$
\begin{equation*}
\int_{0}^{\varepsilon} r^{-1}\left|D_{r}^{2 m-m_{j \sigma \mu}-1} Z_{j \sigma \mu}^{0}-\sum_{j^{\prime}, \sigma^{\prime}, \mu^{\prime}} \beta_{j \sigma \mu}^{j^{\prime} \sigma^{\prime} \mu^{\prime}} D_{r}^{2 m-m_{j^{\prime} \sigma^{\prime} \mu^{\prime}}-1} Z_{j^{\prime} \sigma^{\prime} \mu^{\prime}}^{0}\right|^{2} d r<\infty \tag{19.5}
\end{equation*}
$$

are valid for all indices $j, \sigma$, and $\mu$ that correspond to the operators of system (19.2), which do not belong to system (19.3), then we say that functions $Z_{j \sigma \mu}$ satisfy concordance condition (19.5).
Remark 19.1. The relation $\left\{Z_{j \sigma \mu}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$ is sufficient (but is not necessary) for functions $Z_{j \sigma \mu}$ to satisfy Eqs. (19.5). This follows from [53, Lemma 4.8].
Remark 19.2. In terms of Chap. 2, the concordance condition has the form

$$
\begin{equation*}
D_{\tau_{j \sigma}}^{2 m-m_{j \sigma \mu}-1} \mathbf{Z}_{j \sigma \mu}-\sum_{j^{\prime}, \sigma^{\prime}, \mu^{\prime}} \beta_{j \sigma \mu}^{j^{\prime} \sigma^{\prime} \mu^{\prime}} D_{\tau_{j^{\prime} \sigma^{\prime}}^{2 m-m_{j}}}^{2 m-\sigma^{\prime} \mu^{\prime}-1} \mathbf{Z}_{j^{\prime} \sigma^{\prime} \mu^{\prime}} \in H_{0}^{1}\left(\mathbb{R}^{2}\right) \tag{19.6}
\end{equation*}
$$

where $\mathbf{Z}_{j \sigma \mu} \in W^{2 m-m_{j \sigma \mu}}\left(\mathbb{R}^{2}\right)$ is an extension of the function $Z_{j \sigma \mu}$ to $\mathbb{R}^{2}$, which has a compact support (the corresponding theorems on extension of functions defined in domains with angular points can be found in $[100]$ ). It is easy to show that Eq. (19.5) is equivalent to Eq. (19.6).

Let us show that the following condition is necessary and sufficient condition for some fixed generalized solution $u$ to belong to $W^{2 m}(G)$.
Condition 19.2. Let $u$ be a generalized solution of problem (18.1), (18.2), $\psi_{j \sigma \mu}$ be the right-hand sides in nonlocal conditions (18.5), and $W$ be a vector-valued polynomial from Lemma 18.1. Then the functions $\psi_{j \sigma \mu}-\mathbf{B}_{j \sigma \mu}(y, D) W$ satisfy concordance condition (19.5).
Remark 19.3. 1. The fulfillment of Condition 19.2 depends on the behavior of the function $\mathbf{B}_{i \mu}^{2} u$ near the set (orbit) $\mathcal{K}$. By virtue of Eq. (6.5) (for $l=0$ ), the values of the function $\mathbf{B}_{i \mu}^{2} u$ near the set $\mathcal{K}$ depend on the values of the function $u$ in $G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}$. Therefore, the smoothness of a generalized solution $u$ near the set $\mathcal{K}$ depends on the behavior of $u$ outside the set $\mathcal{K}$.
2. Let us clarify how the fulfillment of condition 19.2 depends on the behavior of the functions $u(y), f_{i \mu}(y),\left(\mathbf{B}_{i \mu}^{2} u\right)(y)$ and the coefficients of the operators $\mathbf{B}_{i \mu}^{0}$ and $\mathbf{B}_{i \mu}^{1}$ near the set $\mathcal{K}$. On the one hand, the vector $W$ from 18.1 is defined by the behavior of the solution $u(y)$ near the set $\mathcal{K}$. On the other hand, the coefficients of the operators $\mathbf{B}_{i \mu}^{0}$ and $\mathbf{B}_{i \mu}^{1}$ at points of the set $\mathcal{K}$ and operators $\mathcal{G}_{j \sigma k s}$ define the constants $\beta_{j \sigma \mu}^{j^{\prime} \sigma^{\prime} \mu^{\prime}}$ from Eq. (19.4) and, hence, the constants from Eq. (19.5). Finally, the derivatives of the functions $f_{i \mu}(y),\left(\mathbf{B}_{i \mu}^{2} u\right)(y)$ and the coefficients of the operators $\mathbf{B}_{i \mu}^{0}$ and $\mathbf{B}_{i \mu}^{1}$ must be coordinated near the set $\mathcal{K}$ in such a way that the absolute values of the corresponding linear combinations of derivatives (of order $2 m-m_{j \sigma \mu}-1$ ) of the functions $\psi_{j \sigma \mu}-\mathbf{B}_{j \sigma \mu}(y, D) W$ are square integrable (with weight $r^{-1}$ ) near the origin.

Theorem 19.1. Let conditions 19.1 and 18.2-18.4 hold and let $u$ be a generalized solution of problem (18.1), (18.2) with the right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{W}^{0}(G, \partial G)$. Then $u \in W^{2 m}(G)$ if and only if condition 19.2 holds.

Proof. 1. Necessity. Let $u \in W^{2 m}(G)$. Consider the function $U=\left(U_{1}, \ldots, U_{N}\right)$ corresponding to the set (orbit) $\mathcal{K}$. Obviously, $U \in \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$. By Lemma 18.1, we have $U=W+U^{\prime}$, where $U^{\prime} \in \mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right)$,
$0<\delta<1$. Since $U^{\prime}=U-W \in \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$, by the Sobolev embedding theorem, we have $\left.D^{\alpha} U^{\prime}\right|_{y=0}=0$, $|\alpha| \leq 2 m-2$. This and Lemma 7.2 imply that the functions $\psi_{j \sigma \mu}-\mathbf{B}_{j \sigma \mu} W=\mathbf{B}_{j \sigma \mu}(y, D) U^{\prime}$ satisfy concordance condition (19.5).
2. Sufficiency. Let condition 19.2 hold. It follows from Eq. (18.13), Lemma 7.4, and Corollary 7.2 that there exists a function $V \in \mathcal{H}_{\delta}^{2 m}(K) \cap \mathcal{W}^{2 m}(K)(\delta$ is the same as in Lemma 18.1) such that

$$
\begin{align*}
\left\{\mathbf{P}_{j}(y, D)\left(U_{j}^{\prime}-V_{j}\right)\right\} & \in \mathcal{H}_{0}^{0}\left(K^{\varepsilon}\right), \\
\left\{\mathbf{B}_{j \sigma \mu}(y, D)\left(U^{\prime}-V\right)\right\} & \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) . \tag{19.7}
\end{align*}
$$

By virtue of Eq. (19.7) and the fact that the strip $1-2 m \leq \operatorname{Im} \lambda \leq 1-2 m+\delta$ contains only the regular eigenvalue $i(1-2 m)$ of the operator $\tilde{\mathcal{L}}(\lambda)$, we can apply Lemma 7.5 . By this lemma, all derivatives of order $2 m$ of the function $U^{\prime}-V$ belong to $\mathcal{W}^{0}\left(K^{\varepsilon}\right)$. This and the relations

$$
U^{\prime}-V \in \mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right) \subset \mathcal{H}_{0}^{2 m-1}\left(K^{\varepsilon}\right) \subset \mathcal{W}^{2 m-1}\left(K^{\varepsilon}\right)
$$

imply that $U^{\prime}-V \in \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$. Combining this relation with Lemma 18.1, we complete the proof.
Note that Theorem 19.1 allows one to determine whether a given fixed solution $u$ is smooth near the set $\mathcal{K}$ only in the case where the asymptotic behavior of the solution $u$ of form (18.12) near the set $\mathcal{K}$ is known (i.e., if we have a vector-valued polynomial $W$ ). Theorem 19.1 clarifies the factors that affect the smoothness of generalized solutions. Below, we will obtain necessary and sufficient conditions of the fact that every generalized condition belongs to $W^{2 m}(G)$.
19.2. Problem with nonlocal conditions. In this subsection, we formulate necessary and sufficient conditions for the preservation of smoothness of generalized solutions. First, we show that the right-hand sides of $f_{i \mu}$ cannot be arbitrary functions from $W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$; they must satisfy concordance conditions.

Denote the set consisting of functions $\left\{f_{i \mu}\right\} \in \mathcal{W}^{2 m-\mathbf{m}-1 / 2}(\partial G)$ such that the functions $f_{j \sigma \mu}$ (see (18.3)) satisfy concordance condition (19.5) by $\tilde{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G)$. Introduce the space

$$
\tilde{\mathcal{S}}^{0}(G, \partial G)=L_{2}(G) \times \tilde{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G) .
$$

Obviously,

$$
\begin{gathered}
\mathcal{S}^{2 m-\mathbf{m}-1 / 2}(\partial G) \cap \tilde{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G)=\hat{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G) \subset \tilde{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G) \subset \mathcal{W}^{2 m-\mathbf{m}-1 / 2}(\partial G), \\
\mathcal{S}^{0}(G, \partial G) \cap \tilde{\mathcal{S}}^{0}(G, \partial G)=\hat{\mathcal{S}}^{0}(G, \partial G) \subset \tilde{\mathcal{S}}^{0}(G, \partial G) \subset \mathcal{W}^{0}(G, \partial G)
\end{gathered}
$$

The smoothness of generalized solutions of problem (18.1), (18.2) can be violated if the right-hand sides in nonlocal conditions (18.2) do not satisfy the concordance condition.

Theorem 19.2. Let conditions 19.1 and 18.2-18.4 hold. Then there exist functions $\left\{f_{0}, f_{i \mu}\right\} \in$ $\mathcal{W}^{0}(G, \partial G),\left\{f_{i \mu}\right\} \notin \tilde{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G)$, and $u \in W^{2 m-1}(G)$ such that $u$ is a generalized solution of problem (18.1), (18.2) with the right-hand side $\left\{f_{0}, f_{i \mu}\right\}$ and $u \notin W^{2 m}(G)$.

Prove the following auxiliary result. Let

$$
\begin{equation*}
\varepsilon^{\prime}=d_{1} \min \left(\varepsilon, \varkappa_{2}\right), \tag{19.8}
\end{equation*}
$$

where $d_{1}$ is defined in Eq. (6.15).
Lemma 19.1. Let condition 19.1 hold and a function $\left\{Z_{j \sigma \mu}\right\} \in \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$ be such that $\operatorname{supp}\left\{Z_{j \sigma \mu}\right\} \subset \mathcal{O}_{\varepsilon / 2}(0),\left.D_{\tau_{j \sigma}}^{\beta} Z_{j \sigma \mu}\right|_{y=0}=0, \beta \leq 2 m-m_{j \sigma \mu}-2$, and functions $Z_{j \sigma \mu}$ do not satisfy concordance condition (19.5). Then there exists a function $U \in \mathcal{H}_{\delta}^{2 m}(K) \subset \mathcal{W}^{2 m-1}(K)$, where $\delta>0$ is arbitrary, such that $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon^{\prime}}(0), U \notin \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$, and $U$ satisfies the relations

$$
\begin{equation*}
\left\{\mathbf{P}_{j}(y, D) U_{j}\right\} \in \mathcal{W}^{0}\left(K^{\varepsilon}\right), \quad\left\{\mathbf{B}_{j \sigma \mu}(y, D) U-Z_{j \sigma \mu}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) . \tag{19.9}
\end{equation*}
$$

Proof. By Lemma 5.1, there exists a sequence of functions $\left\{Z_{j \sigma \mu}^{n}\right\} \in \mathcal{W}^{2 m-\mathbf{m}-1 / 2}(\gamma), n=1,2, \ldots$, such that supp $Z_{j \sigma \mu}^{n} \subset \mathcal{O}_{\varepsilon}(0), Z_{j \sigma \mu}^{n}$ vanish near the origin (hence, they satisfy concordance condition (19.5)), and $\left\{Z_{j \sigma \mu}^{n}\right\} \rightarrow\left\{Z_{j \sigma \mu}\right\}$ in $W^{2 m-\mathbf{m}-1 / 2}(\gamma)$. Taking into account Lemma 5.4, we see that $\left\{Z_{j \sigma \mu}^{n}\right\} \rightarrow$ $\left\{Z_{j \sigma \mu}\right\}$ in $H_{\delta}^{2 m-\mathbf{m}-1 / 2}(\gamma), \delta>0$ is arbitrary. Now we apply Lemma 7.6. By this lemma, there exists a sequence $V^{n}=\left(V_{1}^{n}, \ldots, V_{N}^{n}\right)$ satisfying the following conditions: $V^{n} \in \mathcal{W}^{2 m}\left(K^{d}\right) \cap \mathcal{H}_{\delta}^{2 m}\left(K^{d}\right)$ for every $d>0$,

$$
\begin{equation*}
\mathcal{P}_{j}(D) V_{j}^{n}=0, \quad y \in K_{j}, \quad \mathcal{B}_{j \sigma \mu}(D) V^{n}=Z_{j \sigma \mu}^{n}(y), \quad y \in \gamma_{j \sigma}, \tag{19.10}
\end{equation*}
$$

and the sequence $V^{n}$ converges to the function $V \in \mathcal{H}_{\delta}^{2 m}\left(K^{d}\right)$ in $\mathcal{H}_{\delta}^{2 m}\left(K^{d}\right)$ for every $d>0$. Passing to the limit in Eq. (19.10) (in the spaces $\mathcal{H}_{\delta}^{0}\left(K^{d}\right)$ and $\mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(K^{d}\right)$, respectively), we obtain the equalities

$$
\begin{equation*}
\mathcal{P}_{j}(D) V_{j}=0, \quad y \in K_{j}, \quad \mathcal{B}_{j \sigma \mu}(D) V=Z_{j \sigma \mu}(y), \quad y \in \gamma_{j \sigma} . \tag{19.11}
\end{equation*}
$$

Consider a patch function $\xi \in C_{0}^{\infty}\left(\mathcal{O}_{\varepsilon^{\prime}}(0)\right)$, which is equal to 1 near the origin. Let $U=\xi V$. Obviously, $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon^{\prime}}(0)$ and

$$
U \in \mathcal{H}_{\delta}^{2 m}(K) \subset \mathcal{W}^{2 m-1}(K)
$$

2. Let us show that the function $U$ is as required. Indeed, using the Leibnitz formula, relations (19.11), and Lemmas 5.5 and 5.6, we derive Eq. (19.9).

It remains to show that $U \notin \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$. Suppose the contrary: let $U \in \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$. Then by virtue of the Sobolev embedding theorem and the relation $U \in \mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right)(\delta>0$ is arbitrary), we have $\left.D^{\alpha} U\right|_{y=0}=0,|\alpha| \leq 2 m-2$. It follows from here and Lemma 7.2 that the functions $\mathbf{B}_{j \sigma \mu}(y, D) U$ satisfy concordance condition (19.5). However, in this case, the functions $\mathbf{B}_{j \sigma \mu}(y, D) U-Z_{j \sigma \mu}$ do not satisfy concordance condition (19.5). This contradicts (19.9) (see Remark 19.1).
Proof of Theorem 19.2. 1. Let us construct a generalized solution $u \notin W^{2 m}(G)$ with the support near the set $\mathcal{K}$; in this case, by Eq. (6.5), $\mathbf{B}_{i \mu}^{2} u=0$ (for $l=0$ ).

It was proved in Lemma 7.3 that there exists a function $\left\{Z_{j \sigma \mu}\right\} \in \mathcal{W}^{2 m-\mathbf{m}-1 / 2}(\gamma)$ such that $\operatorname{supp} Z_{j \sigma \mu} \subset \mathcal{O}_{\varepsilon / 2}(0),\left.D_{\tau_{j \sigma}}^{\beta} Z_{j \sigma \mu}\right|_{y=0}=0, \beta \leq 2 m-m_{j \sigma \mu}-2$, and the functions $Z_{j \sigma \mu}$ do not satisfy concordance condition (19.5). By Lemma 19.1, there exists a function $U \in \mathcal{H}_{\delta}^{2 m}(K) \subset \mathcal{W}^{2 m}(K)$ such that $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon^{\prime}}(0), U \notin \mathcal{W}^{2 m}(K)$, and $U$ satisfies relations (19.9). Hence

$$
\left\{\mathbf{P}_{j}(y, D) U_{j}\right\} \in \mathcal{W}^{0}\left(K^{\varepsilon}\right), \quad\left\{\mathbf{B}_{j \sigma \mu}(y, D) U\right\} \in \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)
$$

and the functions $\mathbf{B}_{j \sigma \mu}(y, D) U$ do not satisfy concordance condition (19.5).
2. Introduce a function $u(y)$ such that $u(y)=U_{j}\left(y^{\prime}(y)\right)$ for $y \in \mathcal{O}_{\varepsilon^{\prime}}\left(g_{j}\right)$ and $u(y)=0$ for $y \notin \mathcal{O}_{\varepsilon^{\prime}}(\mathcal{K})$, where $y^{\prime} \mapsto y\left(g_{j}\right)$ is the change of variables, which is inverse to $y \mapsto y^{\prime}\left(g_{j}\right)$ (see Sec. 6.1, Chap. 2). Since supp $u \subset \mathcal{O}_{\varkappa_{1}}(\mathcal{K})$, we have $\mathbf{B}_{i \mu}^{2} u=0$. Hence $u(y)$ is a desired generalized solution of problem (18.1), (18.2).

Theorem 19.2 shows us that it is necessary for the right-hand sides $\left\{f_{0}, f_{i \mu}\right\}$ to belong to the space $\tilde{\mathcal{S}}^{0}(G, \partial G)$ if we want that any generalized solution of problem (18.1), (18.2) be smooth.

Let $v$ be an arbitrary function from $W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$. Consider the change of variables $y \mapsto y^{\prime}\left(g_{j}\right)$ from Sec. 6.1 (see Chap. 2). Introduce the functions

$$
\begin{equation*}
B_{j \sigma \mu}^{v}\left(y^{\prime}\right)=\left(\mathbf{B}_{i \mu}^{2} v\right)\left(y\left(y^{\prime}\right)\right), \quad y^{\prime} \in \gamma_{j \sigma}^{\varepsilon} \tag{19.12}
\end{equation*}
$$

(cf. functions (18.3)). Let us prove that the following condition is necessary and sufficient for any generalized solution to be smooth.
Condition 19.3. (1) For any $v \in W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$, the functions $B_{j \sigma \mu}^{v}$ satisfy concordance condition (19.5).
(2) For any vector-valued polynomial $W$ of degree $2 m-2$, the functions $\mathbf{B}_{j \sigma \mu}(y, D) W$ satisfy concordance condition (19.5).

Note that the fulfillment of condition 19.3 (unlike Condition 19.2) is independent of a special generalized solution. It depends only on the operators $\mathbf{B}_{i \mu}^{0}, \mathbf{B}_{i \mu}^{1}$, and $\mathbf{B}_{i \mu}^{2}$ and on the geometry of the domain $G$ near the set $\mathcal{K}$. This is natural since here we study the smoothness of all generalized solutions (while in Sec. 19.1, we studied the smoothness of a fixed solution).

Theorem 19.3. Let conditions 19.1 and 18.2-18.4 hold. Then the following statements hold.
(1) If condition 19.3 is valid and $u$ is a generalized solution of problem (18.1), (18.2) with the right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \tilde{\mathcal{S}}^{0}(G, \partial G)$, then $u \in W^{2 m}(G)$.
(2) If condition 19.3 is violated, then there exist a right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \tilde{\mathcal{S}}^{0}(G, \partial G)$ and a generalized solution $u$ of problem (18.1), (18.2) such that $u \notin W^{2 m}(G)$.

Proof. 1. Sufficiency. Let condition 19.3 be valid and let $u$ be an arbitrary generalized solution of problem (18.1), (18.2) with the right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \tilde{\mathcal{S}}^{0}(G, \partial G)$. By Eq. (13.21), we have $u \in$ $W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$. Hence the functions $B_{j \sigma \mu}^{u}$ (by condition 19.3) satisfy concordance condition (19.5). Let $W$ be a vector-valued polynomial of degree $2 m-2$ from Lemma 18.1. Using condition 19.3, we see that the functions $\mathbf{B}_{j \sigma \mu}(y, D) W$ satisfy concordance condition (19.5). Since $\left\{f_{i \mu}\right\} \in \tilde{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G)$, we see that the functions $f_{j \sigma \mu}$ satisfy concordance condition (19.5). Hence, the functions $\psi_{j \sigma \mu}=$ $f_{j \sigma \mu}-B_{j \sigma \mu}^{u}$ and $\mathbf{B}_{j \sigma \mu}(y, D) W$ satisfy condition 19.2. Applying Theorem 19.1, we obtain $u \in W^{2 m}(G)$.
2. Necessity. Let condition 19.3 hold. Then there exist a function $v \in W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$ and a vector-valued polynomial $W=\left(W_{1}, \ldots, W_{N}\right)$ of degree $2 m-2$ such that the functions $B_{j \sigma \mu}^{v}+\mathbf{B}_{j \sigma \mu} W$ do not satisfy concordance condition (19.5) (we can consider either $v=0, W \neq 0$, or $v \neq 0, W=0$ ). Let us extend the function $v$ to the domain $G$ such that $v(y)=0$ for $y \in \mathcal{O}_{\varkappa_{1} / 2}(\mathcal{K})$ and $v \in W^{2 m}(G)$.

By Lemma 5.3, there exist polynomials $f_{j \sigma \mu}^{\prime}(r)$ of degree $2 m-m_{j \sigma \mu}-2$ (if $m_{j \sigma \mu}=2 m-1$, then $\left.f_{j \sigma \mu}^{\prime}(r) \equiv 0\right)$ such that

$$
\left\{B_{j \sigma \mu}^{v}+\mathbf{B}_{j \sigma \mu}(y, D) W-f_{j \sigma \mu}^{\prime}\right\} \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \cap \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right),
$$

where $\delta>0$ is arbitrary. Hence

$$
D_{\tau_{j \sigma}}^{\beta}\left(B_{j \sigma \mu}^{v}+\mathbf{B}_{j \sigma \mu}(y, D) W-f_{j \sigma \mu}^{\prime}\right)(0)=0, \quad \beta \leq 2 m-m_{j \sigma \mu}-2 .
$$

Since $D_{r}^{2 m-m_{j \sigma \mu}-1} f_{j \sigma \mu}^{\prime}(r) \equiv 0$, we see that the functions $f_{j \sigma \mu}^{\prime}$ satisfy concordance condition (19.5). Then the functions $B_{j \sigma \mu}^{v}+\mathbf{B}_{j \sigma \mu}(y, D) W-f_{j \sigma \mu}^{\prime}$ do not satisfy concordance condition (19.5).

By Lemma 19.1, there exists a function $U^{\prime} \in \mathcal{H}_{\delta}^{2 m}(K) \subset \mathcal{W}^{2 m-1}(K)$ such that $\operatorname{supp} U^{\prime} \subset \mathcal{O}_{\varepsilon^{\prime}}(0)$, $U^{\prime} \notin \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$ and

$$
\begin{gather*}
\left\{\mathbf{P}_{j}(y, D) U_{j}^{\prime}\right\} \in \mathcal{W}^{0}\left(K^{\varepsilon}\right),  \tag{19.13}\\
\left.\left\{\mathbf{B}_{j \sigma \mu}(y, D) U^{\prime}-\left(f_{j \sigma \mu}^{\prime}-B_{j \sigma \mu}^{v}-\mathbf{B}_{j \sigma \mu}(y, D) W\right)\right)\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)
\end{gather*}
$$

We can rewrite the last relation as follows:

$$
\begin{equation*}
\left\{\mathbf{B}_{j \sigma \mu}(y, D)\left(U^{\prime}+W\right)+B_{j \sigma \mu}^{v}-f_{j \sigma \mu}^{\prime}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) . \tag{19.14}
\end{equation*}
$$

Introduce a function $u^{\prime}(y)$ such that $u^{\prime}(y)=U_{j}^{\prime}\left(y^{\prime}(y)\right)+\xi_{j}(y) W_{j}$ for $y \in \mathcal{O}_{\varepsilon^{\prime}}\left(g_{j}\right)$ and $u^{\prime}(y)=0$ for $y \notin \mathcal{O}_{\varepsilon^{\prime}}(\mathcal{K})$, where $y^{\prime} \mapsto y\left(g_{j}\right)$ is the change of variables inverse to the change $y \mapsto y^{\prime}\left(g_{j}\right)$ from Sec. 6.1 (see Chap. 2), $\xi_{j} \in C_{0}^{\infty}\left(O_{\varepsilon^{\prime}}\left(g_{j}\right)\right), \xi_{j}(y)=1$ for $y \in \mathcal{O}_{\varepsilon^{\prime} / 2}\left(g_{j}\right)$ and $\varepsilon^{\prime}$ is defined in Eq. (19.8). Let us prove that the function $u=u^{\prime}+v$ is as required. Obviously, $u \in W^{2 m-1}(G), u \notin W^{2 m}(G)$, and $u$ satisfies relations (13.21). It follows from $v \in W^{2 m}(G)$ and Eqs. (19.13) that

$$
\mathbf{P}(y, D) u \in L_{2}(G) .
$$

Consider the functions $f_{i \mu}=\mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u$. It follows from $v \in W^{2 m}(G)$, Eqs. (13.21), and inequality (6.5) (with $l=0$ ) that $f_{i \mu} \in W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i} \backslash \overline{\mathcal{O}_{\delta}(\mathcal{K})}\right)$ for any $\delta>0$. Consider
the behavior of $f_{i \mu}$ near the set $\mathcal{K}$. Note that, by virtue of (6.5) (with $l=0$ ), we have $\mathbf{B}_{i \mu}^{2} u^{\prime}=0$. Moreover, $\mathbf{B}_{i \mu}^{0} v+\mathbf{B}_{i \mu}^{1} v=0$ for $y \in \mathcal{O}_{\varkappa_{1} / d_{2}}(\mathcal{K})$. Hence,

$$
\begin{equation*}
f_{i \mu}=\mathbf{B}_{i \mu}^{0} u^{\prime}+\mathbf{B}_{i \mu}^{1} u^{\prime}+\mathbf{B}_{i \mu}^{2} v, \quad y \in \mathcal{O}_{\varkappa_{1} / d_{2}}(\mathcal{K}) . \tag{19.15}
\end{equation*}
$$

Introduce the functions $f_{j \sigma \mu}\left(y^{\prime}\right)=f_{i \mu}\left(y\left(y^{\prime}\right)\right)$, where $y \mapsto y^{\prime}\left(g_{j}\right)$ is the change of variables from Sec. 6.1 (see Chap. 2). Equations (19.15) and (19.14) yield $\left\{f_{j \sigma \mu}-f_{j \sigma \mu}^{\prime}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$. Hence, $\left\{f_{j \sigma \mu}\right\} \in \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$ and the functions $f_{j \sigma \mu}$ (as well as $f_{j \sigma \mu}^{\prime}$ ) satisfy concordance condition (19.5). Thus, $\left\{f_{i \mu}\right\} \in \tilde{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G)$.

### 19.3. Problem with regular nonlocal conditions.

Definition 19.2. A function $v \in W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$ is said to be admissible if there exists a vectorvalued polynomial $W=\left(W_{1}, \ldots, W_{N}\right)$ of degree $2 m-2$ such that

$$
\begin{align*}
& \left.D_{\tau_{j \sigma}}^{\beta}\left(B_{j \sigma \mu}^{v}+\mathbf{B}_{j \sigma \mu}(y, D) W\right)\right|_{y=0}=0, \\
& \beta \leq 2 m-m_{j \sigma \mu}-2, \quad j=1, \ldots, N, \quad \sigma=1,2, \quad \mu=1, \ldots, m . \tag{19.16}
\end{align*}
$$

Any vector $W$ of degree $2 m-2$ satisfying relations (19.16) is called an admissible vector-valued polynomial corresponding to the function $v$.

Remark 19.4. The set of admissible functions is linear. Obviously, the function $v=0$ is admissible and $W=0$ is the admissible vector corresponding to it.

The set of admissible vector-valued polynomials that correspond to an admissible function $v$ form an affine space of the form
$\{W+\tilde{W}: \tilde{W}$ is a vector-valued polynomial of degree $2 m-2$,

$$
\begin{equation*}
\left.\left.D_{\tau_{j \sigma}}^{\beta} \mathbf{B}_{j \sigma \mu}(y, D) \tilde{W}\right|_{y=0}=0, \beta \leq 2 m-m_{j \sigma \mu}-2\right\}, \tag{19.17}
\end{equation*}
$$

where $W$ is a fixed, admissible vector-valued polynomial that corresponds to $v$.
Definition 19.3. The right-hand sides $f_{i \mu}$ in nonlocal conditions (18.2) are said to be regular if
(1) condition 18.1 holds and $\left\{f_{i \mu}\right\} \in \mathcal{S}^{2 m-\mathbf{m}-1 / 2}(\partial G)$ or
(2) condition 19.1 holds and $\left\{f_{i \mu}\right\} \in \hat{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G)$.

The right-hand sides $\psi_{j \sigma \mu}$ in nonlocal conditions (18.5) are said to be regular if
(1) condition 18.1 holds and $\left\{\psi_{j \sigma \mu}\right\} \in \mathcal{S}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$ or
(2) condition 19.1 holds and $\left\{\psi_{j \sigma \mu}\right\} \in \hat{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$.

Thus, the regular right-hand sides $f_{i \mu}\left(\psi_{j \sigma \mu}\right)$ have a zero of a certain order near the set $\mathcal{K}$ (respectively, near the origin). In particular, the right-hand sides $\left\{f_{i \mu}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}(\partial G)$ and $\left\{\psi_{j \sigma \mu}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$ are regular by virtue of the Sobolev embedding theorem and Remark 19.1.

If the right-hand sides $f_{i \mu}$ from nonlocal conditions (18.2) are regular, we also say that the right-hand side $\left\{f_{0}, f_{i \mu}\right\}$ of problem (18.1), (18.2) is regular.

We prove that the following statement is necessary and sufficient for any generalized solution of problem (18.1), (18.2) with a regular right-hand side $\left\{f_{i \mu}\right\} \in \hat{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G)$ to be smooth.
Condition 19.4. The functions $B_{j \sigma \mu}^{v}+\mathbf{B}_{j \sigma \mu}(y, D) W$ satisfy concordance condition (19.5) for any admissible function $v$ and any admissible vector-valued polynomial $W$ (of degree $2 m-2$ ) that corresponds to $v$.

Note that condition 19.4 is weaker than condition 19.3.
Theorem 19.4. Let conditions 19.1 and 18.2-18.4. Then the following conditions hold.
(1) If condition 19.4 holds and a function $u$ is a generalized solution of problem (18.1), (18.2) with a regular right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \hat{\mathcal{S}}^{0}(G, \partial G)$, then $u \in W^{2 m}(G)$.
(2) If condition 19.4 is violated, then there exists a right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{H}_{0}^{0}(G, \partial G)$ and a generalized solution $u$ of problem (18.1), (18.2) such that $u \notin W^{2 m}(G)$.

Proof. 1. Sufficiency. Let condition 19.4 hold, and let $u$ be a generalized solution of problem (18.1), (18.2) with a regular right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \hat{\mathcal{S}}^{0}(G, \partial G)$. By virtue of Eq. (13.21), we have $u \in W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$.

It follows from the conditions on the functions $f_{i \mu}$ that the right-hand sides in nonlocal conditions (18.5) have the following form:

$$
\begin{equation*}
\psi_{j \sigma \mu}=f_{j \sigma \mu}-B_{j \sigma \mu}^{u} \tag{19.18}
\end{equation*}
$$

where $\left\{f_{j \sigma \mu}\right\} \in \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$,

$$
\begin{equation*}
\left.D_{\tau_{j \sigma}}^{\beta} f_{j \sigma \mu}\right|_{y=0}=0, \quad \beta \leq 2 m-m_{j \sigma \mu}-2, \tag{19.19}
\end{equation*}
$$

and $f_{j \sigma \mu}$ satisfies concordance condition (19.5).
Further, let $U=W+U^{\prime}$, where $U^{\prime} \in \mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right)$ and $W$ is a function and a vector-valued polynomial (of degree $2 m-2$ ) from Lemma 18.1. Equations (18.5) and (19.18) yield

$$
\mathbf{B}_{j \sigma \mu}(y, D) U^{\prime}=f_{j \sigma \mu}-\left(B_{j \sigma \mu}^{u}+\mathbf{B}_{j \sigma \mu}(y, D) W\right) .
$$

Since

$$
\left\{B_{j \sigma \mu}^{u}+\mathbf{B}_{j \sigma \mu}(y, D) W-f_{j \sigma \mu}\right\} \in \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right), \quad U^{\prime} \in \mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right),
$$

we have

$$
\left\{B_{j \sigma \mu}^{u}+\mathbf{B}_{j \sigma \mu}(y, D) W-f_{j \sigma \mu}\right\}=\left\{-\mathbf{B}_{j \sigma \mu}(y, D) U^{\prime}\right\} \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \cap \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) .
$$

It follows from here and Eq. (19.19) that

$$
\left.D_{\tau_{j \sigma}}^{\beta}\left(B_{j \sigma \mu}^{u}+\mathbf{B}_{j \sigma \mu}(y, D) W\right)\right|_{y=0}=0, \quad \beta \leq 2 m-m_{j \sigma \mu}-2
$$

i.e., $u$ is an admissible function and $W$ is an admissible vector-valued polynomial corresponding to $u$. Hence, by virtue of Eq. (19.18) and condition 19.4, condition 19.2 holds. Thus Theorem 19.1 implies that $u \in W^{2 m}(G)$.
2. Necessity. Let condition 19.4 be violated. Then there exist a function $v \in W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$ and a vector-valued polynomial $W=\left(W_{1}, \ldots, W_{N}\right)$ of degree $2 m-2$ such that

$$
\left.D_{\tau_{j \sigma}}^{\beta}\left(B_{j \sigma \mu}^{u}+\mathbf{B}_{j \sigma \mu}(y, D) W\right)\right|_{y=0}=0, \quad \beta \leq 2 m-m_{j \sigma \mu}-2,
$$

and the functions $B_{j \sigma \mu}^{v}+\mathbf{B}_{j \sigma \mu}(y, D) W$ do not satisfy concordance condition (19.5).
We must obtain a function $u \in W^{\ell}(G)$ satisfying relations (13.21) and such that $u \notin W^{2 m}(G)$ and

$$
\mathbf{P}(y, D) u \in L_{2}(G), \quad\left\{\mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}(\partial G) .
$$

For this, it suffices to repeat the reasoning of the proof of statement 2 of Theorem 19.3 assuming that $v$ is the above-mentioned function, $W$ be the above-mentioned polynomial, and $f_{j \sigma \mu}^{\prime}(y) \equiv 0$; this is possible by virtue of the relation

$$
B_{j \sigma \mu}^{v}+\mathbf{B}_{j \sigma \mu}(y, D) W \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \cap \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right),
$$

where $\delta>0$ is arbitrary.
19.4. Homogeneous nonlocal conditions. Violation of smoothness of generalized solutions. Here, we consider some situations where the violation of condition 19.4 leads to the violation of smoothness of generalized solutions even in the case of homogeneous conditions.

As well as in Sec. 11 (see Chap. 3) and Sec. 16 (see Chap. 4), we assume that the "local" operators form a normal system (see, e.g., [57, Chap. 2, Sec. 1]).
Condition 19.5. Thr system of operators $\left\{\mathbf{B}_{i \mu}^{0}\right\}_{\mu=1}^{m}$ is normal $\overline{\Gamma_{i}}, i=1, \ldots, N$.
Lemma 19.2. If condition 19.5 holds, then the following assertions are valid.
(1) Let $\left\{f_{i \mu}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}(\partial G)$. Then there exists a function $u_{0} \in H_{0}^{2 m}(G)$ such that

$$
\begin{gather*}
\operatorname{supp} u_{0} \subset \mathcal{O}_{\varkappa_{1}}(\mathcal{K}), \\
\mathbf{B}_{i \mu}^{0} u_{0}=f_{i \mu}(y), \quad y \in \Gamma_{i} \cap \mathcal{O}_{\varkappa_{2}}(\mathcal{K}), \\
\mathbf{B}_{i \mu}^{1} u_{0}=\mathbf{B}_{i \mu}^{2} u_{0}=0 . \tag{19.20}
\end{gather*}
$$

(2) Let $\left\{f_{i \mu}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}(\partial G)$ and $\operatorname{supp} f_{i \mu} \subset \mathcal{O}_{\varkappa_{2}}(\mathcal{K})$. Then there exists a function $u_{0} \in H_{0}^{2 m}(G)$ such that

$$
\begin{gathered}
\operatorname{supp} u_{0} \subset \mathcal{O}_{\varkappa_{2}}(\mathcal{K}), \\
\mathbf{B}_{i \mu}^{0}=f_{i \mu}(y), \quad y \in \Gamma_{i},
\end{gathered}
$$

and relations (19.20) hold.
Proof. 1. Using Lemma 11.1, the partition of unity, and the corresponding patch functions supported in $\mathcal{O}_{\varkappa_{1}}(\mathcal{K})$, we construct a function $u_{0} \in H_{0}^{2 m}(G)$ such that

$$
\begin{gather*}
\operatorname{supp} u_{0} \subset \mathcal{O}_{\varkappa_{1}}(\mathcal{K}),  \tag{19.21}\\
\mathbf{B}_{i \mu}^{0} u_{0}=f_{i \mu}(y), \quad y \in \Gamma_{i} \cap \mathcal{O}_{\varkappa_{2}}(\mathcal{K}),  \tag{19.22}\\
\mathbf{B}_{i \mu}^{1} u_{0}=0 .
\end{gather*}
$$

Equations (19.21) and (6.5) (as $l=0$ ) yield $\mathbf{B}_{i \mu}^{2} u_{0}=0$. Hence, $u_{0}$ is a required function.
2. If supp $f_{i \mu} \subset \mathcal{O}_{\varkappa_{2}}(\mathcal{K})$, then we can assume that $\operatorname{supp} u_{0} \subset \mathcal{O}_{\varkappa_{2}}(\mathcal{K})$. In this case Eqs. (19.22) are valid for $y \in \Gamma_{i}$.

First, we consider the violation of smoothness if the right-hand sides in the nonlocal conditions vanish near the set $\mathcal{K}$.

Corollary 19.1. Let conditions 19.1, 19.5, and 18.2-18.4 hold. If condition 19.4 is violated, then there exist a right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{H}_{0}^{0}(G, \partial G)$, where $f_{i \mu}(y)=0$ for $y \in \Gamma_{i} \cap \mathcal{O}_{\varkappa_{2}}(\mathcal{K})$, and a generalized solution $u$ of problem (18.1), (18.2) such that $u \notin W^{2 m}(G)$.

Proof. The proof of this corollary follows from statement 2 of Theorem 19.4, statement 1 of Lemma 19.2, and the embedding $H_{0}^{2 m}(G) \subset W^{2 m}(G)$.

Now we study the violation of smoothness in the case where the right-hand sides in the nonlocal conditions vanish on the whole boundary of the domain.

Statement 2 of Theorem 19.4 yields the following assertion.
Corollary 19.2. Let conditions 19.1 and 18.2-18.4 hold, but let condition 19.4 be violated. Let $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{H}_{0}^{0}(G, \partial G)$ be a function from statement 2 of Theorem 19.4. Assume that there exists a function $u_{0} \in W^{2 m}(G)$ such that

$$
\begin{equation*}
\mathbf{B}_{i \mu}^{0} u_{0}+\mathbf{B}_{i \mu}^{1} u_{0}+\mathbf{B}_{i \mu}^{2} u_{0}=f_{i \mu}(y), \quad y \in \Gamma_{i} . \tag{19.23}
\end{equation*}
$$

Then there exist a right-hand side $\left\{f_{0}, 0\right\}$, where $f_{0} \in L_{2}(G)$, and a generalized solution $u$ of problem (18.1), (18.2) such that $u \notin W^{2 m}(G)$.

In the general case, this corollary does not give us constructive algorithms of construction of function $u_{0}$ satisfying relations (19.23). However, we can prove the existence of such a function (in some particular cases described in Corollaries 19.3 and 19.4; see also Sec. 22.2).

Corollary 19.3. Assume that the operators $\mathbf{B}_{i \mu}^{2}$ satisfy the condition

$$
\begin{equation*}
\left\|\mathbf{B}_{i \mu}^{2} v\right\|_{W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)} \leq c\|v\|_{W^{2 m}\left(G_{\rho}\right)} \quad \forall v \in W^{2 m}\left(G_{\rho}\right) \tag{19.24}
\end{equation*}
$$

for some $\rho>0$. Let conditions 19.1, 19.5, and 18.2-18.4 hold, but let condition 19.4 be violated. Then Corollary 19.2 is valid.

The proof follows from Corollary 19.2, the embedding $H_{0}^{2 m}(G) \subset W^{2 m}(G)$, and the next lemma.
Lemma 19.3. Let condition 19.5 hold. Let the operators $\mathbf{B}_{i \mu}^{2}$ satisfy condition (19.24) and let $\left\{f_{i \mu}\right\} \in$ $\mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}(\partial G)$. Then there exists a function $u_{0} \in H_{0}^{2 m}(G)$ satisfying (19.23).

Proof. Using Lemma 11.1 and the partition of unity, we construct a function $u_{0} \in H_{0}^{2 m}(G)$ such that

$$
\begin{gather*}
\operatorname{supp} u_{0} \subset \bar{G} \backslash \overline{G_{\rho}},  \tag{19.25}\\
\mathbf{B}_{i \mu}^{0} u_{0}=f_{i \mu}, \quad \mathbf{B}_{i \mu}^{1} u_{0}=0 .
\end{gather*}
$$

By virtue of Eq. (19.25) and (19.24), we have $\mathbf{B}_{i \mu}^{2} u_{0}=0$. Hence, $u_{0}$ satisfies (19.23).
Remark 19.5. Condition (19.24) (which is stricter than condition 6.4) means that the operators $\mathbf{B}_{i \mu}^{2}$ correspond to nonlocal terms supported inside the domain $G$.

Corollary 19.4. Let conditions 19.1, 19.5, and 18.2-18.4 hold. Let condition 19.4 be violated for an admissible function $v$ such that

$$
\begin{equation*}
\operatorname{supp}\left(\mathbf{B}_{i \mu}^{0} v+\mathbf{B}_{i \mu}^{1} v+\mathbf{B}_{i \mu}^{2} v\right) \subset \Gamma_{i} \cap \mathcal{O}_{\varkappa_{2}}(\mathcal{K}) . \tag{19.26}
\end{equation*}
$$

Then Corollary 19.2 is valid.
Proof. If

$$
\operatorname{supp}\left(\mathbf{B}_{i \mu}^{0} v+\mathbf{B}_{i \mu}^{1} v+\mathbf{B}_{i \mu}^{2} v\right) \subset \Gamma_{i} \cap \mathcal{O}_{\varkappa_{2}}(\mathcal{K})
$$

then the function

$$
\left\{f_{i \mu}\right\}=\left\{\mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}(\partial G)
$$

constructed in the proof of statement 2 of Theorem 19.4 (see also proof of statement 2 of Theorem 19.3) has support lying in $\mathcal{O}_{\varkappa_{2}}(\mathcal{K})$. Hence, applying statement 2 of Lemma 19.2, we obtain the function $u_{0}$ satisfying Eq. (19.23). Using Corollary 19.2, we complete the proof.

## 20. Nonlocal Conditions of Special Form. <br> Regular and Zero Right-hand Sides

In this section, we show that, in some cases, the preservation of the smoothness of generalized solutions of problem (18.1), (18.2) is independent of conditions 18.3 and 18.4. We consider regular (see Definition 19.3) or, in particular, zero right-hand sides in the nonlocal conditions. If the right-hand sides are irregular (and $\ell \leq 2 m-2$ ), then conditions 18.3 and 18.4 are necessary for the preservation of smoothness of generalized solutions (see Theorem 21.2 in Sec. 21.2). Obviously, if $\ell=2 m-1$, then conditions 18.3 and 18.4 are absent; in this case, the results of this section follow from the results of the previous sections.
20.1. Nonlocal conditions of a special form. Auxiliary results. Assume that one of the following two conditions holds.

Condition 20.1. (1) $\ell \leq 2 m-2$;
(2) $\mathbf{B}_{j \sigma \mu}(y, D)=\mathcal{B}_{j \sigma \mu}(D)$ for $y \in \gamma_{j \sigma}^{\varepsilon}$;
(3) the set $\Lambda$ contains only regular eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$.

Item 2 in condition 20.1 means that the operators $B_{i \mu s}(y, D)$ are homogeneous and have constant coefficients near the set $\mathcal{K}$.

Condition 20.2. (1) $\ell \leq 2 m-2$;
(2) if $\ell \geq 1, Q$ is a homogeneous polynomial of degree not higher than $\ell-1$, and

$$
\left.\mathcal{B}_{j \sigma \mu}(D) Q\right|_{\gamma_{j \sigma}}=0
$$

for all $j, \sigma, \mu$, then $Q=0$;
(3) the set $\Lambda$ is empty.

Remark 20.1. Item 2 in condition 20.1 holds, for example, in the case of a "local" Dirichlet problem. In this case, if we search for solutions in the space $W^{\ell}(G)$ for $\ell \leq m$, then item 2 in condition 20.2 also holds. Thus, the results of this section generalize results of Kondrat'ev (see [53, Sec. 5]).

Finally, assume that the abstract nonlinear operators $\mathbf{B}_{i \mu}^{2}$ "have a zero of a certain order" at points of the set $\mathcal{K}$.
Condition 20.3. $\left.D_{\tau_{j \sigma}}^{\beta} B_{j \sigma \mu}^{v}\right|_{y=0}=0, \beta \leq 2 m-m_{j \sigma \mu}-2$, for any function $v \in W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$, where $B_{j \sigma \mu}^{v}(y)$ are functions defined in $E q$. (19.12) (if $m_{j \sigma \mu}=2 m-1$, then the corresponding relations are absent).

We prove the following analog of Lemma 18.1.
Lemma 20.1. Let either condition 20.1 or 20.2 hold. Let $U \in \mathcal{W}^{\ell}\left(K^{\varepsilon}\right)$ be a solution of problem (18.4), (18.5) with the right-hand side $\left\{f_{j}, \psi_{j \sigma \mu}\right\} \in \mathcal{S}^{0}\left(K^{\varepsilon}, \gamma^{\varepsilon}\right)$. Then

$$
\begin{equation*}
U=W+U^{\prime} \tag{20.1}
\end{equation*}
$$

where $U^{\prime} \in \mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right)$ for any $\delta>0, W=\left(W_{1}, \ldots, W_{N}\right)$ is a vector-valued polynomial of degree $2 m-2$ such that if condition 20.1 holds, then

$$
\begin{equation*}
\left.\mathcal{B}_{j \sigma \mu}(D) W\right|_{\gamma_{j \sigma}}=0 \tag{20.2}
\end{equation*}
$$

and if condition 20.2 holds, then $W=0$.
Proof. 1. Since $\left\{\psi_{j \sigma \mu}\right\} \in \mathcal{S}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$, it follows from Lemma 5.4 that $\left\{\psi_{j \sigma \mu}\right\} \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$ for all $\delta>0$. In particular, this and the embedding $\mathcal{W}^{0}\left(K^{\varepsilon}\right) \subset \mathcal{H}_{\delta}^{0}\left(K^{\varepsilon}\right)$ (for any $\delta>0$ ) imply that

$$
\begin{equation*}
\left\{\mathbf{P}_{j}(y, D) U_{j}\right\} \in \mathcal{H}_{\delta}^{0}\left(K^{\varepsilon}\right), \quad\left\{\mathbf{B}_{j \sigma \mu}(y, D) U\right\} \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \quad \forall \delta>0 \tag{20.3}
\end{equation*}
$$

Consider a number $\delta>0$ for which the strips

$$
\begin{equation*}
1-\delta \leq \operatorname{Im} \lambda<1, \quad-\delta \leq \operatorname{Im} \lambda<0, \quad \ldots, \quad 1-\ell-\delta \leq \operatorname{Im} \lambda<1-\ell \tag{20.4}
\end{equation*}
$$

do not contain eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ (Lemma 6.1 guarantees the existence of such $\delta$ ).
It follows from Eq. (20.3), the relation $U \in \mathcal{H}_{2 m}^{2 m}\left(K^{\varepsilon}\right)$ (see (15.14)), and Lemmas 5.5 and 5.6 that

$$
\begin{equation*}
\left\{\mathcal{P}_{j}(D) U_{j}\right\} \in \mathcal{H}_{2 m-\delta}^{0}\left(K^{\varepsilon}\right), \quad\left\{\mathcal{B}_{j \sigma \mu}(y, D) U\right\} \in \mathcal{H}_{2 m-\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) . \tag{20.5}
\end{equation*}
$$

Using Eq. (20.5), [26, Theorem 2.2], and the fact that there are no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ in the strip $1-\delta \leq \operatorname{Im} \lambda<1$, we obtain the embedding

$$
\begin{equation*}
U \in \mathcal{H}_{2 m-\delta}^{2 m}\left(K^{\varepsilon}\right) \tag{20.6}
\end{equation*}
$$

Equations (20.3) and (20.6) and Lemmas 5.5 and 5.6 yield

$$
\begin{equation*}
\left\{\mathcal{P}_{j}(D) U_{j}\right\} \in \mathcal{H}_{2 m-1-\delta}^{0}\left(K^{\varepsilon}\right), \quad\left\{\mathcal{B}_{j \sigma \mu}(y, D) U\right\} \in \mathcal{H}_{2 m-1-\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \quad \forall \delta>0 . \tag{20.7}
\end{equation*}
$$

Hence, using Eq. (20.7) and [26, Theorem 2.2], we obtain the equality

$$
\begin{equation*}
U=W^{1}+U^{1} \tag{20.8}
\end{equation*}
$$

where

$$
W^{1}=\sum_{n=1}^{n_{1}} \sum_{l=0}^{l_{1}} r^{i \mu_{n}}(i \ln r)^{l} \varphi_{n l}(\omega),
$$

$\left\{\mu_{1}, \ldots, \mu_{n_{1}}\right\}$ is the set of all eigenvalues from the strip $0 \leq \operatorname{Im} \lambda<1-\delta$ (we must take eigenvalues from the strip $-\delta<\operatorname{Im} \lambda<1-\delta$, but the second strip in Eq. (20.4) does not contain eigenvalues), $\varphi_{n l} \in \mathcal{W}^{2 m}(-\bar{\omega}, \bar{\omega})$ and $U^{1} \in \mathcal{H}_{2 m-1-\delta}^{2 m}\left(K^{\varepsilon}\right) ;$ moreover,

$$
\begin{equation*}
\left.\mathcal{B}_{j \sigma \mu}(D) W^{1}\right|_{\gamma_{j \sigma}}=0 . \tag{20.9}
\end{equation*}
$$

Taking into account the inequality $\operatorname{Re} i \mu_{n} \leq 0$, the relation

$$
W^{1}=U-U^{1} \in \mathcal{W}^{1}\left(K^{\varepsilon}\right)
$$

and [53, Lemma 4.20] we obtain that if condition 20.2 holds, then $W^{1}$ is a homogeneous vector-valued polynomial of degree 0 with respect to variables $y_{1}$ and $y_{2}$ (i.e., a constant vector) and $W^{1}=0$.
2. If $2 m-2>0$, then we take the following step. Using Eqs. (20.3) and (20.8), Lemma 5.5, and the fact that $W^{1}$ is a vector-valued polynomial (in the following step, it is a constant vector), we obtain the following equality:

$$
\begin{equation*}
\left\{\mathcal{P}_{j}(D) U_{j}^{1}\right\}=\left\{\mathbf{P}_{j}(y, D) U_{j}\right\}-\left\{\mathbf{P}(y, D) W_{j}^{1}\right\}+\left\{\left(\mathcal{P}_{j}(D)-\mathbf{P}_{j}(y, D)\right) U_{j}^{1}\right\} \in \mathcal{H}_{2 m-2-\delta}^{0}\left(K^{\varepsilon}\right) . \tag{20.10}
\end{equation*}
$$

If condition 20.1 hold, then, using the equality $\mathbf{B}_{j \sigma \mu}(y, D)=\mathcal{B}_{j \sigma \mu}(D)$ and the relations (20.3), (20.8), and (20.9), we obtain the equality

$$
\begin{equation*}
\left\{\mathcal{B}_{j \sigma \mu}(D) U^{1}\right\}=\left\{\mathbf{B}_{j \sigma \mu}(y, D) U\right\} \in \mathcal{H}_{2 m-2-\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) . \tag{20.11}
\end{equation*}
$$

If condition 20.2 holds, then $W^{1}=0$, i.e., $U=U^{1}$ and, by virtue of Eq. (20.3) and Lemma 5.6, we have

$$
\begin{equation*}
\left\{\mathcal{B}_{j \sigma \mu}(D) U^{1}\right\}=\left\{\mathbf{B}_{j \sigma \mu}(y, D) U\right\}+\left\{\left(\mathcal{B}_{j \sigma \mu}(D)-\mathbf{B}_{j \sigma \mu}(y, D)\right) U^{1}\right\} \in \mathcal{H}_{2 m-2-\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) . \tag{20.12}
\end{equation*}
$$

Repeating this procedure finitely many times, we obtain

$$
\begin{gathered}
U=W^{1}+\cdots+W^{\ell}+U^{\ell} \\
\left.\mathcal{B}_{j \sigma \mu}(D) W^{1}\right|_{\gamma_{j \sigma}}=\cdots=\left.\mathcal{B}_{j \sigma \mu}(D) W^{\ell}\right|_{\gamma_{j \sigma}}=0
\end{gathered}
$$

where $W^{s}, s=1, \ldots, \ell$, is a homogeneous vector-valued polynomial of degree $s-1$ and $U^{\ell} \in$ $\mathcal{H}_{2 m-\ell-\delta}^{2 m}\left(K^{\varepsilon}\right)$. Moreover, if condition 20.2 holds, then $W^{1}=\cdots=W^{\ell}=0$ and

$$
U=U^{\ell} \in \mathcal{H}_{2 m-\ell-\delta}^{2 m}\left(K^{\varepsilon}\right)
$$

3. Similarly to Eqs. (20.10)-(20.12) we can verify that the function $U^{\ell}$ satisfies the relations

$$
\left\{\mathcal{P}(D) U_{j}^{\ell}\right\} \in \mathcal{H}_{2 m-\ell-1-\delta}^{0}\left(K^{\varepsilon}\right), \quad\left\{\mathcal{B}_{j \sigma \mu}(D) U^{\ell}\right\} \in \mathcal{H}_{2 m-\ell-1-\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) .
$$

Repeating the procedure again, we obtain

$$
\begin{gathered}
U=W^{1}+\cdots+W^{2 m-1}+U^{\prime} \\
\left.\mathcal{B}_{j \sigma \mu}(D) W^{\ell+1}\right|_{\gamma_{j \sigma}}=\cdots=\left.\mathcal{B}_{j \sigma \mu}(D) W^{2 m-1}\right|_{\gamma_{j \sigma}}=0
\end{gathered}
$$

where $W^{\ell+1}, \ldots, W^{2 m-1}$ are homogeneous vector-valued polynomials of degrees $\ell, \ldots, 2 m-2$, respectively. They appear since if condition 20.1 holds and $W^{\ell+1}=\cdots=W^{2 m-1}=0$ (by virtue of the fact
that $\Lambda=\varnothing$ ) and if condition 20.2 holds, then the set $\Lambda$ contains only regular eigenvalues. Finally, $U^{\prime} \in \mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right)$.
20.2. Main results. We prove an analog of Theorem 18.1.

Theorem 20.1. Let conditions 18.1 and 20.3 hold. Assume also that conditions 20.1 or 20.2 hold. Let $u$ be a generalized solution of problem (18.1), (18.2) with a regular right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in$ $\mathcal{S}^{0}(G, \partial G)$. Then $u \in W^{2 m}(G)$.

Proof. Since the right-hand sides $f_{i \mu}$ are regular and condition 20.3 holds, we have

$$
\psi_{j \sigma \mu}=f_{j \sigma \mu}-B_{j \sigma \mu}^{u} \in \mathcal{S}^{2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}^{\varepsilon}\right)
$$

(i.e., the right-hand sides $\psi_{j \sigma \mu}$ are regular). Hence, Lemma 20.1 is valid. By virtue of (18.4), (18.5), (20.1), and (20.2), the function $U^{\prime} \in \mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right)$ from Lemma 20.1 satisfies the following relations:

$$
\begin{align*}
& \left\{\mathbf{P}_{j}(y, D) U_{j}^{\prime}\right\}=\left\{f_{j}-\mathbf{P}_{j}(y, D) W_{j}\right\} \in \mathcal{H}_{0}^{0}\left(K^{\varepsilon}\right) \\
& \left\{\mathbf{B}_{j \sigma \mu}(y, D) U^{\prime}\right\} \in \mathcal{H}_{\delta}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \quad \forall \delta>0  \tag{20.13}\\
& \left\{\mathbf{B}_{j \sigma \mu}(y, D) U^{\prime}\right\}=\left\{\psi_{j \sigma \mu}-\mathbf{B}_{j \sigma \mu}(y, D) W\right\} \in \mathcal{W}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)
\end{align*}
$$

(cf. (18.13)). Similarly to the proof of Lemma 18.2 , we obtain from Eq. (20.13) that $U^{\prime} \in \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$; hence, $U \in \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$. Combining this embedding with Eq. (13.21), we complete the proof.

Further we prove an analog of Lemma 19.4, where we study the situation where the line $\operatorname{Im} \lambda=1-2 m$ contains only an eigenvalue $i(1-2 m)$ of the operator $\tilde{\mathcal{L}}(\lambda)$.

The following condition is an analog of condition 19.4.
Condition 20.4. The functions $B_{j \sigma \mu}^{v}$ satisfy concordance condition (19.5) for any function $v \in$ $W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$.
Theorem 20.2. Let conditions 19.1 and 20.3 hold. Assume that either condition 20.1 or condition 20.2 is valid. Then the following assertions are valid.
(1) If condition 20.4 holds and $u$ is a generalized solution of problem (18.1), (18.2) with the regular right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \hat{\mathcal{S}}^{0}(G, \partial G)$, then $u \in W^{2 m}(G)$.
(2) If condition 20.4 is violated, then there exist a right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{H}_{0}^{0}(G, \partial G)$ and a generalized solution $u$ of problem (18.1), (18.2) such that $u \notin W^{2 m}(G)$.

Proof. 1. Sufficiency. Let condition 20.4 hold and let $u$ be a generalized solutions of problem (18.1), (18.2) with the regular right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \hat{\mathcal{S}}^{0}(G, \partial G)$. Since the right-hand sides $f_{i \mu}$ are regular, we see that, by virtue of condition 20.3,

$$
\psi_{j \sigma \mu}=f_{j \sigma \mu}-B_{j \sigma \mu}^{u} \in \mathcal{S}^{2 m-m_{j \sigma \mu}-1 / 2}\left(\gamma_{j \sigma}^{\varepsilon}\right)
$$

Hence, Lemma 20.1 is valid. By Eqs. (18.4), (18.5), (20.1), and (20.2), the function $U^{\prime} \in \mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right)$ from Lemma 20.1 satisfies relations (20.13).

Further, we note that $\left\{f_{i \mu}\right\} \in \hat{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G)$, i.e., the functions $f_{j \sigma \mu}$ satisfy concordance condition (19.5). By condition 20.4, the functions $B_{j \sigma \mu}^{u}$ also satisfy concordance condition (19.5). Hence, the functions $\psi_{j \sigma \mu}=f_{j \sigma \mu}-B_{j \sigma \mu}^{u}$ satisfy concordance condition (19.5).

Let $W$ be a vector-valued polynomial from Lemma 20.1. We show that the functions

$$
\mathbf{B}_{j \sigma \mu}(y, D) W
$$

satisfy concordance condition (19.5). Indeed, if condition 20.1 holds, then, using Eq. (20.2), we have

$$
\left.D_{\tau_{j \sigma}}^{2 m-m_{j \sigma \mu}-1} \mathbf{B}_{j \sigma \mu}(y, D) W\right|_{\gamma_{j \sigma}^{\varepsilon}}=\left.D_{\tau_{j \sigma}}^{2 m-m_{j \sigma \mu}-1} \mathcal{B}_{j \sigma \mu}(D) W\right|_{\gamma_{j \sigma}^{\varepsilon}}=0 .
$$

If condition 20.2 holds, then by Lemma 20.1 we see that $W=0$. Thus, in both cases, the functions $\mathbf{B}_{j \sigma \mu}(y, D) W$ satisfy concordance condition (19.5).

Hence, using the embedding $U^{\prime} \in \mathcal{H}_{\delta}^{2 m}\left(K^{\varepsilon}\right)$ and relations (20.13), we can repeat the reasoning from the proof of the sufficiency of Theorem 19.1. As a result, we obtain $U^{\prime} \in \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$; hence, $U \in \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$. This and Eq. (13.21) imply that $u \in W^{2 m}(G)$.
2. Necessity. Assume that condition 20.4 is violated. Then there exists a function $v \in W^{2 m}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$ such that the functions $B_{j \sigma \mu}^{v}$ do not satisfy concordance condition (19.5). Let us extend the function $v$ to the domain $G$ such that $v(y)=0$ for $y \in \mathcal{O}_{\varkappa_{1} / 2}(\mathcal{K})$ and $v \in W^{2 m}(G)$.

By Lemma 19.1, there exists a function $U^{\prime} \in \mathcal{H}_{\delta}^{2 m}(K) \subset \mathcal{W}^{2 m-1}(K)$ such that $\operatorname{supp} U^{\prime} \subset \mathcal{O}_{\varepsilon^{\prime}}(0)$, $U^{\prime} \notin \mathcal{W}^{2 m}\left(K^{\varepsilon}\right)$ and

$$
\begin{gather*}
\left\{\mathbf{P}_{j}(y, D) U_{j}^{\prime}\right\} \in \mathcal{W}^{0}\left(K^{\varepsilon}\right)  \tag{20.14}\\
\left\{\mathbf{B}_{j \sigma \mu}(y, D) U^{\prime}+B_{j \sigma \mu}^{v}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) . \tag{20.15}
\end{gather*}
$$

We introduce a function $u^{\prime}(y)$ such that $u^{\prime}(y)=U_{j}^{\prime}\left(y^{\prime}(y)\right)$ for $y \in \mathcal{O}_{\varepsilon^{\prime}}\left(g_{j}\right)$ and $u^{\prime}(y)=0$ for $y \notin \mathcal{O}_{\varepsilon^{\prime}}(\mathcal{K})$, where $y^{\prime} \mapsto y\left(g_{j}\right)$ is the change of variables inverse to the change $y \mapsto y^{\prime}\left(g_{j}\right)$ from Sec. 6.1 (see Chap. 2), $\xi_{j} \in C_{0}^{\infty}\left(O_{\varepsilon^{\prime}}\left(g_{j}\right)\right), \xi_{j}(y)=1$ for $y \in \mathcal{O}_{\varepsilon^{\prime} / 2}\left(g_{j}\right)$, and $\varepsilon^{\prime}$ is defined in Eq. (19.8). Prove that the function $u=u^{\prime}+v$ is as required. Obviously, $u \in W^{2 m-1}(G)$ and $u \notin W^{2 m}(G)$ and $u$ satisfies relations (13.21). It follows from embedding $v \in W^{2 m}(G)$ and relations (20.14) that

$$
\mathbf{P}(y, D) u \in L_{2}(G)
$$

Let us consider functions

$$
f_{i \mu}=\mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u .
$$

The embedding $v \in W^{2 m}(G)$, Eqs. (13.21), and inequality (6.5) (for $l=0$ ) imply that

$$
f_{i \mu} \in W^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i} \backslash \overline{\mathcal{O}_{\delta}(\mathcal{K})}\right)
$$

for all $\delta>0$. Consider the behavior of $f_{i \mu}$ near the set $\mathcal{K}$. Note that, by virtue of (6.5) (for $l=0$ ), we have $\mathbf{B}_{i \mu}^{2} u^{\prime}=0$. Further, $\mathbf{B}_{i \mu}^{0} v+\mathbf{B}_{i \mu}^{1} v=0$ for $y \in \mathcal{O}_{\varkappa_{1} / d_{2}}(\mathcal{K})$. Hence,

$$
\begin{equation*}
f_{i \mu}=\mathbf{B}_{i \mu}^{0} u^{\prime}+\mathbf{B}_{i \mu}^{1} u^{\prime}+\mathbf{B}_{i \mu}^{2} v, \quad y \in \mathcal{O}_{\varkappa_{1} / d_{2}}(\mathcal{K}) . \tag{20.16}
\end{equation*}
$$

Introduce functions $f_{j \sigma \mu}\left(y^{\prime}\right)=f_{i \mu}\left(y\left(y^{\prime}\right)\right)$, where $y \mapsto y^{\prime}\left(g_{j}\right)$ is the change of variables from Sec. 6.1 (see Chap. 2). Equations (20.16) and (20.15) imply that $\left\{f_{j \sigma \mu}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right)$. Thus, $\left\{f_{i \mu}\right\} \in$ $\mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}(\partial G)$.

Remark 20.2. There exist analogs of Corollaries 19.1-19.4. To prove them, one must use item 2 of Theorem 20.2 instead of item 2 of Theorem 19.4).

## 21. Violation of Smoothness of Generalized Solutions

21.1. Simultaneous violation of conditions 18.1 and 19.1 or violation of condition 18.2. The situation declared in the title of this subsection is equivalent to the following condition.
Condition 21.1. The strip $1-2 m \leq \operatorname{Im} \lambda<1-\ell$ contains an irregular eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$.

Show that in this case the smoothness of generalized solutions can be violated for any operators $\mathbf{B}_{i \mu}^{2}$.

## Theorem 21.1.

(1) Let condition 21.1 hold. Then there exist a right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{H}_{0}^{0}(G, \partial G)$ and a generalized $u$ of problem (18.1), (18.2) such that $u \notin W^{2 m}(G)$.
(2) Let conditions 21.1 and 19.5 hold. Then statement (1) holds with $f_{i \mu}=0$.

Proof. 1. Let $\lambda=\lambda_{0}$ be an irregular eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda), 1-2 m \leq \operatorname{Im} \lambda_{0}<1-\ell$. Consider the function

$$
\begin{equation*}
V=r^{i \lambda_{0}} \sum_{l=0}^{l_{0}} \frac{1}{l!}(i \ln r)^{l} \varphi^{\left(l_{0}-l\right)}(\omega) \in \mathcal{W}^{\ell}\left(K^{d}\right) \quad \forall d>0 \tag{21.1}
\end{equation*}
$$

where $\varphi^{(0)}, \ldots, \varphi^{(\varkappa-1)}$ is a Jordan chain of length $\varkappa \geq 1$ of the operator $\tilde{\mathcal{L}}(\lambda)$; it consists of an eigenvector and adjoint vectors corresponding to an eigenvalue $\lambda_{0}$. The number $l_{0}, 0 \leq l_{0} \leq \varkappa-1$, from the definition of the function $V$ is such that the function $V$ is not a vector-valued polynomial with respect to the variables $y_{1}$ and $y_{2}$. Such an $l_{0}$ exists since $\lambda_{0}$ is an irregular eigenvalue (if $\operatorname{Im} \lambda$ is not an integer or $\operatorname{Im} \lambda$ is an integer but $\operatorname{Re} \lambda \neq 0$, we can take $l_{0}=0$ ).

Since $V$ is not a vector-valued polynomial, we see, according to [53, Lemma 4.20], that

$$
\begin{equation*}
V \notin \mathcal{W}^{2 m}\left(K^{d}\right) \quad \forall d>0 \tag{21.2}
\end{equation*}
$$

It follows from [26, Lemma 2.1] that

$$
\begin{equation*}
\mathcal{P}_{j}(D) V_{j}=0,\left.\quad \mathcal{B}_{j \sigma \mu}(D) V\right|_{\gamma_{j \sigma}}=0 \tag{21.3}
\end{equation*}
$$

Using inequalities (21.3) and the Taylor expansions for the coefficients of the operators $\mathbf{P}_{j}(y, D)$ and $\mathbf{B}_{j \sigma \mu}(y, D)$, we obtain the following embeddings:

$$
\begin{equation*}
\left\{\mathbf{P}_{j}(y, D) V_{j}-P_{j}\right\} \in \mathcal{W}^{0}\left(K^{\varepsilon}\right), \quad\left\{\mathbf{B}_{j \sigma \mu}(y, D) V-P_{j \sigma \mu}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right), \tag{21.4}
\end{equation*}
$$

where $P_{j}$ is a linear combination of terms of the form

$$
r^{i \lambda_{0}-2 m+1}(i \ln r)^{l} \varphi(\omega), \quad \ldots, \quad r^{i \lambda_{0}-2 m+k_{0}}(i \ln r)^{l} \varphi(\omega)
$$

$P_{j \sigma \mu}$ is a linear combination of terms of the form

$$
r^{i \lambda_{0}-m_{j \sigma \mu}+1}(i \ln r)^{l}, \quad \ldots, \quad r^{i \lambda_{0}-m_{j \sigma \mu}+k_{0}}(i \ln r)^{l}
$$

$\varphi(\omega)$ is infinitely differentiable vector-valued functions, and the number $k_{0} \in \mathbb{N}$ is such that

$$
\begin{equation*}
-\operatorname{Im} \lambda_{0}-2 m+k_{0} \leq-1, \quad-\operatorname{Im} \lambda_{0}-2 m+k_{0}+1>-1 . \tag{21.5}
\end{equation*}
$$

Obviously, if inequalities (21.5) hold for $k_{0}=0$, i.e., $1-2 m \leq \operatorname{Im} \lambda_{0}<2-2 m$, we can assume that $P_{j}=0$ and $P_{j \sigma \mu}=0$.

Applying [26, Lemma 4.3], we construct a function

$$
\begin{equation*}
V^{\prime}=\sum_{k=1}^{k_{0}} \sum_{l=0}^{l^{\prime}} r^{i \lambda_{0}+k}(i \ln r)^{l_{k}} \varphi_{k l}(\omega) \in W^{\ell}\left(K^{d}\right) \quad \forall d>0 \tag{21.6}
\end{equation*}
$$

such that

$$
\begin{equation*}
\left\{\mathbf{P}_{j}(y, D) V_{j}^{\prime}-P_{j}\right\} \in \mathcal{W}^{0}\left(K^{\varepsilon}\right), \quad\left\{\mathbf{B}_{j \sigma \mu}(y, D) V^{\prime}-P_{j \sigma \mu}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \tag{21.7}
\end{equation*}
$$

Consider a patch function $\xi \in C_{0}^{\infty}\left(\mathcal{O}_{\varepsilon^{\prime}}(0)\right)$, which is equal to 1 near the origin, where $\varepsilon^{\prime}$ is defined in (19.8). Let $U=\xi\left(V-V^{\prime}\right)$. Obviously, $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon^{\prime}}(0)$; hence,

$$
\begin{equation*}
\operatorname{supp} \mathbf{B}_{j \sigma \mu}(y, D) U \subset \overline{\gamma_{j \sigma}} \cap \mathcal{O}_{\varkappa_{2}}(0) \tag{21.8}
\end{equation*}
$$

It follows from (21.1), (21.6), and (21.2) that

$$
\begin{equation*}
U \in \mathcal{W}^{\ell}(K), \quad U \notin \mathcal{W}^{2 m}\left(K^{d}\right) \quad \forall d>0 . \tag{21.9}
\end{equation*}
$$

Moreover, by (21.4) and (21.7) we have

$$
\begin{equation*}
\left\{\mathbf{P}_{j}(y, D) U_{j}\right\} \in \mathcal{W}^{0}\left(K^{\varepsilon}\right), \quad\left\{\mathbf{B}_{j \sigma \mu}(y, D) U\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) . \tag{21.10}
\end{equation*}
$$

2. Consider a function $u(y)$ such that $u(y)=U_{j}\left(y^{\prime}(y)\right)$ for $y \in \mathcal{O}_{\varepsilon^{\prime}}\left(g_{j}\right)$ and $u(y)=0$ for $y \notin \mathcal{O}_{\varepsilon^{\prime}}(\mathcal{K})$, where $y^{\prime} \mapsto y\left(g_{j}\right)$ is the change of variables that is inverse to the change $y \mapsto y^{\prime}\left(g_{j}\right)$ from Sec. 6.1 (see Chap. 2). The function $u$ is as required. Indeed, by (21.9) we have $u \notin W^{2 m}(G)$. By virtue of
inequality (6.5) (as $l=0$ ), we see that $\mathbf{B}_{i \mu}^{2} u=0 \operatorname{since} \operatorname{supp} u \subset \mathcal{O}_{\varkappa_{1}}(\mathcal{K})$. It follows from the equality $\mathbf{B}_{i \mu}^{2} u=0$ and relations (21.10) that the function $u$ satisfies the relations

$$
\begin{gather*}
\mathbf{P}(y, D) u \in L_{2}(G), \quad \mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u \in H_{0}^{2 m-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right),  \tag{21.11}\\
\operatorname{supp}\left(\mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u\right) \subset \overline{\Gamma_{i} \cap \mathcal{O}_{\varkappa_{2}}(\mathcal{K}) .}
\end{gather*}
$$

Statement 1 is proved.
Statement 2 follows from Eq. (21.11) and statement 2 of Lemma 19.2.
21.2. Violation of condition 18.3 or 18.4. We have considered all possible cases of location of eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ for $\ell=2 m-1$. It remains to consider the case where $\ell \leq 2 m-2$ and either condition 18.3 or condition 18.4 is violated.

Theorem 21.2. Let condition 18.2 hold, but let either condition 18.3 or condition 18.4 be violated. Then there exist a right-hand side $\left\{f_{0}, f_{i \mu}^{1}+f_{i \mu}^{2}\right\} \in \mathcal{W}^{0}(G, \partial G)$ and a generalized solution $u$ of problem (18.1), (18.2) such that $u \notin W^{2 m}(G)$, where $f_{i \mu}^{1}$ is a polynomial of degree $\leq 2 m-m_{i \mu}-2$ in a neighborhood of a point $g \in \overline{\Gamma_{i}} \cap \mathcal{K}$ and $\left\{f_{i \mu}^{2}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}(\partial G)$.

Proof. 1. According to item 2 of Remark 18.3, for some natural $s$ from the set $\{\ell, \ldots, 2 m-2\}$ and some (nonzero) vector $\left\{c_{j \sigma \mu}\right\} \in C_{s}$, one can find a function $V$ of the form (18.8) such that

$$
\begin{gather*}
V \in \mathcal{W}^{\ell}\left(K^{d}\right), \quad V \notin \mathcal{W}^{2 m}\left(K^{d}\right) \quad \forall d>0  \tag{21.12}\\
\mathcal{P}_{j}(D) V_{j}=0,\left.\quad \mathcal{B}_{j \sigma \mu}(D) V\right|_{\gamma_{j \sigma}}=c_{j \sigma \mu} r^{s-m_{j \sigma \mu}} \tag{21.13}
\end{gather*}
$$

Using inequality (21.13) and the Taylor expansion for the coefficients of the operators $\mathbf{P}_{j}(y, D)$ and $\mathbf{B}_{j \sigma \mu}(y, D)$, we obtain the embeddings

$$
\begin{align*}
\left\{\mathbf{P}_{j}(y, D) V_{j}-P_{j}\right\} & \in \mathcal{W}^{0}\left(K^{\varepsilon}\right), \\
\left\{\mathbf{B}_{j \sigma \mu}(y, D) V-c_{j \sigma \mu} r^{s-m_{j \sigma \mu}}-P_{j \sigma \mu}\right\} & \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right), \tag{21.14}
\end{align*}
$$

where the functions $P_{j}$ and $P_{j \sigma \mu}$ have the same form as in Eq. (21.4).
Similarly to the proof of Theorem 21.1, we construct a function $V^{\prime}$ of the form (21.6) (where $i \lambda_{0}$ must be replaced by $s$ ) satisfying relations (21.7).

Consider a patch function $\xi \in C_{0}^{\infty}\left(\mathcal{O}_{\varepsilon^{\prime}}(0)\right)$, which is equal to 1 near the origin, where $\varepsilon^{\prime}$ is defined in Eq. (19.8). Let $U=\xi\left(V-V^{\prime}\right)$. Obviously, $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon^{\prime}}(0)$ and

$$
\begin{equation*}
U \in \mathcal{W}^{\ell}(K), \quad U \notin \mathcal{W}^{2 m}\left(K^{d}\right) \quad \forall d>0 . \tag{21.15}
\end{equation*}
$$

Moreover, by virtue of Eqs. (21.14) and (21.7) we have

$$
\begin{equation*}
\left\{\mathbf{P}_{j}(y, D) U_{j}\right\} \in \mathcal{W}^{0}\left(K^{\varepsilon}\right), \quad\left\{\mathbf{B}_{j \sigma \mu}(y, D) U-c_{j \sigma \mu} r^{s-m_{j \sigma \mu}}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}\left(\gamma^{\varepsilon}\right) \tag{21.16}
\end{equation*}
$$

Note that, since $\left\{c_{j \sigma \mu}\right\} \in C_{s}$, the function $c_{j \sigma \mu} r^{s-m_{j \sigma \mu}}$ either vanishes (in particular, it vanishes if $(j, \sigma, \mu) \in J_{s}$ ) or is a monomial of degree $s-m_{j \sigma \mu}$ (i.e., the degree of the monomial is not greater than $\left.2 m-m_{j \sigma \mu}-2\right)$.
2. Consider a function $u(y)$ such that $u(y)=U_{j}\left(y^{\prime}(y)\right)$ for $y \in \mathcal{O}_{\varepsilon^{\prime}}\left(g_{j}\right)$ and $u(y)=0$ for $y \notin \mathcal{O}_{\varepsilon^{\prime}}(\mathcal{K})$, where $y^{\prime} \mapsto y\left(g_{j}\right)$ is the change of the variables inverse to the change $y \mapsto y^{\prime}\left(g_{j}\right)$ from Sec. 6.1 (see Chap. 2). The function $u$ is a required function. Indeed, $u \notin W^{2 m}(G)$ according to Eq. (21.15). By virtue of inequality (6.5) (for $l=0$ ), we have $\mathbf{B}_{i \mu}^{2} u=0$ since supp $u \subset \mathcal{O}_{\varkappa_{1}}(\mathcal{K})$. It follows from $\mathbf{B}_{i \mu}^{2} u=0$ and relations (21.16) that the function $u$ satisfies the relations

$$
\mathbf{P}(y, D) u \in L_{2}(G), \quad \mathbf{B}_{i \mu}^{0} u+\mathbf{B}_{i \mu}^{1} u+\mathbf{B}_{i \mu}^{2} u=f_{i \mu}^{1}+f_{i \mu}^{2},
$$

where $f_{i \mu}^{1}$ is a polynomial ${ }^{11}$ of degree not greater than $2 m-m_{i \mu}-2$ in a neighborhood of a point $g \in \overline{\Gamma_{i}} \cap \mathcal{K}$ and $\left\{f_{i \mu}^{2}\right\} \in \mathcal{H}_{0}^{2 m-\mathbf{m}-1 / 2}(\partial G)$.

Remark 21.1. Recall that the space $\tilde{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G)$ was introduced in Sec. 19.2 in the case where the line $\operatorname{Im} \lambda=1-2 m$ contains a unique eigenvalue $i(1-2 m)$. According to Theorem 19.2, the smoothness of generalized solutions can be violated if the right-hand side $\left\{f_{i \mu}\right\} \in \mathcal{W}^{2 m-\mathbf{m}-1 / 2}(\partial G)$ does not belong to $\tilde{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G)$. Theorem 21.2 shows that if either condition 18.3 or condition 18.4 is violated, then the smoothness of generalized solutions can be violated even for the right-hand sides $\left\{f_{i \mu}\right\} \in \tilde{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G)$. This happens since the right-hand sides $\left\{f_{i \mu}^{1}+f_{i \mu}^{2}\right\}$ in the statement of Theorem 21.2 belong to $\tilde{\mathcal{S}}^{2 m-\mathbf{m - 1 / 2}}(\partial G)$ (cf. Remark 19.1).

On the other hand, the fact that the violation of the smoothness in Theorem 21.2 occurs for nonzero (and even irregular, i.e., not belonging to $\hat{\mathcal{S}}^{2 m-\mathbf{m}-1 / 2}(\partial G)$ ) right-hand sides $\left\{f_{i \mu}\right\}$ is substantial. By Theorems 20.1 and 20.2, if we consider only regular right-hand sides, the smoothness of solutions can be preserved even when condition 18.3 or 18.4 is violated.

## 22. Example

We give an example illustrating the results of this chapter. In this example, the set $\mathcal{K}$ consists of some orbits; therefore, when we refer to theorems from the previous sections, we must use obvious generalizations to this case.

### 22.1. Problem with inhomogeneous conditions.

22.1.1. Statement of the problem. Let $\partial G \backslash \mathcal{K}=\Gamma_{1} \cup \Gamma_{2}$, where $\Gamma_{i}$ are open (in the topology of $\partial G$ ) curves of class $C^{\infty}$ and $\mathcal{K}=\overline{\Gamma_{1}} \cap \overline{\Gamma_{2}}=\{g, h\}$, where $g$ and $h$ are the endpoints of the curves $\overline{\Gamma_{1}}$ and $\overline{\Gamma_{2}}$. Assume that the domain $G$ coincides with a plane angle of spread $\pi$ in a neighborhood of every point $g$ and $h$. Thus, the boundary $G$ is infinitely smooth. Consider the following nonlocal problem in the domain $G$ (cf. example in Sec. 6.2, Chap. 2):

$$
\begin{array}{rlrl}
\Delta u & =f_{0}(y), \quad y \in G, & \\
\left.u\right|_{\Gamma_{1}}+\left.b_{1}(y) u\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}}+\left.a(y) u(\Omega(y))\right|_{\Gamma_{1}}=f_{1}(y), & & y \in \Gamma_{1}, \\
\left.u\right|_{\Gamma_{2}}+\left.b_{2}(y) u\left(\Omega_{2}(y)\right)\right|_{\Gamma_{2}}=f_{2}(y), & y \in \Gamma_{2}, \tag{22.2}
\end{array}
$$

where $b_{1}, b_{2}$, and $a$ are real-valued, infinitely differentiable functions, $\Omega_{i}(\Omega)$ is a diffeomorphism of class $C^{\infty}$ that maps a neighborhood $\mathcal{O}_{i}$ (respectively, $\mathcal{O}_{1}$ ) of the curve $\Gamma_{i}$ (respectively, $\Gamma_{1}$ ) to the set $\Omega_{i}\left(\mathcal{O}_{i}\right)$ (respectively, $\Omega\left(\mathcal{O}_{1}\right)$ ) such that $\Omega_{i}\left(\Gamma_{i}\right) \subset G, \Omega_{i}(g)=g, \Omega_{i}(h)=h$, and the transformation $\Omega_{i}$ near the points $g$ and $h$ is a rotation of the boundary $\Gamma_{i}$ by the angle $\pi / 2$ inwards the domain $G$ (respectively, $\Omega\left(\Gamma_{1}\right) \subset G, \overline{\Omega\left(\Gamma_{1}\right)} \cap\{g, h\}=\varnothing$, but the approach of the curve $\Omega\left(\overline{\Gamma_{1}}\right)$ to the boundary $\partial G$ is arbitrary); see Fig. 22.1.

To write nonlocal conditions (22.2) in the form (18.2), we choose a sufficiently small number $\varepsilon$ such that the sets $\overline{\mathcal{O}_{\varepsilon}(g)}$ and $\overline{\mathcal{O}_{\varepsilon}(h)}$ do not intersect with the curve $\overline{\Omega\left(\Gamma_{1}\right)}$.

Consider a function $\zeta \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ such that $\zeta(y)=1$ for $y \in \mathcal{O}_{\varepsilon / 2}(\mathcal{K})$ and $\operatorname{supp} \zeta \subset \mathcal{O}_{\varepsilon}(\mathcal{K})$. Let us introduce the operators

$$
\begin{aligned}
& \mathbf{B}_{i}^{1} u=\left.\zeta(y) b_{i}(y) u\left(\Omega_{i}(y)\right)\right|_{\Gamma_{i}}, \\
& \mathbf{B}_{1}^{2} u=\left.(1-\zeta(y)) b_{1}(y) u\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}}+\left.a(y) u(\Omega(y))\right|_{\Gamma_{1}}, \\
& \mathbf{B}_{2}^{2} u=\left.(1-\zeta(y)) b_{2}(y) u\left(\Omega_{2}(y)\right)\right|_{\Gamma_{2}} .
\end{aligned}
$$

[^10]

Fig. 22.1. The domain $G$ with the boundary $\partial G=\Gamma_{1} \cup \Gamma_{2} \cup\{g, h\}$.
In this example, the set $\mathcal{K}$ is formed by two orbits. The first orbit consists of the point $g$ and the second consists of the point $h$. Since the support of $\zeta$ is located near a neighborhood of the set $\mathcal{K}$, we can assume that the transformations $\Omega_{i}$ from the definition of the operators $\mathbf{B}_{i \mu}^{1}$ are also defined in a neighborhood of the set $\mathcal{K}$ and satisfy condition 6.3. It is easy to see that the operators $\mathbf{B}_{i \mu}^{2}$ satisfy condition 6.4 with $\varkappa_{1}=\varepsilon / 2$ and some $\varkappa_{2}<\varkappa_{1}$ and $\rho$.

Here, we will use spaces of vector-valued functions introduced in Eqs. (5.3) and (5.21) for $N=2$.
Consider a model problem corresponding to the point $g$ (a model problem corresponding to the point $h$ can be considered similarly). Assume that the point $g$ coincides with the origin, $g=0$, and the axis $O y_{1}$ is directed inwards the domain $G$ orthogonally to the boundary. Consider the sets

$$
\begin{aligned}
K^{\varepsilon} & =\left\{y \in \mathbb{R}^{2}: 0<r<\varepsilon,|\omega|<\pi / 2\right\}, \\
\gamma_{\sigma}^{\varepsilon} & =\left\{y \in \mathbb{R}^{2}: 0<r<\varepsilon, \omega=(-1)^{\sigma} \pi / 2\right\} .
\end{aligned}
$$

We choose a small $\varepsilon$ such that $\mathcal{O}_{\varepsilon}(0) \cap G=K^{\varepsilon}$. The model problem takes the form

$$
\begin{align*}
\Delta U=F(y), & y \in K^{\varepsilon},  \tag{22.3}\\
U(y)+b_{\sigma}(y) U\left(\mathcal{G}_{\sigma} y\right)=\psi_{\sigma}(y), & y \in \gamma_{\sigma}^{\varepsilon}, \quad \sigma=1,2, \tag{22.4}
\end{align*}
$$

where

$$
\mathcal{G}_{\sigma}=\left(\begin{array}{cc}
0 & (-1)^{\sigma} \\
(-1)^{\sigma+1} & 0
\end{array}\right)
$$

is the operator of the rotation by the angle $(-1)^{\sigma+1} \pi / 2$,

$$
F(y)=f_{0}(y), \quad y \in K^{\varepsilon}, \quad \psi_{\sigma}(y)=f_{\sigma}(y)-B_{\sigma}^{u}(y), \quad y \in \gamma_{\sigma}^{\varepsilon} .
$$

Moreover,

$$
B_{1}^{u}(y)=a(y) u(\Omega(y)), \quad y \in \gamma_{1}^{\varepsilon / 2}, \quad B_{2}^{u}(y)=0, \quad y \in \gamma_{2}^{\varepsilon / 2}
$$

since $(1-\zeta(y)) b_{\sigma}(y) u\left(\Omega_{\sigma}(y)\right)=0$ for $y \in \gamma_{\sigma}^{\varepsilon / 2}, \sigma=1,2$.
The eigenproblem has the form

$$
\begin{gather*}
\varphi^{\prime \prime}(\omega)-\lambda^{2} \varphi(\omega)=0, \quad|\omega|<\pi / 2  \tag{22.5}\\
\varphi(-\pi / 2)+b_{1}(0) \varphi(0)=0, \quad \varphi(\pi / 2)+b_{2}(0) \varphi(0)=0 . \tag{22.6}
\end{gather*}
$$

Introduce the notation $I_{1}=(-\infty,-2] \cup(0, \infty)$ and $I_{2}=(-2,0)$. We can directly verify that eigenvalues of problem (22.5), (22.6) can be arranged relative to the strip $-1 \leq \operatorname{Im} \lambda<0$ as follows:

Case $1\left(b_{1}(0)+b_{2}(0) \in I_{1}\right)$. The strip $-1 \leq \operatorname{Im} \lambda<0$ does not contain eigenvalues.
Case $2\left(b_{1}(0)+b_{2}(0)=0\right)$. The strip $-1 \leq \operatorname{Im} \lambda<0$ contains a unique eigenvalue $\lambda=-i$; this eigenvalue is regular.

Case $3\left(b_{1}(0)+b_{2}(0) \in I_{2}\right)$. The strip $-1 \leq \operatorname{Im} \lambda<0$ contains an irregular eigenvalue

$$
\lambda=2 \pi^{-1} i \arctan \frac{\sqrt{4-\left(b_{1}(0)+b_{2}(0)\right)^{2}}}{b_{1}(0)+b_{2}(0)}
$$

### 22.1.2. Case 1.

Theorem 22.1. Let $b_{1}(0)+b_{2}(0) \in I_{1}$ and $b_{1}(h)+b_{2}(h) \in I_{1}$. Let $u \in W^{1}(G)$ be a generalized solution of problem (22.1), (22.2) with the right-hand side

$$
\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{W}^{0}(G, \partial G)
$$

Then $u \in W^{2}(G)$.
Proof. In this case, the strip $-1 \leq \operatorname{Im} \lambda<0$ does not contain eigenvalues of problem (22.5), (22.6) (as well as eigenvalues of a similar problem corresponding to the point $h$ ). Hence, the theorem follows from Theorem 18.1.

Note that in this case we do not impose any restrictions on the coefficients $b_{i}$ and $a$ and on the right-hand sides $f_{i \mu}$.
22.1.3. Case 2. Assume that

$$
b_{1}(h)+b_{2}(h) \in I_{1} .
$$

In this case, concordance condition (19.5) is considered only near the origin. Let us write this condition for problem (22.1), (22.2). Denote by $\tau_{\sigma}$ the vector with coordinates $\left(0,(-1)^{\sigma}\right)$. Then ${ }^{12}$

$$
\begin{gathered}
\frac{\partial}{\partial \tau_{\sigma}}=(-1)^{\sigma} \frac{\partial}{\partial y_{2}} \\
\frac{\partial}{\partial \tau_{1}}\left(U(y)+b_{1}(0) U\left(\mathcal{G}_{1} y\right)\right)=-U_{y_{2}}(y)+b_{1}(0) U_{y_{1}}\left(\mathcal{G}_{1} y\right) \\
\frac{\partial}{\partial \tau_{2}}\left(U(y)+b_{2}(0) U\left(\mathcal{G}_{2} y\right)\right)=U_{y_{2}}(y)+b_{2}(0) U_{y_{1}}\left(\mathcal{G}_{2} y\right)
\end{gathered}
$$

Hence,

$$
\hat{\mathcal{B}}_{\sigma}(D) U=(-1)^{\sigma} U_{y_{2}}+b_{\sigma}(0) U_{y_{1}}, \quad \sigma=1,2 .
$$

Since $b_{1}(0)+b_{2}(0)=0$, we see that the operators $\hat{\mathcal{B}}_{1}(D)$ and $\hat{\mathcal{B}}_{2}(D)$ are linearly dependent:

$$
\hat{\mathcal{B}}_{1}(D)+\hat{\mathcal{B}}_{2}(D)=0 .
$$

Thus, concordance condition (19.5) for the functions $Z_{\sigma} \in W^{3 / 2}\left(\gamma_{\sigma}^{\varepsilon}\right)$ has the form

$$
\begin{equation*}
\int_{0}^{\varepsilon} r^{-1}\left|\frac{\partial Z_{1}}{\partial y_{2}}\right|_{y=(0,-r)}-\left.\left.\frac{d Z_{2}}{d y_{2}}\right|_{y=(0, r)}\right|^{2} d r<\infty . \tag{22.7}
\end{equation*}
$$

Taking into account Eq. (22.7), we denote by $\tilde{\mathcal{S}}^{3 / 2}(\partial G)$ the set of all functions $\left\{f_{i \mu}\right\} \in \mathcal{W}^{3 / 2}(\partial G)$ such that

$$
\begin{equation*}
\int_{0}^{\varepsilon} r^{-1}\left|\frac{\partial f_{1}}{\partial y_{2}}\right|_{y=(0,-r)}-\left.\left.\frac{\partial f_{2}}{\partial y_{2}}\right|_{y=(0, r)}\right|^{2} d r<\infty \tag{22.8}
\end{equation*}
$$

Let $\tilde{\mathcal{S}}^{0}(G, \partial G)=\tilde{\mathcal{S}}^{0}(G, \partial G)$.
By Theorem 19.2, the embedding $\left\{f_{i \mu}\right\} \in \tilde{\mathcal{S}}^{3 / 2}(\partial G)$ is necessary for any generalized solution of problem (22.1), (22.2) to belong to $W^{2}(G)$.
Theorem 22.2. Let $b_{1}(0)+b_{2}(0)=0$ and $b_{1}(h)+b_{2}(h) \in I_{1}$. Then the following statements hold.

[^11](1) If
\[

$$
\begin{gather*}
a(0)=0,\left.\quad \frac{\partial a}{\partial y_{2}}\right|_{y=0}=0  \tag{22.9}\\
\int_{0}^{\varepsilon} r^{-1}\left|\frac{\partial b_{1}}{\partial y_{2}}\right|_{y=(0,-r)}-\left.\left.\frac{\partial b_{2}}{\partial y_{2}}\right|_{y=(0, r)}\right|^{2} d r<\infty \tag{22.10}
\end{gather*}
$$
\]

and the function $u \in W^{1}(G)$ is a generalized solution od problem (22.1), (22.2) with the righthand side $\left\{f_{0}, f_{i \mu}\right\} \in \tilde{\mathcal{S}}^{0}(G, \partial G)$, then $u \in W^{2}(G)$.
(2) If condition (22.9)-(22.10) is violated, then there exist a right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \tilde{\mathcal{S}}^{0}(G, \partial G)$ and a generalized solution $u \in W^{1}(G)$ of problem (22.1), (22.2) such that $u \notin W^{2}(G)$.

Proof. 1. By Theorem 19.3, it suffices to show that condition (22.9)-(22.10) is equivalent to condition 19.3.

For any function $v \in W^{2}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$, we denote $v_{\Omega}(y)=v(\Omega(y)), y \in \Gamma_{1}$. Then

$$
B_{1}^{v}(y)=a(y) v_{\Omega}(y), \quad y \in \gamma_{1}^{\varepsilon / 2}, \quad B_{2}^{v}(y)=0, \quad y \in \gamma_{2}^{\varepsilon / 2} .
$$

Hence, the functions $B_{\sigma}^{v}$ satisfy concordance condition (22.7) if and only if

$$
\begin{equation*}
\left.\int_{0}^{\varepsilon / 2} r^{-1}\left|\frac{\partial\left(a v_{\Omega}\right)}{\partial y_{2}}\right|_{y=(0,-r)}\right|^{2} d r=\left.\int_{0}^{\varepsilon / 2} r^{-1}\left|\left(\frac{\partial a}{\partial y_{2}} v_{\Omega}+a \frac{\partial v_{\Omega}}{\partial y_{2}}\right)\right|_{y=(0,-r)}\right|^{2} d r<\infty \tag{22.11}
\end{equation*}
$$

We take $\varepsilon / 2$ instead of $\varepsilon$ as the upper limit of integration since in this case the functions $B_{\sigma}^{v}$ are simpler; obviously, the replacement of $\varepsilon$ by $\varepsilon / 2$ does not influence the convergence of the integral.

Prove that condition (22.11) is equivalent to Eq. (22.9). Let (22.11) hold. Choose a function $v$ such that $v_{\Omega}(y)=y_{2}$ near the origin; then

$$
\left.\frac{\partial\left(a v_{\Omega}\right)}{\partial y_{2}}\right|_{y=0}=a(0)
$$

Since the function $\partial\left(a v_{\Omega}\right) / \partial y_{2}$ is continuous near the origin, we have from the last relation and Eq. (22.11) that $a(0)=0$. Similarly, substituting a function $v$ such that $v_{\Omega}(y)=1$ near the origin into Eq. (22.11), we obtain the equality

$$
\left.\frac{\partial a}{\partial y_{2}}\right|_{y=0}=0
$$

Conversely, let Eq. (22.9) hold. By virtue of the smoothness of the transformation $\Omega$, we have

$$
v_{\Omega}, \frac{\partial v_{\Omega}}{\partial y_{2}} \in W^{1 / 2}\left(\gamma_{1}^{\varepsilon}\right) \subset H_{1}^{1 / 2}\left(\gamma_{1}^{\varepsilon}\right)
$$

for any function $v \in W^{2}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$. This, Eq. (22.9), and Lemma 5.6 imply that $\partial\left(a v_{\Omega}\right) / \partial y_{2} \in H_{0}^{1 / 2}\left(\gamma_{1}^{\varepsilon}\right)$. Hence, by [53, Lemma 4.8], Eq. (22.11) is valid. Thus, we proved that item 1 of condition 19.3 is equivalent to condition (22.9).
2. Item 2 of condition 19.3 is fulfilled if and only if the functions $C+b_{1}(y) C$ and $C+b_{2}(y) C$ satisfy concordance condition (22.7) for any constant $C$. This is equivalent to Eq. (22.10).

Thus, in case 2 the smoothness of generalized solutions depends on the first derivatives of the coefficients $b_{1}$ and $b_{2}$ near the origin and on the coefficient $a$ and its first derivative at the origin.

### 22.1.4. Case 3.

Theorem 22.3. Let $b_{1}(0)+b_{2}(0) \in I_{2}$ or $b_{1}(h)+b_{2}(h) \in I_{2}$. Then there exist a right-hand side $\left\{f_{0}, 0\right\}$, where $f_{0} \in L_{2}(G)$, and a generalized solution $u \in W^{1}(G)$ of problem (22.1), (22.2) such that $u \notin W^{2}(G)$.

Proof. The strip $-1 \leq \operatorname{Im} \lambda<0$ contains an irregular eigenvalue of problem (22.5), (22.6) (or of a similar problem corresponding to the point $h$ ). Hence, the theorem follows from Theorem 21.1.

Thus, in case 3 the smoothness of generalized solutions can be violated independently of the behavior of the coefficient $a$ and the derivatives of the coefficients $b_{1}$ and $b_{2}$ near the point $g$.
22.2. Problem with regular and zero right-hand sides in nonlocal conditions. Consider problem (22.1), (22.2) with regular and zero right-hand sides in boundary conditions. By Theorems 22.1 and 22.3 , the smoothness of generalized solutions is preserved in case 1 and can be violated in case 3. Case 2 (the "boundary" case) is of most interest.
22.2.1. Problem with regular right-hand sides. Assume that

$$
b_{1}(h)+b_{2}(h) \in I_{1} .
$$

Theorem 22.4. Let $b_{1}(0)+b_{2}(0)=0$ and $b_{1}(h)+b_{2}(h) \in I_{1}$. Then the following statements hold.
(1) If

$$
\begin{equation*}
a(0)=0,\left.\quad \frac{\partial a}{\partial y_{2}}\right|_{y=0}=0 \tag{22.12}
\end{equation*}
$$

and the function $u \in W^{1}(G)$ is a generalized solution of problem (22.1), (22.2) with the righthand side $\left\{f_{0}, f_{i \mu}\right\} \in \tilde{\mathcal{S}}^{0}(G, \partial G)$, where $f_{i \mu}(0)=0$, then $u \in W^{2}(G)$.
(2) If condition (22.12) is violated, then there exist a right-hand side $\left\{f_{0}, f_{i \mu}\right\} \in \mathcal{H}^{0}(G, \partial G)$, where $f_{i \mu}(y)=0$ near the origin and a generalized solution $u \in W^{1}(G)$ of problem (22.1), (22.2) such that $u \notin W^{2}(G)$.
Proof. 1. By Theorem 19.4 and Corollary 19.1, it suffices to prove that condition (22.12) is equivalent to condition 19.4.

By Definition 19.2, the function $v \in W^{2}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)$ is acceptable if and only if there exist constants $C$ and $C_{h}$ such that

$$
\begin{array}{r}
a(0) v_{\Omega}(0)+C+b_{1}(0) C=0, \quad C+b_{2}(0) C=0, \\
a(h) v_{\Omega}(h)+C_{h}+b_{1}(h) C_{h}=0, \quad C_{h}+b_{2}(h) C_{h}=0, \tag{22.13}
\end{array}
$$

where $v_{\Omega}(y)=v(\Omega(y)), y \in \Gamma_{1}$.
Let $\xi \in C^{\infty}\left(\mathbb{R}^{2}\right)$ be a patch function such that

$$
\operatorname{supp} \xi \subset \mathcal{O}_{\delta}(\Omega(0)), \quad \xi(y)=1, \quad y \in \mathcal{O}_{\delta / 2}(\Omega(0))
$$

where $\delta>0$ is so small that $\Omega(h) \notin \mathcal{O}_{\delta}(\Omega(0))$. Since $b_{1}(h)+b_{2}(h) \in I_{1}$, we see that one must consider concordance condition (19.5) only near the origin. Hence, if $v$ is an acceptable function, $C$ and $C_{h}$ are acceptable constants corresponding to the function $v$, and condition 19.4 is valid (respectively, violated) for $v$ and $C$, then $\xi v$ is also an acceptable function, $C$ and 0 are acceptable vectors corresponding to $\xi v$, and condition 19.4 is valid (respectively, violated) for $\xi v$ and $C$. Thus, it suffices to consider only functions $v$ with supports in $\mathcal{O}_{\delta}(\Omega(0))$ (i.e., functions $v_{\Omega}$ with supports near the origin); moreover, we can assume that $C_{h}=0$.

First, we consider the case where $b_{2}(0) \neq-1$. In this case, by (22.13), the function $v$ with support in $\mathcal{O}_{\delta}(\Omega(0))$ is acceptable if and only if

$$
\begin{equation*}
a(0) v_{\Omega}(0)=0 . \tag{22.14}
\end{equation*}
$$

The corresponding set of acceptable constants contains a unique constant $C=0$ (recall that $C_{h}$ by assumption also vanish). Hence, condition 19.4 is valid if and only if the relation

$$
\begin{equation*}
\left.\int_{0}^{\varepsilon / 2} r^{-1}\left|\frac{\partial\left(a v_{\Omega}\right)}{\partial y_{2}}\right|_{y=(0,-r)}\right|^{2} d r=\left.\int_{0}^{\varepsilon / 2} r^{-1}\left|\left(\frac{\partial a}{\partial y_{2}} v_{\Omega}+a \frac{\partial v_{\Omega}}{\partial y_{2}}\right)\right|_{y=(0,-r)}\right|^{2} d r<\infty \tag{22.15}
\end{equation*}
$$

is valid for any function $v_{\Omega}$ satisfying Eq. (22.14). Assume that Eq. (22.12) holds. Then any function $v$ with support in $\mathcal{O}_{\delta}(\Omega(0))$ is acceptable (since $a(0)=0$ ). Repeating the reasoning from the proof of Theorem 22.2, we obtain (22.15).

Conversely, assume that Eq. (22.15) holds for any function $v_{\Omega}$ satisfying (22.14). Obviously, a function $v$ such that $v_{\Omega}(y)=y_{2}$ near the origin satisfies (22.14). Substituting the function $v_{\Omega}$ to (22.15), we obtain that $a(0)=0$ (cf. the proof of Theorem 22.2). Hence, any function $v$ with support in $\mathcal{O}_{\delta}(\Omega(0))$ is acceptable. Substituting $v_{\Omega}(y)=1$ to Eq. (22.15), we obtain $\left.\left(\partial a / \partial y_{2}\right)\right|_{y=0}=0$.
2. It remains to consider the case where $b_{2}(0)=-1$; in this case, $b_{1}(0)=1$. Then by virtue of (22.13), any function $v$ with support in $\mathcal{O}_{\delta}(\Omega(0))$ is acceptable; the corresponding set of acceptable constants contains a unique constant $C=-a(0) v_{\Omega}(0) / 2$ (we still assume that the constant $C_{h}$ vanishes). Hence, condition 19.4 is valid if and only if the relation

$$
\begin{align*}
& \int_{0}^{\varepsilon / 2} r^{-1}\left|\frac{\partial\left(a v_{\Omega}\right)}{\partial y_{2}}\right|_{y=(0,-r)}+\left.C\left(\left.\frac{\partial b_{1}}{\partial y_{2}}\right|_{y=(0,-r)}-\left.\frac{\partial b_{2}}{\partial y_{2}}\right|_{y=(0, r)}\right)\right|^{2} d r \\
& \quad=\int_{0}^{\varepsilon / 2} r^{-1}\left|\left(\frac{\partial a}{\partial y_{2}} v_{\Omega}+a \frac{\partial v_{\Omega}}{\partial y_{2}}\right)\right|_{y=(0,-r)}-\left.\frac{a(0) v_{\Omega}(0)}{2}\left(\left.\frac{\partial b_{1}}{\partial y_{2}}\right|_{y=(0,-r)}-\left.\frac{\partial b_{2}}{\partial y_{2}}\right|_{y=(0, r)}\right)\right|^{2} d r<\infty \tag{22.16}
\end{align*}
$$

is valid for any function $v$ with support in $\mathcal{O}_{\delta}(\Omega(0))$. Assume that condition (22.12) holds. Then we see (similarly to the previous reasoning) that Eq. (22.15) holds for any function $v_{\Omega}$. Hence, Eq. (22.16) also holds for any function $v_{\Omega}$ (since $a(0)=0$ ).

Conversely, assume that Eq. (22.16) holds. Let us substitute into Eq. (22.16) a function $v$ such that $v_{\Omega}(y)=y_{2}$ near the origin. Since $v_{\Omega}(0)=0$ and $\left.\left(\partial v_{\Omega} / \partial y_{2}\right)\right|_{y=0}=1$, we obtain from Eq. (22.16) (similarly to the previous reasoning) that $a(0)=0$. Hence, relation (22.16) coincides with (22.15). Then, repeating the previous reasoning, we obtain that $\left.\left(\partial a / \partial y_{2}\right)\right|_{y=0}=0$.

Obviously, condition (22.12) is weaker than condition (22.9)-(22.10): there are no restrictions on the behavior of the coefficients $b_{1}$ and $b_{2}$ in condition (22.12). The absence of these restrictions is "compensated" by the fact that the right-hand sides in nonlocal conditions are regular, i.e., $\left\{f_{i \mu}\right\} \in$ $\tilde{\mathcal{S}}^{3 / 2}(\partial G)$ and $f_{i \mu}(0)=0$.
22.2.2. Problem with zero right-hand sides. It follows from statement 1 of Theorem 22.4 that in the case of zero right-hand sides, condition (22.12) is sufficient for any generalized solution to be smooth. Let us prove that this condition is also necessary in the following cases (see Figs. 22.2-22.4).

$$
\begin{align*}
\text { Case A : } & \left.\operatorname{supp} a\left(\Omega^{-1}(y)\right)\right|_{\Omega\left(\Gamma_{1}\right)} \subset G ; \\
\text { Case B : } & \Omega(0) \in G, \quad \Omega(0) \notin \Omega_{1}\left(\Gamma_{1}\right) \cup \Omega_{2}\left(\Gamma_{2}\right) \\
\text { Case C : } & \Omega(0) \in \Gamma_{1}, \quad \Omega(\Omega(0)) \notin \Omega_{1}\left(\Gamma_{1}\right) \cup \Omega_{2}\left(\Gamma_{2}\right),  \tag{22.17}\\
& a(\Omega(0)) \neq 0 . \tag{22.18}
\end{align*}
$$

Corollary 22.1. Let $b_{1}(0)+b_{2}(0)=0$ and $b_{1}(h)+b_{2}(h) \in I_{1}$. Let one of cases $\mathrm{A}, \mathrm{B}$, or C hold. If condition (22.12) is violated, then there exist a right-hand side $\left\{f_{0}, 0\right\}$, where $f_{0} \in L_{2}(G)$, and a generalized solution $u \in W^{1}(G)$ of problem (22.1), (22.2) such that $u \notin W^{2}(G)$.


Fig. 22.2. Case A.


Fig. 22.3. Case B.


Fig. 22.4. Case B.

Proof. 1. First, we assume that Case A holds. It follows from the continuity of the transformations $\Omega_{i}$ and $\Omega$ that the operators $\mathbf{B}_{i \mu}^{2}$ satisfy condition (19.24) for all $\rho$ such that

$$
0<\rho<\operatorname{dist}\left(\left.\operatorname{supp} a\left(\Omega^{-1}(y)\right)\right|_{\Omega\left(\Gamma_{1}\right)}, \partial G\right)
$$

Therefore, the corollary follows from Corollary 19.3.
2. Now we assume Case B holds. As above, we can assume that condition 19.4 is violated for an acceptable function $v$ with support in arbitrary small $\delta$-neighborhood $\mathcal{O}_{\delta}(\Omega(0))$ of a point $\Omega(0)$. We can choose so small $\delta$ that

$$
\left.v(y)\right|_{\Gamma_{i}} \equiv 0,\left.\quad v\left(\Omega_{i}(y)\right)\right|_{\Gamma_{i}}=0,\left.\quad \operatorname{supp} v(\Omega(y))\right|_{\Gamma_{1}} \subset \Gamma_{1} \cap \mathcal{O}_{\varkappa_{2}}(0) .
$$

Thus, the function $v$ satisfies Eqs. (19.26), and the corollary follows from Corollary 19.4.
3. Finally, we assume that Case C holds. Again, we assume that condition 19.4 is violated for an acceptable function $v$ with support in $\mathcal{O}_{\delta}(\Omega(0))$. By relations (22.17), we can choose a number $\delta$ such that

$$
\begin{gather*}
\left.v\left(\Omega_{i}(y)\right)\right|_{\Gamma_{i}} \equiv 0,  \tag{22.19}\\
\left.\operatorname{supp} v(\Omega(y))\right|_{\Gamma_{1}} \subset \Gamma_{1} \cap \mathcal{O}_{\varkappa_{2}}(0) . \tag{22.20}
\end{gather*}
$$

Let $f_{i \mu}$ be functions from statement 2 of Theorem 19.4 constructed according to the scheme from the proof of Theorem 19.4. It follows from Eqs. (22.19) and (22.20) that

$$
\operatorname{supp} f_{1} \subset \Gamma_{1} \cap\left(\mathcal{O}_{\varkappa_{2}}(0) \cup \mathcal{O}_{\delta}(\Omega(0))\right), \quad \operatorname{supp} f_{2} \subset \Gamma_{2} \cap \mathcal{O}_{\varkappa_{2}}(0) .
$$

Assume that we have constructed a function $u_{1} \in H_{0}^{2}(G)$ such that

$$
\begin{gather*}
\left.u_{1}\right|_{\Gamma_{i}}+\mathbf{B}_{i \mu}^{1} u_{1}+\mathbf{B}_{i \mu}^{2} u_{1}=f_{i}(y), \quad y \in \Gamma_{i} \backslash \mathcal{O}_{\varkappa_{2}}(\mathcal{K}), \quad i=1, \ldots, N,  \tag{22.21}\\
\left.u_{1}\right|_{\Gamma_{i}}+\mathbf{B}_{i \mu}^{1} u_{1}+\mathbf{B}_{i \mu}^{2} u_{1}=0, \quad y \in \Gamma_{i} \cap \mathcal{O}_{\varkappa_{2}}(\mathcal{K}), \quad i=1, \ldots, N . \tag{22.22}
\end{gather*}
$$

Then the corollary follows from Lemma 19.2 and Corollary 19.2.
Construct the function $u_{1}$. For this, we consider a function $u_{1 \Omega} \in W^{2}\left(\mathcal{O}_{\delta}(\Omega(0))\right)$ with support in $\mathcal{O}_{\delta}(\Omega(0))$ such that

$$
u_{1 \Omega}(y)=f_{1}(y) / a(y), \quad y \in \Gamma_{1} \cap \mathcal{O}_{\delta}(\Omega(0)),
$$

where $\delta$ is so small that $a(y) \neq 0$ for $y \in \overline{\mathcal{O}_{\delta}(\Omega(0))}$ (the existence of such $\delta$ follows from Eq. (22.18) and the continuity of $a(y)$ ).

Let $u_{1}(y)=u_{1 \Omega}\left(\Omega^{-1}(y)\right)$ for $y \in \Omega\left(\mathcal{O}_{\delta}(\Omega(0))\right)$ and $u_{1}(y)=0$ for $y \notin \Omega\left(\mathcal{O}_{\delta}(\Omega(0))\right)$. We choose so small $\delta$ that

$$
\Gamma_{i} \cap \Omega\left(\mathcal{O}_{\delta}(\Omega(0))\right)=\varnothing, \quad \Omega_{i}\left(\Gamma_{i}\right) \cap \Omega\left(\mathcal{O}_{\delta}(\Omega(0))\right)=\varnothing, \quad \mathcal{O}_{\delta}(\Omega(0)) \cap \mathcal{O}_{\varkappa_{2}}(0)=\varnothing
$$

(the existence of such $\delta$ follows from Eq. (22.17) and the continuity of the transformation $\Omega$ ). Then

$$
\begin{gathered}
\left.u_{1}\right|_{\Gamma_{i}}=0,\left.\quad u_{1}\left(\Omega_{i}(y)\right)\right|_{\Gamma_{i}}=0, \\
a(y) u_{1}(\Omega(y))=f_{1}(y), \quad y \in \Gamma_{1} \backslash \mathcal{O}_{\varkappa_{2}}(0), \\
u_{1}(\Omega(y))=0, \quad y \in \Gamma_{1} \cap \mathcal{O}_{\varkappa_{2}}(0) .
\end{gathered}
$$

Thus, the function $u_{1}$ satisfies Eqs. (22.21) and (22.22). The corollary is proved.

## Chapter 6

## FELLER SEMIGROUPS

## AND TWO-DIMENSIONAL DIFFUSION PROCESSES

## 23. Nonlocal Problems in Spaces of Continuous Functions

23.1. Preliminary information. In this subsection, we recall the notions of a Feller semigroup and its generator and formulate the Hille-Yosida theorem in an appropriate form.

Let $\mathcal{X}$ be a closed subspace in $C(\bar{G})$ containing at least one nonnegative function.
Definition 23.1. A strongly elliptic semigroup of operators $\mathbf{T}_{t}: \mathcal{X} \rightarrow \mathcal{X}$ is called a Feller semigroup on $\mathcal{X}$ if:
(1) $\left\|\mathbf{T}_{t}\right\| \leq 1, t \geq 0$;
(2) $\mathbf{T}_{t} u \geq 0$ for all $t \geq 0$ and $u \in \mathcal{X}, u \geq 0$.

Definition 23.2. A linear operator $\mathbf{P}: \mathrm{D}(\mathbf{P}) \subset \mathcal{X} \rightarrow \mathcal{X}$ is called a generator (infinitesimal generating operator) of a strongly continuous semigroup $\left\{\mathbf{T}_{t}\right\}$ if

$$
\mathbf{P} u=\lim _{t \rightarrow+0} \frac{\mathbf{T}_{t} u-u}{t}, \quad \mathrm{D}(\mathbf{P})=\{u \in \mathcal{X}: \text { a limit in } \mathcal{X} \text { exists }\} .
$$

Theorem 23.1 (Hille-Yosida theorem, see [101, Theorem 9.3.1]).
(1) Let $\mathbf{P}: \mathrm{D}(\mathbf{P}) \subset \mathcal{X} \rightarrow \mathcal{X}$ be a generator of a Feller semigroup on $\mathcal{X}$. Then the following assertions hold:
(a) the domain $\mathrm{D}(\mathbf{P})$ is dense in $\mathcal{X}$;
(b) for any $q>0$, the operator $q \mathbf{I}-\mathbf{P}$ has a bounded inverse operator $(q \mathbf{I}-\mathbf{P})^{-1}: \mathcal{X} \rightarrow \mathcal{X}$ and $\left\|(q \mathbf{I}-\mathbf{P})^{-1}\right\| \leq 1 / q ;$
(c) the operator $(q \mathbf{I}-\mathbf{P})^{-1}: \mathcal{X} \rightarrow \mathcal{X}, q>0$, is nonnegative.
(2) If $\mathbf{P}: \mathcal{X} \rightarrow \mathcal{X}$ is a linear operator satisfying the condition (a) and there exists a constant $q_{0} \geq 0$ such that conditions (b) and (c) hold for $q>q_{0}$, then $\mathbf{P}$ is a generator of some Feller semigroup on $\mathcal{X}$, which is uniquely defined by the operator $\mathbf{P}$.
23.2. Statement of nonlocal problems. Let $p_{j k}, p_{j} \in C^{\infty}\left(\mathbb{R}^{2}\right)$ be real-valued functions and let $p_{j k}=p_{k j}, j, k=1,2$. In this chapter, we consider a second-order differential operator

$$
\begin{equation*}
\mathbf{P}(y, D) u=\sum_{j, k=1}^{2} p_{j k}(y) u_{y_{j} y_{k}}(y)+\sum_{j=1}^{2} p_{j}(y) u_{y_{j}}(y)+p_{0}(y) u(y) . \tag{23.1}
\end{equation*}
$$

Condition 23.1. (1) There exists a constant $c>0$ such that

$$
\sum_{j, k=1}^{2} p_{j k}(y) \xi_{j} \xi_{k} \geq c|\xi|^{2}
$$

for $y \in \bar{G}$ and $\xi=\left(\xi_{1}, \xi_{2}\right) \in \mathbb{R}^{2}$.
(2) $p_{0}(y) \leq 0, y \in \bar{G}$.

Let $\Omega_{i s}, i=1, \ldots, N, s=1, \ldots, S_{i}$, be diffeomorphisms of class $C^{\infty}$ satisfying condition 6.3.
Let us introduce the operator

$$
\mathbf{B}_{i} u=\sum_{s=1}^{S_{i}} b_{i s}(y) u\left(\Omega_{i s}(y)\right)
$$

for $y \in \Gamma_{i} \cap \mathcal{O}_{\varepsilon}(\mathcal{K})$ and $\mathbf{B}_{i} u=0$ for $y \in \Gamma_{i} \backslash \mathcal{O}_{\varepsilon}(\mathcal{K})$, where $b_{i s} \in C^{\infty}\left(\mathbb{R}^{2}\right)$ are real-valued, $\operatorname{supp} b_{i s} \subset$ $\mathcal{O}_{\varepsilon}(\mathcal{K})$.

Condition 23.2. The following relations hold:

$$
\begin{gather*}
b_{i s}(y) \geq 0, \quad \sum_{s=1}^{S_{i}} b_{i s}(y) \leq 1, \quad y \in \overline{\Gamma_{i}} ;  \tag{23.2}\\
\sum_{s=1}^{S_{i}} b_{i s}(g)+\sum_{s=1}^{S_{j}} b_{j s}(g)<2, \quad g \in \overline{\overline{\Gamma_{i}}} \cap \overline{\Gamma_{j}} \subset \mathcal{K}, \quad \text { if } i \neq j \text { and } \overline{\Gamma_{i}} \cap \overline{\Gamma_{j}} \neq \varnothing . \tag{23.3}
\end{gather*}
$$

Let us study the nonlocal problem

$$
\begin{gather*}
\mathbf{P}(y, D) u-q u=f(y), \quad y \in G  \tag{23.4}\\
\left.u\right|_{\Gamma_{i}}-\mathbf{B}_{i} u=0, \quad y \in \Gamma_{i}, \quad i=1, \ldots, N,
\end{gather*}
$$

where $q \geq 0$, and the same problem with inhomogeneous nonlocal conditions.
Before we consider problem (23.4) in spaces of continuous functions, we study it in weight spaces. Consider the bounded operator

$$
\mathbf{L}(q)=\mathbf{L}_{l+1-\delta}(q): H_{l+1-\delta}^{l+2}(G) \rightarrow \mathcal{H}_{l+1-\delta}^{l}(G, \partial G)
$$

defined by the formula

$$
\mathbf{L}(q) u=\left\{\mathbf{P}(y, D) u-q u,\left.u\right|_{\Gamma_{i}}-\mathbf{B}_{i} u\right\}, \quad q>0,
$$

where $H_{l+1-\delta}^{l+2}(G)$ and $\mathcal{H}_{l+1-\delta}^{l}(G, \partial G)$ are the spaces defined in Eq. (5.21) with norms (5.22) depending on a parameter $q>0$.

In Sec. 23.3, we prove the following result.
Theorem 23.2. Let conditions 23.1 and 23.2 hold and let $l \geq 0$ be fixed. Then for every sufficiently small $\delta \geq 0$, there exists $q_{1}>0$ such that the operator $\mathbf{L}(q)$ has a bounded inverse operator for $q \geq q_{1}$ and

$$
\begin{equation*}
c_{1}\|\mathbf{L}(q) u\|_{\mathcal{H}_{l+1-\delta}^{l}(G, \partial G)} \leq\|u\|_{H_{l+1-\delta}^{l+2}(G)} \leq c_{2}\|\mathbf{L}(q) u\|_{\mathcal{H}_{l+1-\delta}^{l}(G, \partial G)}, \quad q \geq q_{1} \tag{23.5}
\end{equation*}
$$

where $c_{1}, c_{2}>0$ are independent of $u$ and $q$.
Corollary 23.1 (local a priori estimate). Let $G_{1}$ and $G_{2}$ be subdomains of the domain $G$ such that $G_{1} \subset G_{2}$ and $\operatorname{dist}\left(\partial G_{1} \backslash \partial G, \partial G_{2} \backslash \partial G\right)>0$. Then for all $q \geq q_{1}$, the following a priori estimate holds:

$$
\begin{equation*}
\|u\|_{H_{a}^{l+2}\left(G_{1}\right)} \leq c\left(\|\mathbf{P}(y, D) u-q u\|_{H_{a}^{l}\left(G_{2}\right)}+\sum_{i=1}^{N}\left\|\left.u\right|_{\Gamma_{i}}\right\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i} \cap \overline{G_{2}}\right)}+q^{-1 / 2}\|u\|_{H_{a}^{l+2}\left(G_{2}\right)}\right), \tag{23.6}
\end{equation*}
$$

where $H_{a}^{l+2}\left(G_{j}\right)=H_{a}^{l+2}\left(G_{j}, \mathcal{K}\right), H_{a}^{l}\left(G_{2}\right)=H_{a}^{l}\left(G_{2}, \mathcal{K}\right), H_{a}^{l+3 / 2}\left(\Gamma_{i} \cap \overline{G_{2}}\right)=H_{a}^{l+3 / 2}\left(\Gamma_{i} \cap \overline{G_{2}}, \mathcal{K}\right), q_{1}>0$ is sufficiently large, and $c>0$ is independent of $u$ and $q$. If $\Gamma_{i} \cap \overline{G_{2}}=\varnothing$, then in Eq. (23.6) the term $\left\|\left.u\right|_{\Gamma_{i}}\right\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i} \cap \overline{G_{2}}\right)}$ is absent.

Proof. Consider a patch function $\zeta \in C^{\infty}\left(\mathbb{R}^{2}\right)$ such that $\zeta(y)=1$ for $y \in \overline{G_{1}}$ and $\zeta(y)=0$ for $y \in G \backslash \overline{G_{2}}$. Applying Theorem 23.2 for $\mathbf{B}_{i}=0$ and using the Leibnitz formula, we obtain the inequality

$$
\begin{aligned}
&\|u\|_{H_{a}^{l+2}\left(G_{1}\right)} \leq\|\zeta u\|_{H_{a}^{l+2}(G)} \\
& \leq k_{1}\left(\|\mathbf{P}(y, D)(\zeta u)-q \zeta u\|_{H_{a}^{l}(G)}+\sum_{i=1}^{N}\left\|\left.(\zeta u)\right|_{\Gamma_{i} i}\right\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i}\right)}\right) \\
& \leq k_{2}\left(\|\mathbf{P}(y, D) u-q u\|_{H_{a}^{l}\left(G_{2}\right)}+\sum_{i=1}^{N}\left\|\left.u\right|_{\Gamma_{i}}\right\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i} \cap \overline{G_{2}}\right)}+\|u\|_{H_{a}^{l+1}\left(G_{2}\right)}\right)
\end{aligned}
$$

where $k_{1}, k_{2}>0$ are independent of $u$ and $q$. On the other hand, according to [41, Lemma 7.1], we have

$$
\|u\|_{H_{a}^{l+1}\left(G_{2}\right)} \leq q^{-1 / 2}\|u\|_{H_{a}^{l+2}\left(G_{2}\right)} .
$$

The corollary follows from these two inequalities.
23.3. Nonlocal problems in weight spaces. Denote by $u_{j}(y)$ the function $u(y)$ for $y \in \mathcal{O}_{\varepsilon_{1}}\left(g_{j}\right)$. If $g_{j} \in \overline{\Gamma_{i}}, y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right)$, and $\Omega_{i s}(y) \in \mathcal{O}_{\varepsilon_{1}}\left(g_{k}\right)$, then we denote by $u_{k}\left(\Omega_{i s}(y)\right)$ the function $u\left(\Omega_{i s}(y)\right)$. Then nonlocal problem (23.4) has the following form in a $\varepsilon$-neighborhood of the orbit $\mathcal{K}$ :

$$
\begin{gathered}
\mathbf{P}(y, D) u_{j}-q u_{j}=f(y), \quad y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap G \\
u_{j}(y)-\sum_{s=1}^{S_{i}} b_{i s}(y) u_{k}\left(\Omega_{i s}(y)\right)=0, \quad y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right) \cap \Gamma_{i}, \quad i \in\left\{1 \leq i \leq N: g_{j} \in \overline{\Gamma_{i}}\right\}, \quad j=1, \ldots, N .
\end{gathered}
$$

Let $y \mapsto y^{\prime}\left(g_{j}\right)$ be the change of variables from Sec. 6.1. Introduce the functions $U_{j}\left(y^{\prime}\right)=u\left(y\left(y^{\prime}\right)\right)$ and $F_{j}\left(y^{\prime}\right)=f\left(y\left(y^{\prime}\right)\right)$ for $y^{\prime} \in K_{j}^{\varepsilon}$, where $\sigma=1(\sigma=2)$, if the transformation $y \mapsto y^{\prime}\left(g_{j}\right)$ maps $\Gamma_{i}$ to the ray $\gamma_{j 1}\left(\gamma_{j 2}\right)$ of the angle $K_{j}$. Let us re-denote $y^{\prime}=y$. Then, according to condition 6.3 problem (23.4) has the form

$$
\begin{array}{ll}
\mathbf{P}_{j}(y, D) U_{j}-q U_{j}=F_{j}(y), & y \in K_{j}^{\varepsilon} ; \\
U_{j}(y)-\sum_{k=1}^{N} \sum_{s=1}^{S_{j \sigma k}} B_{j \sigma k s}(y) U_{k}\left(\mathcal{G}_{j \sigma k s} y\right)=0, & y \in \gamma_{j \sigma}^{\varepsilon}, \tag{23.7}
\end{array}
$$

where $\mathbf{P}_{j}(y, D)$ is a second-order elliptic differential operator with real-valued coefficients of class $C^{\infty}$; moreover, the principal homogeneous part of the operator $\mathbf{P}_{j}(0, D)$ is the Laplace operator $\Delta$; $B_{j \sigma k s}(y)$ are smooth functions; $\mathcal{G}_{j \sigma k s}$ is the operator of rotation by the angle $\omega_{j \sigma k s}$ and dilation with the scale factor $\chi_{j \sigma k s}>0$; moreover, $\left|(-1)^{\sigma} \omega_{j}+\omega_{j \sigma k s}\right|<\omega_{k}$.

According to [41], we "freeze" the coefficients of problem (23.7) at the point $y=0$, replace the operators $\mathbf{P}_{j}(0, D)$ by their principal homogeneous parts, and set $q=1$. Thus, we consider the following problem:

$$
\begin{array}{ll}
\Delta U_{j}-U_{j}=F_{j}(y), & y \in K_{j} ; \\
\mathcal{B}_{j \sigma} U \equiv U_{j}(y)-\sum_{k=1}^{N} \sum_{s=1}^{S_{j \sigma k}} b_{j \sigma k s} U_{k}\left(\mathcal{G}_{j \sigma k s} y\right)=0, & y \in \gamma_{j \sigma},
\end{array}
$$

where $U=\left(U_{1}, \ldots, U_{N}\right)$ and $b_{j \sigma k s}=B_{j \sigma k s}(0)$. It follows from condition 23.2 that

$$
\begin{equation*}
b_{j \sigma k s} \geq 0, \quad \sum_{k=1}^{N} \sum_{s=1}^{S_{j \sigma k}} b_{j \sigma k s} \leq 1, \quad \sum_{k=1}^{N}\left(\sum_{s=1}^{S_{j 1 k}} b_{j 1 k s}+\sum_{s=1}^{S_{j 2 k}} b_{j 2 k s}\right)<2 . \tag{23.9}
\end{equation*}
$$

We consider problem (23.8) in weight spaces with inhomogeneous weights. Denote by $E_{a}^{l}\left(K_{j}\right)$ the completion of the set $C_{0}^{\infty}\left(\overline{K_{j}} \backslash\{0\}\right)$ with respect to the norm

$$
\|v\|_{E_{a}^{l}\left(K_{j}\right)}=\left(\sum_{|\alpha| \leq l} \int_{K_{j}}|y|^{2 a}\left(|y|^{2(|\alpha|-l)}+1\right)\left|D^{\alpha} v(y)\right|^{2} d y\right)^{1 / 2},
$$

where $l \geq 0$ is integer and $a \in \mathbb{R}$. Denote by $E_{a}^{l-1 / 2}\left(\gamma_{j \sigma}\right)$ (where $l \geq 1$ is integer) the trace space on $\gamma_{j \sigma}$ (with the infimum norm). Introduce the spaces of vector-valued functions

$$
\mathcal{E}_{a}^{l+2}(K)=\prod_{j=1}^{N} E_{a}^{l+2}\left(K_{j}\right), \quad \mathcal{E}_{a}^{l}(K, \gamma)=\prod_{j=1}^{N}\left(E_{a}^{l}\left(K_{j}\right) \times \prod_{\sigma=1,2} E_{a}^{l+3 / 2}\left(\gamma_{j \sigma}\right)\right) .
$$

Let us consider the operator

$$
\mathcal{L}: \mathcal{E}_{1-\delta}^{2}(K) \rightarrow \mathcal{E}_{1-\delta}^{0}(K, \gamma)
$$

defined by the formula

$$
\mathcal{L} U=\left\{\Delta U_{j}-U_{j}, \mathcal{B}_{j \sigma} U\right\} .
$$

We prove that the operator $\mathcal{L}$ is an isomorphism for all sufficiently small $\delta \geq 0$. For this, we consider an analytic operator-valued function

$$
\tilde{\mathcal{L}}(\lambda): \mathcal{W}_{2}^{2}(-\bar{\omega}, \bar{\omega}) \rightarrow \mathcal{W}^{0}[-\bar{\omega}, \bar{\omega}]
$$

defined by the formula (cf. (6.18))

$$
\tilde{\mathcal{L}}(\lambda) \varphi=\left\{\varphi_{j}^{\prime \prime}-\lambda^{2} \varphi_{j}, \varphi_{j}\left((-1)^{\sigma} \omega_{j}\right)-\sum_{k, s}\left(\chi_{j \sigma k s}\right)^{i \lambda} b_{j \sigma k s} \varphi_{k}\left((-1)^{\sigma} \omega_{j}+\omega_{j \sigma k s}\right)\right\} .
$$

Lemma 23.1. Let conditions 23.1 and 23.2 hold. Then there are no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ on the line $\operatorname{Im} \lambda=0$.
Proof. 1. Assume that $\lambda_{0} \neq 0$ is a real eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$ (the case where $\lambda_{0}=0$ is simpler and can be considered similarly). Let $\varphi(\omega)$ be a corresponding eigenvector; we represent it in the form $\varphi(\omega)=\varphi^{1}(\omega)+i \varphi^{2}(\omega)$, where $\varphi^{1}(\omega)$ and $\varphi^{2}(\omega)$ are real-valued functions of class $C^{\infty}$. It easy to see that the function $U=r^{i \lambda_{0}} \varphi(\omega)=e^{i \lambda_{0} \ln r} \varphi(\omega)$ is a solution of the problem

$$
\begin{equation*}
\Delta U_{j}=0, \quad y \in K_{j} ; \quad \mathcal{B}_{j \sigma} U=0, \quad y \in \gamma_{j \sigma} . \tag{23.10}
\end{equation*}
$$

We represent the function $U$ in form $U=V+i W$, where

$$
V=\cos \left(\lambda_{0} \ln r\right) \varphi^{1}(\omega)-\sin \left(\lambda_{0} \ln r\right) \varphi^{2}(\omega), \quad W=\cos \left(\lambda_{0} \ln r\right) \varphi^{2}(\omega)+\sin \left(\lambda_{0} \ln r\right) \varphi^{1}(\omega)
$$

Since the coefficients in Eq. (23.10) are real, we see that $V$ (and $W$ ) is a solution of the problem

$$
\begin{equation*}
\Delta V_{j}=0, \quad y \in K_{j} ; \quad \mathcal{B}_{j \sigma} V=0, \quad y \in \gamma_{j \sigma} . \tag{23.11}
\end{equation*}
$$

Assume that

$$
M=\max _{j=1, \ldots, N} \sup _{y \in K_{j}}\left|V_{j}(y)\right| .
$$

We prove that $M=0$. Assume the contrary: let $M>0$.
2. If $\left|V_{j}\left(y^{0}\right)\right|=M$ for some $j$ and $y^{0} \in K_{j}$, then $V_{j}(y) \equiv M$ according to the maximum principle. From the nonlocal conditions in Eq. (23.11)) we obtain

$$
\begin{equation*}
M=\left|V_{j}\left(y^{0}\right)\right|=\left|V_{j}\right|_{\gamma_{j \sigma}} \mid \leq M \sum_{k, s} b_{j \sigma k s}, \quad \sigma=1,2 . \tag{23.12}
\end{equation*}
$$

However, $0 \leq \sum_{k, s} b_{j \sigma k s}<1$ for $\sigma=1$ or 2 according to conditions (23.9), which contradicts Eq. (23.12).
3. Let $\left|V_{j}\left(y^{0}\right)\right|=M$ for some $j, \sigma=1$ or 2 and $y^{0} \in \gamma_{j \sigma}$. In this case, taking into account Eq. (23.9), from the nonlocal conditions in (23.11) we obtain the inequality

$$
\begin{equation*}
M=\left|V_{j}\left(y^{0}\right)\right| \leq \sum_{k, s} b_{j \sigma k s}\left|V_{k}\left(\mathcal{G}_{j \sigma k s} y^{0}\right)\right| \leq M \tag{23.13}
\end{equation*}
$$

for $\sigma=1$ or 2. Hence, the inequalities in Eq. (23.13) become equalities, but then

$$
\sum_{k, s} b_{j \sigma k s}=1, \quad\left|V_{k}\left(\mathcal{G}_{j \sigma k s} y^{0}\right)\right|=M
$$

at least for one pair $(k, s)$. By the above, this is impossible since $\mathcal{G}_{j \sigma k s} y^{0} \in K_{k}$.
4. Finally, assume that there exists a sequence $\left\{y^{s}\right\}_{s=1}^{\infty} \subset K_{j}$ such that $\left|V_{j}\left(y^{s}\right)\right| \rightarrow M$ for some $j$ for $\left|y^{s}\right| \rightarrow 0$ or $\left|y^{s}\right| \rightarrow \infty$.

Note that the function $V_{j}$ is periodic with respect to $\ln r$, i.e., the function $V_{j}$ is completely defined by its values on the set

$$
\hat{K}_{j}=\overline{K_{j}} \cap\left\{1 \leq r \leq e^{2 \pi / \lambda_{0}}\right\} .
$$

Since the set $\hat{K}_{j}$ is compact, we see that there exists a sequence $\left\{\hat{y}^{s}\right\}_{s=1}^{\infty} \subset \hat{K}_{j}$ such that $\left|V_{j}\left(\hat{y}^{s}\right)\right| \rightarrow$ $M$ for $\hat{y}^{s} \rightarrow \hat{y}$, where $\hat{y} \in \hat{K}_{j}$. It follows from the continuity of $V_{j}(y)$ on the compact $\hat{K}_{j}$ that $\left|V_{j}(\hat{y})\right|=M$. We again obtain a contradiction with the above.
5. It follows from items 1-4 of this proof that $M=0$. Hence, $V=0$, i.e., $\varphi^{1}(\omega)=\varphi^{2}(\omega)=0$.

Lemma 23.2. Let conditions 23.1 and 23.2 hold. Then the operator $\mathcal{L}: \mathcal{E}_{1}^{2}(K) \rightarrow \mathcal{E}_{1}^{0}(K, \gamma)$ is an isomorphism.
Proof. 1. First, we prove that the operator $\mathcal{L}: \mathcal{E}_{1}^{2}(K) \rightarrow \mathcal{E}_{1}^{0}(K, \gamma)$ is a Fredholm operator and ind $\mathcal{L}=0$. Consider the family of operators $\mathcal{L}_{t}: \mathcal{E}_{1}^{2}(K) \rightarrow \mathcal{E}_{1}^{0}(K, \gamma)$ defined by the formula

$$
\mathcal{L}_{t} U=\left\{\Delta U_{j}-U_{j},\left.U_{j}\right|_{\gamma_{j \sigma}}-\left.t \sum_{k, s} b_{j \sigma k s} U_{k}\left(\mathcal{G}_{j \sigma k s} y\right)\right|_{\gamma_{j \sigma}}\right\}, \quad 0 \leq t \leq 1
$$

Analogously to the operator $\tilde{\mathcal{L}}(\lambda)$, we introduce operators $\tilde{\mathcal{L}}_{t}(\lambda)$. By Lemma 23.1 there are no eigenvalues of the operators $\tilde{\mathcal{L}}_{t}(\lambda)$ on the line $\operatorname{Im} \lambda=0$. Hence, $\mathcal{L}_{t}$ are Fredholm operators (see [24, Theorem 9.1]). By virtue of the homotopic stability of the index of Fredholm operators, we have ind $\mathcal{L}_{t}=$ const, $t \in[0,1]$. Since the local operator $\mathcal{L}_{0}$ is an isomorphism (see, e.g., [24, Sec. 10.3]), we have ind $\mathcal{L}=\operatorname{ind} \mathcal{L}_{0}=0$.
2. It remains to prove that $\operatorname{dim} \operatorname{ker} \mathcal{L}=0$. Let $U \in \mathcal{E}_{1}^{2}(K)$ be a real-valued solution of the problem

$$
\begin{equation*}
\Delta U_{j}=U_{j}, \quad y \in K_{j} ; \quad \mathcal{B}_{j \sigma} U=0, \quad y \in \gamma_{j \sigma} \tag{23.14}
\end{equation*}
$$

By the theorem on the internal smoothness, the functions $U_{j}$ are infinitely differentiable in $K_{j}$. Prove that $U_{j}$ are continuous on $\overline{K_{j}}$.

Since there are no eigenvalues $\tilde{\mathcal{L}}(\lambda)$ on the line $\operatorname{Im} \lambda=0$, it follows from [88] that there exists a number $\delta \in[0,1]$ such that there is no more than a finite set of eigenvalues $\left\{\lambda_{k}\right\}$ of the operator $\tilde{\mathcal{L}}(\lambda)$
(at that, $-1-\delta<\operatorname{Im} \lambda_{k}<0$ ) on the strip $-1-\delta \leq \operatorname{Im} \lambda \leq 0$. Taking into account the fact that $U_{j} \in E_{1}^{2}\left(K_{j}\right) \subset H_{1}^{2}\left(K_{j}\right)$ is a solution of problem (23.14) with the right-hand sides $U_{j} \in E_{1}^{2}\left(K_{j}\right) \subset$ $H_{-\delta}^{0}\left(K_{j}\right)$ and using [26, Theorem 2.2] (on the asymptotic behavior of solutions of nonlocal problems), we obtain the relation

$$
\begin{equation*}
U=\sum_{k} \sum_{q=1}^{J_{k}} \sum_{m=0}^{\varkappa_{q k}-1} c_{k}^{(m, q)} W_{k}^{(m, q)}+U^{\prime}, \quad W_{k}^{(m, q)}(\omega, r)=r^{i \lambda_{k}} \sum_{\nu=0}^{m} \frac{1}{\nu!}(i \ln r)^{\nu} \varphi_{k}^{(m-\nu, q)}(\omega) \tag{23.15}
\end{equation*}
$$

where $\varphi_{k}^{(0, q)}, \ldots, \varphi_{k}^{\left(\varkappa_{q k}-1, q\right)} \in \prod_{j} C^{\infty}\left(\left[-\omega_{j}, \omega_{j}\right]\right)$ is a Jordan chain corresponding to the eigenvalue $\lambda_{k}$, $c_{k}^{(m, q)}$ are constants, and $U_{j}^{\prime} \in H_{-\delta}^{2}\left(K_{j}\right)$. It follows from here and the Sobolev embedding theorem that $U_{j}$ are continuous on $\overline{K_{j}}$ and $U_{j}(0)=0$.

Now we show that

$$
\begin{equation*}
\left|U_{j}(y)\right| \rightarrow 0 \quad \text { as } \quad|y| \rightarrow \infty \tag{23.16}
\end{equation*}
$$

If $U \in \mathcal{E}_{1}^{2}(K)$, then $U \in \mathcal{E}_{1}^{0}(K)$. This, the fact that $U$ is a solution of homogeneous problem (23.14), and [24, Theorem 3.2] imply that $U \in \mathcal{E}_{3}^{2}(K)$. Fixing arbitrary large $a \geq 1$ and repeating these reasonings, we obtain $U \in \mathcal{E}_{a}^{2}(K)$. Assuming that $V(\omega, r)=U\left(\omega, r^{-1}\right)$ and using the Sobolev embedding theorem and the arbitrariness of $a$, we see that the function $V_{j}(y)$ is continuous at the origin and $\left|V_{j}(y)\right| \rightarrow 0$ as $|y| \rightarrow 0$. Equation (23.16) follows from here.
3. Introduce the notation

$$
M=\max _{j=1, \ldots, N} \sup _{y \in \overline{K_{j}}}\left|U_{j}(y)\right|
$$

Let us show that $M=0$. Assume the contrary: let $M>0$. By the properties of $U_{j}$ proved above, any function $\left|U_{j}(y)\right|$ has a maximum at some point $y^{0} \in \overline{K_{j}} \backslash\{0\}$. If $\left|U_{j}\left(y^{0}\right)\right|=M$ for some $j$ and $y^{0} \in K_{j}$, then $U_{j}(y) \equiv$ const by the maximum principle. Using the differential equation in (23.14), we obtain $M \equiv\left|U_{j}\right|=\left|\Delta U_{j}\right|=0$.

If $\left|U_{j}\left(y^{0}\right)\right|=M$ for $y^{0} \in \gamma_{j \sigma}$, where $\sigma=1$ or 2 , then, using the nonlocal conditions in (23.14) and conditions (23.9), we obtain the inequality

$$
\begin{equation*}
M=\left|U_{j}\left(y^{0}\right)\right| \leq \sum_{k, s} b_{j \sigma k s}\left|U_{k}\left(\mathcal{G}_{j \sigma k s} y^{0}\right)\right| \leq M \tag{23.17}
\end{equation*}
$$

Thus, inequalities in Eq. (23.17) become equalities, and it follows from here that

$$
\sum_{k, s} b_{j \sigma k s}=1, \quad\left|U_{k}\left(\mathcal{G}_{j \sigma k s} y^{0}\right)\right|=M
$$

at least for one pair $(k, s)$. Nevertheless, $\mathcal{G}_{j \sigma k s} y^{0} \in K_{k}$, which is impossible by the above.
Corollary 23.2. Let conditions 23.1 and 23.2 hold. Then there is $\delta_{0}>0$ such that the operator

$$
\mathcal{L}: \mathcal{E}_{l+1-\delta}^{l+2}(K) \rightarrow \mathcal{E}_{l+1-\delta}^{l}(K, \gamma), \quad l=0,1,2, \ldots,
$$

is an isomorphism for $0 \leq \delta \leq \delta_{0}$.
Proof. By Lemma 23.2, the operator $\mathcal{L}: \mathcal{E}_{1}^{2}(K) \rightarrow \mathcal{E}_{\tilde{L}}^{0}(K, \gamma)$ is an isomorphism. On the other hand, by Lemma 23.1 and the discontinuity of the spectrum $\tilde{L}(\lambda)$ (see [88]), there exists $\delta_{0}>0$ such that there are no eigenvalues $\tilde{\mathcal{L}}(\lambda)$ in the strip $-\delta_{0} \leq \operatorname{Im} \lambda \leq 0$. Similarly (see [64, Chap. 6, Proposition 2.8]), we can show that the operator $\mathcal{L}: \mathcal{E}_{1-\delta}^{2}(K) \rightarrow \mathcal{E}_{1-\delta}^{0}(K, \gamma)$ is an isomorphism for $0 \leq \delta \leq \delta_{0}$. Then, by [24, Theorem 9.2, 9.3], the operator $\mathcal{L}: \mathcal{E}_{l+1-\delta}^{l+2}(K) \rightarrow \mathcal{E}_{l+1-\delta}^{l}(K, \gamma)$ is also an isomorphism.
Proof of Theorem 23.2. The theorem follows from Corollary 23.2 and [41, Theorem 8.1].
23.4. Nonlocal problems in spaces of continuous functions. Take a number $\delta \in[0,1]$ such that there are no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ either in the strip $-\delta \leq \operatorname{Im} \lambda \leq 0$ or on the line $\operatorname{Im} \lambda=-1-\delta$. The existence of such a number follows from Lemma 23.1 and the discreteness of the spectrum of $\tilde{L}(\lambda)$ (see Lemma 6.1).

Let $q_{1}$ be a number from Theorem 23.2. First, we construct an analog of a barrier function for nonlocal problems. Consider the following auxiliary problem:

$$
\begin{equation*}
\mathbf{P}(y, D) v-q_{1} v=0, \quad y \in G ;\left.\quad v\right|_{\Gamma_{i}}-\mathbf{B}_{i} v=1, \quad y \in \Gamma_{i}, \quad i=1, \ldots, N . \tag{23.18}
\end{equation*}
$$

Lemma 23.3. Let conditions 23.1 and 23.2 hold. Then problem (23.18) has a bounded solution $v \in C^{\infty}(\bar{G} \backslash \mathcal{K})$ such that $\inf _{y \in \bar{G} \backslash \mathcal{K}} v(y)>0$.
Proof. 1. Consider the model problem

$$
\begin{equation*}
\Delta W_{j}^{1}=0, \quad y \in K_{j}^{\varepsilon} ; \quad W_{j}^{1}(y)-\sum_{k, s} b_{j \sigma k s} W_{k}^{1}\left(\mathcal{G}_{j \sigma k s} y\right)=1, \quad y \in \gamma_{j \sigma}^{\varepsilon} . \tag{23.19}
\end{equation*}
$$

Find a solution of problem (23.19) in the form

$$
\begin{equation*}
W_{j}^{1}=\varphi_{j}(\omega), \quad|\omega|<\omega_{j}, \quad j=1, \ldots, N \tag{23.20}
\end{equation*}
$$

Obviously, the functions $\varphi_{1}(\omega), \ldots, \varphi_{N}(\omega)$ satisfy the relations

$$
\begin{equation*}
\varphi_{j}^{\prime \prime}(\omega)=0, \quad|\omega|<\omega_{j} ; \quad \varphi_{j}\left((-1)^{\sigma} \omega_{j}\right)-\sum_{k, s} b_{j \sigma k s} \varphi_{k}\left((-1)^{\sigma} \omega_{j}+\omega_{j \sigma k s}\right)=1 \tag{23.21}
\end{equation*}
$$

or, equivalently, $\tilde{\mathcal{L}}(0) \varphi=\left\{\tilde{F}_{j}, \tilde{F}_{j \sigma}\right\}$, where $\tilde{F}_{j}=0$ and $\tilde{F}_{j \sigma}=1$. By Lemma 23.1, the number $\lambda=0$ is not an eigenvalue $\tilde{\mathcal{L}}(\lambda)$. Since $\tilde{\mathcal{L}}(\lambda)$ is a Fredholm operator and it has a zero index (see. [88]), there exists a unique (real-valued) solution $\varphi \in \prod_{j} C^{\infty}\left(\left[-\omega_{j}, \omega_{j}\right]\right)$ of problem (23.21). Obviously, the functions $\varphi_{j}(\omega)$ are linear. Using the nonlocal conditions in Eq. (23.21) and relation (23.9), it is easy to verify that $\varphi_{j}(\omega)>0$ for $\omega \in\left[-\omega_{j}, \omega_{j}\right]$.
2. Consider a function $\xi \in C^{\infty}\left(\mathbb{R}^{2}\right)$ such that $\xi(y)=1$ for $y \in \mathcal{O}_{\varepsilon / 2}(\mathcal{K})$ and $\operatorname{supp} \xi \subset \mathcal{O}_{\varepsilon}(\mathcal{K})$.

Find a solution $v$ of the initial problem (23.18) in the form

$$
\begin{equation*}
v(y)=w^{1}(y)+v^{1}(y), \quad y \in G, \tag{23.22}
\end{equation*}
$$

where $w^{1}(y)=\xi(y) W_{j}^{1}\left(y^{\prime}(y)\right), y \in \mathcal{O}_{\varepsilon}\left(g_{j}\right), g_{j} \in \mathcal{K}, y^{\prime} \mapsto y\left(g_{j}\right)$ is a transformation inverse to the transformation $y \mapsto y^{\prime}\left(g_{j}\right)$ from Sec. 23.2, and the function $w^{1}$ is extended by zero to $G \backslash \mathcal{O}_{\varepsilon}(\mathcal{K}) ; v^{1}$ is an unknown function.

It follows from Eqs. (23.18) and (23.22) that $v^{1}$ satisfies relations

$$
\begin{equation*}
\mathbf{P}(y, D) v^{1}-q_{1} v^{1}=f^{1}(y), \quad y \in G ;\left.\quad v^{1}\right|_{\Gamma_{i}}-\mathbf{B}_{i} v^{1}=f_{i}^{1}(y), \quad y \in \Gamma_{i}, \tag{23.23}
\end{equation*}
$$

where

$$
\begin{equation*}
f^{1}=-\mathbf{P}(y, D) w^{1}+q_{1} w^{1}, \quad f_{i}^{1}=1-\left.w^{1}\right|_{\Gamma_{i}}+\left.\mathbf{B}_{i} w^{1}\right|_{\Gamma_{i}} . \tag{23.24}
\end{equation*}
$$

Let $V_{j}^{1}\left(y^{\prime}\right)=v^{1}\left(y\left(y^{\prime}\right)\right), F_{j}\left(y^{\prime}\right)=f^{1}\left(y\left(y^{\prime}\right)\right)$, and $F_{j \sigma}\left(y^{\prime}\right)=f_{i}^{1}\left(y\left(y^{\prime}\right)\right), y^{\prime} \in K_{j}^{\varepsilon}$, where $y \mapsto y^{\prime}\left(g_{j}\right)$ is the transformation from Sec. 23.2, $g_{j} \in \mathcal{K} \cap \overline{\Gamma_{i}}$. Denote $y^{\prime}=y$. Then by virtue of (23.19) and (23.24) we have

$$
\begin{array}{ll}
F_{j}(y)=\left(\Delta-\mathbf{P}_{j}(y, D)\right) W_{j}^{1}+q_{1} W_{j}^{1}, & y \in K_{j}^{\varepsilon / 2} \\
F_{j \sigma}(y)=\sum_{k, s}\left(B_{j \sigma k s}(y)-b_{j \sigma k s}\right) W_{k}^{1}\left(\mathcal{G}_{j \sigma k s} y\right), & y \in K_{j}^{\varepsilon / 2} \tag{23.25}
\end{array}
$$

where $\mathbf{P}_{j}(y, D)$ and $B_{j \sigma k s}(y)$ are the same as in Eq. (23.7). Taking into account the fact that the principal homogeneous parts of the operators $\mathbf{P}(y, D)_{j}(0, D)$ coincide with the Laplace operator and
$B_{j \sigma k s}(0)=b_{j \sigma k s}$ and using the Taylor expansion, from representation (23.20) and relations (23.25) we obtain

$$
F_{j} \in H_{l+1-\delta}^{l}\left(K_{j}^{\varepsilon / 2}\right), \quad F_{j \sigma} \in H_{l+1-\delta}^{l+3 / 2}\left(\gamma_{j \sigma}^{\varepsilon / 2}\right),
$$

i.e., $\left\{f^{1}, f_{i}^{1}\right\} \in \mathcal{H}_{l+1-\delta}^{l}(G, \partial G)$. Hence, by Theorem 23.2, there exists a unique solution $v^{1} \in H_{l+1-\delta}^{l+2}(G)$ of problem (23.23). Since $l \geq 0$ is arbitrary, by the Sobolev embedding theorem, the function $v$ defined in Eq. (23.22) belongs to $C^{\infty}(\bar{G} \backslash \mathcal{K})$. Obviously, it is a solution of the initial problem (23.18).
3. We prove that $v^{1} \in C(\bar{G})$ and $v^{1}(y)=0$ for $y \in \mathcal{K}$. By virtue of Eq. (23.23), the functions $V_{j}^{1}(y)$ satisfy the following relations:

$$
\begin{array}{ll}
\Delta V_{j}^{1}=F_{j}^{1}(y)+F_{j}(y), & y \in K_{j}^{\varepsilon / 2} \\
V_{j}^{1}(y)-\sum_{k, s} b_{j \sigma k s} V_{k}^{1}\left(\mathcal{G}_{j \sigma k s} y\right)=F_{j \sigma}^{1}(y)+F_{j \sigma}(y), & y \in \gamma_{j \sigma}^{\varepsilon / 2} \tag{23.26}
\end{array}
$$

where $F_{j}^{1}=\left(\Delta-\mathbf{P}_{j}(y, D)\right) V_{j}^{1}+q_{1} V_{j}^{1}$ and

$$
F_{j \sigma}^{1}=\sum_{k, s}\left(B_{j \sigma k s}(y)-b_{j \sigma k s}\right) V_{k}^{1}\left(\mathcal{G}_{j \sigma k s} y\right) .
$$

Taking into account the fact that the principal homogeneous parts of the operators $\mathbf{P}(y, D)_{j}(0, D)$ coincide with the Laplace operator and $B_{j \sigma k s}(0)=b_{j \sigma k s}$ and using the Taylor expansion, we can represent the right-hand side of problem (23.26) in the following form:

$$
\begin{equation*}
F_{j}^{1}+F_{j}=F_{j}^{1}+F_{j}^{2}+r^{-1} \psi_{j}(\omega), \quad F_{j \sigma}^{1}+F_{j \sigma}=F_{j \sigma}^{1}+F_{j \sigma}^{2}+\psi_{j \sigma} r, \tag{23.27}
\end{equation*}
$$

where $\psi_{j} \in C^{\infty}\left(\left[-\omega_{j}, \omega_{j}\right]\right), F_{j}^{1}+F_{j}^{2} \in H_{-\delta}^{0}\left(K_{j}^{\varepsilon / 2}\right)$ and $\psi_{j \sigma} \in \mathbb{R}, F_{j \sigma}^{1}+F_{j \sigma}^{2} \in H_{-\delta}^{3 / 2}\left(\gamma_{j \sigma}^{\varepsilon / 2}\right)$.
Obtain the asymptotic expansions of the functions $V_{j}^{1}$. We denote by $\left\{\lambda_{k}\right\}$ the finite set of eigenvalues $\tilde{\mathcal{L}}(\lambda)$ concentrated in the strip $-1-\delta<\operatorname{Im} \lambda<-\delta$. Then, applying [26, Theorem 2.2] and [26, Lemma 4.3] to problem (23.26) with right-hand side (23.27), we obtain

$$
\begin{equation*}
V^{1}=r \sum_{\nu=0}^{\varkappa} \frac{1}{\nu!}(i \ln r)^{\nu} u^{(\nu)}(\omega)+\sum_{k} \sum_{q=1}^{J_{k}} \sum_{m=0}^{\varkappa_{q k}-1} c_{k}^{(m, q)} W_{k}^{(m, q)}+V^{2}, \quad y \in K_{j}^{\varepsilon / 2} \tag{23.28}
\end{equation*}
$$

where $u^{(\nu)} \in \prod_{j} C^{\infty}\left(\left[-\omega_{j}, \omega_{j}\right]\right)$, the functions $W_{k}^{(m, q)}$ have the same form as in Eq. (23.15), $c_{k}^{(m, q)}$ are some constants, and $V_{j}^{2} \in H_{-\delta}^{2}\left(K_{j}^{\varepsilon / 2}\right)$. It follows from formula (23.28) and the Sobolev embedding theorem that $V_{j}^{1} \in C\left(\overline{K_{j}^{\varepsilon / 2}}\right)$ and $V_{j}^{1}(0)=0$. Hence, $v^{1} \in C(\bar{G})$ and $v^{1}(0)=0$ for $y \in \mathcal{K}$. In particular, it follows from here that the function $v=v^{1}+w^{1}$ is bounded.
4. It remains to show that $m>0$, where $m=\inf _{y \in \bar{G} \backslash \mathcal{K}} v(y)$. Assume the contrary: let $m \leq 0$. Consider a sequence $\left\{y^{k}\right\} \subset \bar{G} \backslash \mathcal{K}$ such that $v\left(y^{k}\right) \rightarrow m$ for $k \rightarrow \infty$. Since the sequence $\left\{y^{k}\right\}$ is bounded, it contains a convergent subsequence (which we denote $\left\{y^{k}\right\}$ ). Let $y^{k} \rightarrow y^{0}$ for $k \rightarrow \infty$, where $y^{0} \in \bar{G}$.

Using the maximum principle, the nonlocal conditions in Eq. (23.18), and relations (23.2), it is easy to verify that $y^{0} \notin \bar{G} \backslash \mathcal{K}$. Assume that $y^{0} \in \mathcal{K}$. By item 1 , we can find a constant $A>0$ such that $w^{1}(y) \geq A$ in some neighborhood of a point $y^{0}$ (except for a point $y^{0}$ where the function $w^{1}$ may not be defined). On the other hand, we have proved in item 3 that $v^{1}\left(y^{0}\right)=0$. Hence, $v(y) \geq A / 2$ in some neighborhood of $y^{0}$ (perhaps, except for the point $y^{0}$ ). In this case, the sequence $\left\{v\left(y^{k}\right)\right\}$ cannot converge to a nonpositive number $m$.

Let us consider the problem

$$
\begin{equation*}
\mathbf{P}(y, D) u-q u=0, \quad y \in G ;\left.\quad u\right|_{\Gamma_{i}}-\mathbf{B}_{i} u=\psi_{i}(y), \quad y \in \Gamma_{i}, \quad i=1, \ldots, N . \tag{23.29}
\end{equation*}
$$

The following theorem is fundamental in the study of Feller semigroups.

Theorem 23.3. Let conditions 23.1 and 23.2 hold and let $q \geq q_{1}$. Then for any $\psi=\left\{\psi_{i}\right\} \in \mathcal{C}_{\mathcal{K}}(\partial G)$, there exists a unique solution $u \in C(\bar{G}) \cap C^{\infty}(G)$ of problem (23.29). Moreover, $u(y)=0$ for $y \in \mathcal{K}$ and the following estimate holds:

$$
\begin{equation*}
\|u\|_{C(\bar{G})} \leq c_{1}\|\psi\|_{\mathcal{C}_{\mathcal{K}}(\partial G)}, \tag{23.30}
\end{equation*}
$$

where $c_{1}>0$ is independent of $\psi$ and $q$.
Proof. 1. We prove the theorem for infinitely smooth functions $\psi_{i}$ that vanish in some neighborhood of the sets $\overline{\Gamma_{i}} \cap \mathcal{K}$. Passing to the limit, we obtain the general case. For the functions $\psi_{i}$ mentioned, we have $\psi_{i} \in H_{-\delta}^{3 / 2}\left(\Gamma_{i}\right)$. Hence, by Theorem 23.2 , there exists a unique solution $u \in H_{1-\delta}^{2}(G)$ of problem (23.29). By virtue (13.22), $u \in C^{\infty}(\bar{G} \backslash \mathcal{K})$. Let $\left\{\lambda_{k}\right\}$ be a finite set of eigenvalues $\tilde{\mathcal{L}}(\lambda)$ concentrated in the strip $-1-\delta<\operatorname{Im} \lambda<-\delta$. Then, according to [26, Theorem 2.2] (theorem on the asymptotic behavior of solutions of nonlocal problems), the function $u$ can be re presented in the following form near the point $g_{j} \in \mathcal{K}(j=1, \ldots, N)$ :

$$
u(y)=\sum_{k} \sum_{q=1}^{J_{k}} \sum_{m=0}^{\varkappa_{q k}-1} c_{k}^{(m, q)} W_{k j}^{(m, q)}+u^{\prime}(y), \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right),
$$

where $c_{k}^{(m, q)}$ are some constants, the functions $W_{k j}^{(m, q)}(\omega, r)$ have the same form as the components of the vector $W_{k}^{(m, q)}(\omega, r)$ in Eq. (23.15) ( $\omega$ and $r$ are the polar coordinates with pole at the point $g_{j}$ ), and $u^{\prime} \in H_{-\delta}^{2}(G)$. Thus, applying the Sobolev embedding theorem, we see that $u \in C(\bar{G})$ and

$$
\begin{equation*}
u(y)=0, \quad y \in \mathcal{K} . \tag{23.31}
\end{equation*}
$$

2. Prove estimate (23.30). Assume that $M=\|\psi\|_{\mathcal{C}_{\mathcal{K}}(\partial G)}$ and $M>0$.

Denote $w_{ \pm}(y)=M v(y) \pm u(y)$, where $v(y)$ is the function from Lemma 23.3. By Eqs. and (23.29), the functions $w_{ \pm}$satisfy the following relations:

$$
\begin{array}{ll}
\mathbf{P}(y, D) w_{ \pm}-q w_{ \pm}=M\left(q_{1}-q\right) v(y), & y \in G,  \tag{23.18}\\
w_{ \pm} \mid \Gamma_{i}-\mathbf{B}_{i} w_{ \pm}=M \pm \psi_{i}(y), & y \in \Gamma_{i}, \quad i=1, \ldots, N .
\end{array}
$$

Since $q_{1} \leq q, v(y)>0, y \in G$ (by Lemma 23.3) and $M \geq \pm \psi_{i}$, we have

$$
\begin{equation*}
\mathbf{P}(y, D) w_{ \pm}-q w_{ \pm} \leq 0, \quad y \in G,\left.\quad w_{ \pm}\right|_{\Gamma_{i}}-\mathbf{B}_{i} w_{ \pm} \geq 0, \quad y \in \Gamma_{i}, \quad i=1, \ldots, N . \tag{23.32}
\end{equation*}
$$

Let us show that $m_{ \pm}=\inf _{y \in \bar{G} \backslash \mathcal{K}} w_{ \pm}(y) \geq 0$. Assume the contrary: let $m_{ \pm}<0$. As well as in item 4 of the proof of Lemma 23.3, we consider a sequence $\left\{y^{k}\right\} \subset \bar{G} \backslash \mathcal{K}$ such that $y^{k} \rightarrow y^{0}$ and $w_{ \pm}\left(y_{k}\right) \rightarrow m_{ \pm}$ as $k \rightarrow \infty$, where $y^{0} \in \bar{G}$. The following three cases are possible: $y^{0} \in G, y^{0} \in \Gamma_{i}$ for some $i$, and $y^{0} \in \mathcal{K}$.

Let $y^{0} \in G$. Since $w_{ \pm}(y)$ is continuous in $G$, we see that it has a negative minimum $m$ inside the domain. It follows from the first inequality in Eq. (23.32) and the maximum principle that $w_{ \pm}(y)=m_{ \pm}$ as $y \in G$. Taking into account condition 23.1, we obtain

$$
\mathbf{P}(y, D) w_{ \pm}\left(y^{0}\right)-q w_{ \pm}\left(y^{0}\right)=p_{0}\left(y^{0}\right) m_{ \pm}-q m_{ \pm} \geq-q m_{ \pm}>0,
$$

which contradicts the first inequality in Eq. (23.32).
Let $y^{0} \in \Gamma_{i}$ for some $i$. Then from Eq. (23.32) and (23.2) we obtain the following inequality:

$$
\begin{equation*}
m_{ \pm}=w_{ \pm}\left(y^{0}\right) \geq \sum_{s=1}^{S_{i}} b_{i s}\left(y^{0}\right) w_{ \pm}\left(\Omega_{i s}\left(y^{0}\right)\right) \geq m_{ \pm} \sum_{s=1}^{S_{i}} b_{i s}\left(y^{0}\right) \geq m_{ \pm} \tag{23.33}
\end{equation*}
$$

Hence, inequalities (23.33) become equalities, i.e.,

$$
\sum_{s=1}^{S_{i}} b_{i s}\left(y^{0}\right)=1, \quad w_{ \pm}\left(\Omega_{i s}\left(y^{0}\right)\right)=m_{ \pm}
$$

for some $s$, i.e., the function $w_{ \pm}(y)$ has a negative minimum at an internal point of $\Omega_{i s}\left(y^{0}\right) \in G$. However, this is impossible.

Finally, we assume that $y^{0} \in \mathcal{K}$. By Lemma 23.3, $m=\inf _{y^{\prime} \in \bar{G} \backslash \mathcal{K}} v\left(y^{\prime}\right)>0$, and we obtain the inequality

$$
M v(y) \geq M m>0, \quad y \in \bar{G} \backslash \mathcal{K} .
$$

The last inequality and Eq. (23.31) imply that

$$
w_{ \pm}(y)=M v(y) \pm u(y) \geq M m / 2>0
$$

in some neighborhood of $y^{0}$ (except for the point $y^{0}$, where $w_{ \pm}(y)$ may not be defined). Hence, the sequence $\left\{w_{ \pm}\left(y^{k}\right)\right\}$ cannot converge to a negative number $m_{ \pm}$.

Hence, we have proved that $\inf _{y \in \bar{G} \backslash \mathcal{K}} w_{ \pm}(y) \geq 0$; therefore,

$$
|u(y)| \leq M v(y) \leq M \sup _{y^{\prime} \in \bar{G} \backslash \mathcal{K}} v\left(y^{\prime}\right), \quad y \in \bar{G} \backslash \mathcal{K} .
$$

Since the function $u(y)$ is continuous in $\bar{G}$, from the last inequality we obtain estimate (23.30), where $c_{1}=\sup _{y^{\prime} \in \bar{G} \backslash \mathcal{K}} v\left(y^{\prime}\right)$. Obviously, the constant $c_{1}>0$ is independent of $\psi$ and $q$.

Consider the nonlocal problem

$$
\begin{equation*}
\mathbf{P}(y, D) u-q u=f_{0}(y), \quad y \in G ;\left.\quad u\right|_{\Gamma_{i}}-\mathbf{B}_{i} u=\psi_{i}(y), \quad y \in \Gamma_{i}, \quad i=1, \ldots, N . \tag{23.34}
\end{equation*}
$$

The following result follows from Theorems 23.2 and 23.3 and the asymptotic properties of solutions of nonlocal problems [26].

Corollary 23.3. Let conditions 23.1 and 23.2 hold. Then we can find a number $q_{1}>0$ such that for any $f_{0} \in C(\bar{G}), \psi=\left\{\psi_{i}\right\} \in \mathcal{C}_{\mathcal{K}}(\partial G)$, and $q \geq q_{1}$, there exists a unique solution $u \in C_{\mathcal{K}}(\bar{G}) \cap W_{\text {loc }}^{2}(G)$ of problem (23.34). Moreover, if $f_{0}=0$, then $u \in C_{\mathcal{K}}(\bar{G}) \cap C^{\infty}(G)$ and the following estimate holds:

$$
\begin{equation*}
\|u\|_{C_{\mathcal{K}}(\bar{G})} \leq c_{1}\|\psi\|_{\mathcal{C}_{\mathcal{K}}(\partial G)} \tag{23.35}
\end{equation*}
$$

where $c_{1}>0$ is independent of $\psi$ and $q$.

## 24. Bounded Perturbations of Diffusion Processes

24.1. Problems with nonlocal terms whose supports lie near conjugation points. In the sequel, we will need the following maximum principle.

Maximum Principle 24.1 ( [22, Theorem 9.6]). Let $D \subset \mathbb{R}^{2}$ be a bounded or unbounded domain and let condition 23.1 hold for the domain $D$. If the function $u \in C(D)$ has a positive maximum at a point $y^{0} \in D$ and $d^{13} \mathbf{P}(y, D) u \in C(D)$, then $\mathbf{P}(y, D) u\left(y^{0}\right) \leq 0$.

Let us formulate some auxiliary results that will be helpful in the next sections.
Let $u \in C^{\infty}(G) \cap C_{\mathcal{K}}(\bar{G})$ be a solution of problem (23.34) with $f_{0}=0$ and $\psi=\left\{\psi_{i}\right\} \in \mathcal{C}_{\mathcal{K}}(\partial G)$. Denote $u=\mathbf{S}_{q} \psi$. By Corollary 23.3, the operator

$$
\mathbf{S}_{q}: \mathcal{C}_{\mathcal{K}}(\partial G) \rightarrow C_{\mathcal{K}}(\bar{G}), \quad q \geq q_{1}
$$

is bounded and $\left\|\mathbf{S}_{q}\right\| \leq c_{1}$, where $c_{1}>0$ is independent of $q$.
Lemma 24.1. Let conditions 23.1 and 23.2 hold. Let $Q_{1}$ and $Q_{2}$ be closed sets such that $Q_{1} \subset \partial G$, $Q_{2} \subset \bar{G}$, and $Q_{1} \cap Q_{2}=\varnothing$, and let $q \geq q_{1}$. Then the following inequality holds for all $\psi \in \mathcal{C}_{\mathcal{K}}(\partial G)$ such that $\left.\operatorname{supp}\left(\mathbf{S}_{q} \psi\right)\right|_{\partial G} \subset Q_{1}$ :

$$
\left\|\mathbf{S}_{q} \psi\right\|_{C\left(Q_{2}\right)} \leq \frac{c_{2}}{q}\|\psi\|_{\mathcal{C}_{\mathcal{K}}(\partial G)}, \quad q \geq q_{1}
$$

[^12]where $c_{2}>0$ is independent of $\psi$ and $q$.
Proof. Using [19, Lemma 1.3] ${ }^{14}$ and Corollary 23.3. We obtain
\[

$$
\begin{equation*}
\left\|\mathbf{S}_{q} \psi\right\|_{C\left(Q_{2}\right)} \leq \frac{k}{q}\left\|\left.\left(\mathbf{S}_{q} \psi\right)\right|_{\partial G}\right\|_{C(\partial G)} \leq \frac{k}{q}\left\|\mathbf{S}_{q} \psi\right\|_{C(\bar{G})} \leq \frac{k c_{1}}{q}\|\psi\|_{\mathcal{C}_{\mathcal{K}}(\partial G)}, \quad q \geq q_{1}, \tag{24.1}
\end{equation*}
$$

\]

where the number $q_{1}$ from Corollary 23.3 is assumed to be sufficiently large (so large that [19, Lemma 1.3] is valid for $\left.q \geq q_{1}\right)$; the number $k=k\left(q_{1}\right)$ is independent of $\psi$ and $q$.

Lemma 24.2. Let conditions 23.1 and 23.2 hold, $Q_{1}$ and $Q_{2}$ be the same as in Lemma 24.1, and let $q \geq q_{1}$. Assume that $Q_{2} \cap \mathcal{K}=\varnothing$. Then for all $\psi \in \mathcal{C}_{\mathcal{K}}(\partial G)$ such that $\operatorname{supp} \psi \subset Q_{1}$, the following inequality holds:

$$
\left\|\mathbf{S}_{q} \psi\right\|_{C\left(Q_{2}\right)} \leq \frac{c_{3}}{q}\|\psi\|_{\mathcal{C}_{\mathcal{K}}\left(Q_{1}\right)}, \quad q \geq q_{1}
$$

where $c_{3}>0$ is independent of $\psi$ and $q$.
Proof. 1. Consider a number $\sigma>0$ such that

$$
\begin{equation*}
\operatorname{dist}\left(Q_{1}, Q_{2}\right)>3 \sigma, \quad \operatorname{dist}\left(\mathcal{K}, Q_{2}\right)>3 \sigma \tag{24.2}
\end{equation*}
$$

Introduce a function $\xi \in C^{\infty}\left(\mathbb{R}^{2}\right)$ such that $0 \leq \xi(y) \leq 1, \xi(y)=1$ for $\operatorname{dist}\left(y, Q_{2}\right) \leq \sigma$ and $\xi(y)=0$ for $\operatorname{dist}\left(y, Q_{2}\right) \geq 2 \sigma$.

Consider the auxiliary problem

$$
\begin{equation*}
\mathbf{P}(y, D) v-q v=0, \quad y \in G ; \quad v(y)=\xi(y) u(y), \quad y \in \partial G, \tag{24.3}
\end{equation*}
$$

where $u=\mathbf{S}_{q} \psi \in C_{\mathcal{K}}(\bar{G})$. Applying Corollary 23.3 (with $\mathbf{B}_{i}=0$ ), we see that there exists a unique solution $v \in C^{\infty}(G) \cap C_{\mathcal{K}}(\bar{G})$ of Problem (24.3). It follows from the maximum principle 24.1 and definition of the function $\xi$ that

$$
\begin{equation*}
\|v\|_{C(\bar{G})} \leq\|\xi u\|_{C(\partial G)} \leq \max _{i=1, \ldots, N}\left\|\left.u\right|_{Q_{2,2 \sigma} \cap \overline{\Gamma_{i}}}\right\|_{C\left(Q_{2,2 \sigma} \cap \overline{\Gamma_{i}}\right)}, \tag{24.4}
\end{equation*}
$$

where $Q_{2,2 \sigma}=\left\{y \in \partial G: \operatorname{dist}\left(y, Q_{2}\right) \leq 2 \sigma\right\}$.
Since $\operatorname{supp} \psi \cap Q_{2,2 \sigma}=\varnothing$, we see that

$$
\begin{equation*}
u-\mathbf{B}_{i} u=0, \quad y \in Q_{2,2 \sigma} \cap \overline{\Gamma_{i}} . \tag{24.5}
\end{equation*}
$$

Since $\mathbf{B}_{i} u=0$ for $y \notin \mathcal{O}_{\varepsilon}(\mathcal{K})$, it follows from Eq. (24.5) that

$$
\begin{equation*}
u(y)=0, \quad y \in\left[Q_{2,2 \sigma} \cap \overline{\Gamma_{i}}\right] \backslash \mathcal{O}_{\varepsilon}(\mathcal{K}) . \tag{24.6}
\end{equation*}
$$

Using (24.4)-(24.6), the definition of the operators $\mathbf{B}_{i}$, and condition 23.2, we obtain the inequality

$$
\begin{align*}
&\|v\|_{C(\bar{G})} \leq \max _{i=1, \ldots, N}\left\|\left.u\right|_{Q_{2,2 \sigma} \cap \overline{\Gamma_{i}} \cap \overline{\mathcal{O}_{\varepsilon}(\mathcal{K})}}\right\|_{C\left(Q_{2,2 \sigma} \cap \overline{\Gamma_{i}} \cap \overline{\mathcal{O}_{\varepsilon}(\mathcal{K})}\right)} \\
& \leq \max _{i=1, \ldots, N} \max _{s=1, \ldots, S_{i}}\left\|\left.u\right|_{\Omega_{i s}\left(Q_{2,2 \sigma} \cap \overline{\Gamma_{i}} \cap \overline{\mathcal{O}_{\varepsilon}(\mathcal{K})}\right)}\right\|_{C\left(\Omega_{i s}\left(Q_{2,2 \sigma} \cap \overline{\Gamma_{i}} \cap \overline{\left.\mathcal{O}_{\varepsilon}(\mathcal{K})\right)}\right)\right.} . \tag{24.7}
\end{align*}
$$

Since $Q_{2,2 \sigma} \cap \mathcal{K}=\varnothing$ (see Eq. (24.2)), it follows from the definition of the transformations $\Omega_{\text {is }}$ that

$$
\left.\Omega_{i s}\left(Q_{2,2 \sigma} \cap \overline{\Gamma_{i}} \cap \overline{\mathcal{O}_{\varepsilon}(\mathcal{K})}\right)\right) \subset G .
$$

Hence, using inequalities (24.7) and Lemma 24.1, where the sets $\partial G$ and $\left.\Omega_{i s}\left(Q_{2,2 \sigma} \cap \overline{\Gamma_{i}} \cap \overline{\mathcal{O}_{\varepsilon}(\mathcal{K})}\right)\right)$ are taken instead of $Q_{1}$ and $Q_{2}$, we have

$$
\begin{equation*}
\|v\|_{C(\bar{G})} \leq \frac{c_{2}}{q}\|\psi\|_{\mathcal{C}_{\mathcal{K}}(\partial G)} \tag{24.8}
\end{equation*}
$$

[^13]2. Let $w=u-v$. Obviously, the function $w$ satisfies the relations
$$
\mathbf{P}(y, D) w-q w=0, \quad y \in G ; \quad w(y)=u(y)-v(y)=0, \quad y \in Q_{2, \sigma} .
$$

Using Lemma 24.1 (with $\mathbf{B}_{i}=0$ ), where the set $\overline{\partial G \backslash Q_{2, \sigma}}$ is taken instead of $Q_{1}$, and taking into account the fact that $\left.w\right|_{\partial G}=\left.(1-\xi) u\right|_{\partial G}$, we obtain the inequality

$$
\|w\|_{C\left(Q_{2}\right)} \leq \frac{c_{2}}{q}\left\|\left.w\right|_{\partial G}\right\|_{C(\partial G)} \leq \frac{c_{2}}{q}\|u\|_{C(\bar{G})} .
$$

It follows from the last inequality and Corollary 23.3 that

$$
\|w\|_{C\left(Q_{2}\right)} \leq \frac{c_{2} c_{1}}{q}\|\psi\|_{\mathcal{C}_{\mathcal{K}}(\partial G)}
$$

Combining this estimate with Eq. (24.8), we complete the proof.
24.2. Bounded perturbations of elliptic operators and their properties. Consider a linear operator $\mathbf{P}_{1}$ satisfying the following condition.
Condition 24.1. An operator $\mathbf{P}_{1}: C(\bar{G}) \rightarrow C(\bar{G})$ is bounded and, if a function $u \in C(\bar{G})$ has a positive maximum at a point $y^{0} \in G$, then $\mathbf{P}_{1} u\left(y^{0}\right) \leq 0$.

Here, the operator $\mathbf{P}_{1}$ is a bounded perturbation of an unbounded elliptic operator in spaces of continuous functions (cf. [19, 20]).

The next result follows from conditions 23.1 and 24.1 and the maximum principle 24.1.
Lemma 24.3. Let conditions 23.1 and 24.1 hold. If a function $u \in C(\bar{G})$ has a positive maximum at a point $y^{0} \in G$ and $\mathbf{P}(y, D) u \in C(G)$, then $\mathbf{P}(y, D) u\left(y^{0}\right)+\mathbf{P}_{1} u\left(y^{0}\right) \leq 0$.

In this paper, we consider the following nonlocal conditions in the nontransversal case:

$$
\begin{equation*}
b(y) u(y)+\int_{\bar{G}}[u(y)-u(\eta)] \mu(y, d \eta)=0, \quad y \in \partial G \tag{24.9}
\end{equation*}
$$

where $b(y) \geq 0$ and $\mu(y, \cdot)$ is a nonnegative Borel measure at $\bar{G}$.
Let $\mathcal{N}=\{y \in \partial G: \mu(y, \bar{G})=0\}$ and $\mathcal{M}=\partial G \backslash \mathcal{N}$. Assume that $\mathcal{N}$ and $\mathcal{M}$ are Borel sets.
Condition 24.2. $\mathcal{K} \subset \mathcal{N}$.
Introduce the function $b_{0}(y)=b(y)+\mu(y, \bar{G})$.
Condition 24.3. $b_{0}(y)>0$ for $y \in \partial G$.
By 24.2 and 24.3, we can write condition (24.9) in the following form:

$$
\begin{equation*}
u(y)-\int_{\bar{G}} u(\eta) \mu_{i}(y, d \eta)=0, \quad y \in \Gamma_{i} ; \quad u(y)=0, \quad y \in \mathcal{K}, \tag{24.10}
\end{equation*}
$$

where $\mu_{i}(y, \cdot)=\frac{\mu(y, \cdot)}{b_{0}(y)}, y \in \Gamma_{i}$. By the definition of the function $b_{0}(y)$, we have

$$
\begin{equation*}
\mu_{i}(y, \bar{G}) \leq 1, \quad y \in \Gamma_{i} \tag{24.11}
\end{equation*}
$$

For any set $Q$, denote by $\chi_{Q}(y)$ a function that equals 1 in $Q$ and vanishes in $\mathbb{R}^{2} \backslash Q$.
Let $b_{i s}(y)$ and $\Omega_{i s}$ be the same as above. Introduce the measures $\delta_{i s}$ as follows:

$$
\delta_{i s}(y, Q)= \begin{cases}b_{i s}(y) \chi_{Q}\left(\Omega_{i s}(y)\right), & y \in \Gamma_{i} \cap \mathcal{O}_{\varepsilon}(\mathcal{K}) \\ 0, & y \in \Gamma_{i} \backslash \mathcal{O}_{\varepsilon}(\mathcal{K})\end{cases}
$$

where $Q$ is an arbitrary Borel set.

We study the measures $\mu_{i}(y, \cdot)$ represented in the form

$$
\begin{equation*}
\mu_{i}(y, \cdot)=\sum_{s=1}^{S_{i}} \delta_{i s}(y, \cdot)+\alpha_{i}(y, \cdot)+\beta_{i}(y, \cdot), \quad y \in \Gamma_{i} \tag{24.12}
\end{equation*}
$$

where $\alpha_{i}(y, \cdot)$ and $\beta_{i}(y, \cdot)$ are nonnegative Borel measures satisfying the following conditions.
For any Borel measure $\mu(y, \cdot)$, a closed set

$$
\operatorname{spt} \mu(y, \cdot)=\bar{G} \backslash \bigcup_{V \in T}\{V \in T: \mu(y, V \cap \bar{G})=0\}
$$

where $T$ is the set of all open sets in $\mathbb{R}^{2}$, is called the support of the measure $\mu(y, \cdot)$.
Condition 24.4. There exist numbers $\varkappa_{1}>\varkappa_{2}>0$ and $\sigma>0$ such that
(1) $\operatorname{spt} \alpha_{i}(y, \cdot) \subset \bar{G} \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K})$ for $y \in \Gamma_{i}$,
(2) $\operatorname{spt} \alpha_{i}(y, \cdot) \subset \overline{G_{\sigma}}$ for $y \in \Gamma_{i} \backslash \mathcal{O}_{\varkappa_{2}}(\mathcal{K})$,
where $\mathcal{O}_{\varkappa_{1}}(\mathcal{K})=\left\{y \in \mathbb{R}^{2}: \operatorname{dist}(y, \mathcal{K})<\varkappa_{1}\right\}$ and $G_{\sigma}=\{y \in G: \operatorname{dist}(y, \partial G)<\sigma\}$.
Condition 24.5. $\beta_{i}(y, \mathcal{M})<1$ for $y \in \Gamma_{i} \cap \mathcal{M}, i=1, \ldots, N$.
Remark 24.1. Condition 24.5 is weaker than similar conditions 2.2 in [19] and 3.2 in [20]. It is needed in these conditions that the inequality $\mu_{i}(y, \mathcal{M})<1$ hold for $y \in \Gamma_{i} \cap \mathcal{M}$.

Remark 24.2. One can show that if conditions 24.3-24.5 are valid, then

$$
b(y)+\mu(y, \bar{G} \backslash\{y\})>0, \quad y \in \partial G
$$

i.e., boundary condition (24.9) is given at every point of the boundary.

Using relations (24.12), we write nonlocal conditions (24.10) in the form

$$
\begin{equation*}
u(y)-\mathbf{B}_{i} u(y)-\mathbf{B}_{\alpha i} u(y)-\mathbf{B}_{\beta i} u(y)=0, \quad y \in \Gamma_{i} ; \quad u(y)=0, \quad y \in \mathcal{K}, \tag{24.13}
\end{equation*}
$$

where the operators $\mathbf{B}_{i}$ are defined in Sec. 23.2,

$$
\mathbf{B}_{\alpha i} u(y)=\int_{\bar{G}} u(\eta) \alpha_{i}(y, d \eta), \quad \mathbf{B}_{\beta i} u(y)=\int_{\bar{G}} u(\eta) \beta_{i}(y, d \eta), \quad y \in \Gamma_{i} .
$$

Let us introduce the space

$$
C_{B}(\bar{G})=\{u \in C(\bar{G}): u \text { satisfies }(24.9)\} .
$$

Obviously, we can use conditions (24.10) or (24.13) in the definition of the space $C_{B}(\bar{G})$. It follows from the definition of the space $C_{B}(\bar{G})$ and condition 24.2 that

$$
\begin{equation*}
C_{B}(\bar{G}) \subset C_{\mathcal{N}}(\bar{G}) \subset C_{\mathcal{K}}(G) \tag{24.14}
\end{equation*}
$$

Lemma 24.4. Let conditions 23.1, 23.2, and 24.1-24.5 hold. Let the function $u \in C_{B}(\bar{G})$ have a positive maximum at a point $y^{0} \in \bar{G}$ and let $\mathbf{P}(y, D) u \in C(G)$. Then there is a point $y^{1} \in G$ such that $u\left(y^{1}\right)=u\left(y^{0}\right)$ and $\mathbf{P}(y, D) u\left(y^{1}\right)+\mathbf{P}_{1} u\left(y^{1}\right) \leq 0$.
Proof. 1. If $y^{0} \in G$, then the lemma follows from Lemma 24.3. Let $y^{0} \in \partial G$. Assume that the lemma does not hold, i.e., $u\left(y^{0}\right)>u(y)$ for all $y \in G$.

Since $u\left(y^{0}\right)>0$ and $u \in C_{B}(\bar{G}) \subset C_{\mathcal{N}}(\bar{G})$, we have $y^{0} \in \mathcal{M}$. Let $y^{0} \in \Gamma_{i} \cap \mathcal{M}$ for some $i$. If $\mu_{i}\left(y^{0}, G\right)>0$, then, taking into account Eq. (24.11), we obtain

$$
u\left(y^{0}\right)-\int_{\bar{G}} u(\eta) \mu_{i}\left(y^{0}, d \eta\right) \geq \int_{G}\left[u\left(y^{0}\right)-u(\eta)\right] \mu_{i}\left(y^{0}, d \eta\right)>0 .
$$

This contradicts Eq. (24.10). Hence, $\operatorname{spt} \mu_{i}\left(y^{0}, \cdot\right) \subset \partial G$. It follows from here, Eq. (24.12), and condition 24.4 (item 1) that

$$
\begin{equation*}
b_{i s}\left(y^{0}\right)=0, \quad \operatorname{spt} \alpha_{i}\left(y^{0}, \cdot\right) \subset \partial G \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K}), \quad \operatorname{spt} \beta_{i}\left(y^{0}, \cdot\right) \subset \partial G . \tag{24.15}
\end{equation*}
$$

2. Assume that $\alpha_{i}\left(y^{0}, \partial G \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K})\right)=0$. In this case, by Eq. (24.15), we have

$$
\begin{equation*}
\alpha_{i}\left(y^{0}, \bar{G}\right)=0 \tag{24.16}
\end{equation*}
$$

Further, from Eqs. (24.12), (24.15), and (24.16) and condition 24.5 we obtain

$$
\mu_{i}\left(y^{0}, \cdot\right)=\beta_{i}\left(y^{0}, \cdot\right), \quad \operatorname{spt} \beta_{i}\left(y^{0}, \cdot\right) \subset \partial G, \quad \beta_{i}\left(y^{0}, \mathcal{M}\right)<1
$$

Hence the following relations hold for $u \in C_{B}(\bar{G}) \subset C_{\mathcal{N}}(\bar{G})$ :

$$
u\left(y^{0}\right)-\int_{\bar{G}} u(\eta) \mu_{i}\left(y^{0}, d \eta\right)=u\left(y^{0}\right)-\int_{\mathcal{M}} u(\eta) \beta_{i}\left(y^{0}, d \eta\right) \geq u\left(y^{0}\right)-u\left(y^{0}\right) \beta_{i}\left(y^{0}, \mathcal{M}\right)>0
$$

This contradicts Eq. (24.10).
The contradiction means that $\alpha_{i}\left(y^{0}, \partial G \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K})\right)>0$. Thus, taking into account condition 24.4 (item 2), we have $y^{0} \in \mathcal{O}_{\varkappa_{2}}(\mathcal{K})$.
3. Let us show that there is a point

$$
\begin{equation*}
y^{\prime} \in \partial G \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K}) \tag{24.17}
\end{equation*}
$$

such that $u\left(y^{\prime}\right)=u\left(y^{0}\right)$. Assume the contrary: $u\left(y^{0}\right)>u(y)$ for $y \in \partial G \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K})$. Then, using Eqs. (24.11), (24.12), and (24.15), we obtain the inequalities

$$
\begin{equation*}
u\left(y^{0}\right)-\int_{\bar{G}} u(\eta) \mu_{i}\left(y^{0}, d \eta\right) \geq \int_{\bar{G}}\left[u\left(y^{0}\right)-u(\eta)\right] \mu_{i}\left(y^{0}, d \eta\right) \geq \int_{\partial G \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\left[u\left(y^{0}\right)-u(\eta)\right] \alpha_{i}\left(y^{0}, d \eta\right)>0 \tag{24.18}
\end{equation*}
$$

since $\alpha_{i}\left(y^{0}, \partial G \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K})\right)>0$. Inequality (24.18) contradicts Eq. (24.10). Hence, the function $u$ has a positive maximum at some point $y^{\prime} \in \partial G \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K})$. Repeating reasonings of items 1 and 2 of the proof, we obtain that $y^{\prime} \in \mathcal{O}_{\varkappa_{2}}(\mathcal{K})$. This contradicts to Eq. (24.17).

Thus, we have proved the existence of a point $y^{1} \in G$ such that $u\left(y^{1}\right)=u\left(y^{0}\right)$. Applying Lemma 24.3, we can also prove the inequality $\mathbf{P}(y, D) u\left(y^{1}\right)+\mathbf{P}_{1} u\left(y^{1}\right) \leq 0$.
Corollary 24.1. Let conditions 23.1, 23.2, and 24.1-24.5 hold. Let $u \in C_{B}(\bar{G})$ be a solution of the equation

$$
q u(y)-\mathbf{P}(y, D) u(y)-\mathbf{P}_{1} u(y)=f_{0}(y), \quad y \in G
$$

where $q>0$ and $f_{0} \in C(\bar{G})$. Then

$$
\begin{equation*}
\|u\|_{C(\bar{G})} \leq \frac{1}{q}\left\|f_{0}\right\|_{C(\bar{G})} \tag{24.19}
\end{equation*}
$$

Proof. Let $\max _{y \in \bar{G}}|u(y)|=u\left(y^{0}\right)>0$ for some $y^{0} \in \bar{G}$. By Lemma 24.4, there exists a point $y^{1} \in G$ such that $u\left(y^{1}\right)=u\left(y^{0}\right)$ and $\mathbf{P}(y, D) u\left(y^{1}\right)+\mathbf{P}_{1} u\left(y^{1}\right) \leq 0$. Hence

$$
\|u\|_{C(\bar{G})}=u\left(y^{0}\right)=u\left(y^{1}\right)=\frac{1}{q}\left(\mathbf{P}(y, D) u\left(y^{1}\right)+\mathbf{P}_{1} u\left(y^{1}\right)+f_{0}\left(y^{1}\right)\right) \leq \frac{1}{q}\left\|f_{0}\right\|_{C(\bar{G})} .
$$

If $\max _{y \in \bar{G}}|u(y)|=-u\left(y^{0}\right)>0$, then, applying the above reasonings to the solution $v(y)=-u(y)$ of the equation $q v-\mathbf{P}(y, D) v-\mathbf{P}_{1} v=-f_{0}$, we again obtain (24.19).
24.3. Reducing to an operator equation on the boundary. In this section, we impose additional restrictions on nonlocal operators. These restrictions will allow us to reduce nonlocal elliptic problems to operator equations on the boundary.

Note that if $u \in C_{\mathcal{N}}(\bar{G})$, then the function $\mathbf{B}_{i} u$ is continuous on $\Gamma_{i}$ and can be extended to a continuous function on $\overline{\Gamma_{i}}$ that belongs to $C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)$; we will denote it by $\mathbf{B}_{i} u$. Assume that the operators $\mathbf{B}_{\alpha i}$ and $\mathbf{B}_{\beta i}$ have similar properties.

Condition 24.6. For any $u \in C_{\mathcal{N}}(\bar{G})$, the functions $\mathbf{B}_{\alpha i} u$ and $\mathbf{B}_{\beta i} u$ can be extended to $\overline{\Gamma_{i}}$ such that the resulting functions (we will denote them by $\mathbf{B}_{\alpha i} u$ and $\mathbf{B}_{\beta i} u$, respectively) belong to $C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)$.

The following lemma directly follows from the definition of nonlocal operators.
Lemma 24.5. Let conditions 6.3, 23.2, 24.2, 24.3, and 24.6 hold. Then the operators $\mathbf{B}_{i}$ and $\mathbf{B}_{\alpha i}$, $\mathbf{B}_{\beta i}: C_{\mathcal{N}}(\bar{G}) \rightarrow C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)$ are bounded and

$$
\begin{gathered}
\left\|\mathbf{B}_{i} u\right\|_{C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)} \leq\|u\|_{C_{\mathcal{N}}(\bar{G})}, \quad\left\|\mathbf{B}_{\alpha i} u\right\|_{C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)} \leq\|u\|_{C_{\mathcal{N}}\left(\bar{G} \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K})\right)}, \\
\left\|\mathbf{B}_{\beta i} u\right\|_{C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)} \leq\|u\|_{C_{\mathcal{N}}(\bar{G})}, \quad\left\|\mathbf{B}_{\alpha i} u+\mathbf{B}_{\beta i} u\right\| \leq\|u\|_{C_{\mathcal{N}}(\bar{G})}, \\
\left\|\mathbf{B}_{i} u+\mathbf{B}_{\alpha i} u+\mathbf{B}_{\beta i} u\right\| \leq\|u\|_{C_{\mathcal{N}}(\bar{G})} .
\end{gathered}
$$

Introduce the operators

$$
\begin{equation*}
\mathbf{B}=\left\{\mathbf{B}_{i}\right\}: C_{\mathcal{N}}(\bar{G}) \rightarrow \mathcal{C}_{\mathcal{N}}(\partial G), \quad \mathbf{B}_{\alpha \beta}=\left\{\mathbf{B}_{\alpha i}+\mathbf{B}_{\beta i}\right\}: C_{\mathcal{N}}(\bar{G}) \rightarrow \mathcal{C}_{\mathcal{N}}(\partial G), \tag{24.20}
\end{equation*}
$$

where $\mathcal{C}_{\mathcal{N}}(\partial G)$ is defined in Eq. (5.1).
Using the operator $\mathbf{S}_{q}$ defined in Sec. 24.1, we introduce the bounded operator

$$
\begin{equation*}
\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}: \mathcal{C}_{\mathcal{N}}(\partial G) \rightarrow \mathcal{C}_{\mathcal{N}}(\partial G), \quad q \geq q_{1} . \tag{24.21}
\end{equation*}
$$

Since $\mathbf{S}_{q} \psi \in C_{\mathcal{N}}(\bar{G})$ for $\psi \in \mathcal{C}_{\mathcal{N}}(\partial G)$, we see that the operator in (24.21) is well defined.
Further, we will formulate sufficient conditions that guarantee the existence of a bounded operator $\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)^{-1}: \mathcal{C}_{\mathcal{N}}(\partial G) \rightarrow \mathcal{C}_{\mathcal{N}}(\partial G)$.

Let us represent the measures $\beta_{i}(y, \cdot)$ in the form

$$
\begin{equation*}
\beta_{i}(y, \cdot)=\beta_{i}^{1}(y, \cdot)+\beta_{i}^{2}(y, \cdot), \tag{24.22}
\end{equation*}
$$

where $\beta_{i}^{1}(y, \cdot)$ and $\beta_{i}^{2}(y, \cdot)$ are nonnegative Borel measures. Describe them. For any $p>0$, we consider a covering of the set $\overline{\mathcal{M}}$ by $p$-neighborhoods of all its points. We denote by $\mathcal{M}_{p}$ some finite covering. Obviously, $\mathcal{M}_{p}$ is an open Borel set. Further, for any $p>0$, we consider a patch function $\hat{\zeta}_{p} \in C^{\infty}\left(\mathbb{R}^{2}\right)$ such that $0 \leq \hat{\zeta}_{p}(y) \leq 1, \hat{\zeta}_{p}(y)=1$ for $y \in \mathcal{M}_{p / 2}$, and $\hat{\zeta}_{p}(y)=0$ for $y \notin \mathcal{M}_{p}$. Let $\tilde{\zeta}_{p}=1-\hat{\zeta}_{p}$. Introduce the operators

$$
\hat{\mathbf{B}}_{\beta i}^{1} u(y)=\int_{\bar{G}} \hat{\zeta}_{p}(\eta) u(\eta) \beta_{i}^{1}(y, d \eta), \quad \tilde{\mathbf{B}}_{\beta i}^{1} u(y)=\int_{\bar{G}} \tilde{\zeta}_{p}(\eta) u(\eta) \beta_{i}^{1}(y, d \eta), \quad \mathbf{B}_{\beta i}^{2} u(y)=\int_{\bar{G}} u(\eta) \beta_{i}^{2}(y, d \eta) .
$$

Condition 24.7. For all $i=1, \ldots, N$ we have:
(1) the operators $\hat{\mathbf{B}}_{\beta i}^{1}, \tilde{\mathbf{B}}_{\beta i}^{1}: C_{\mathcal{N}}(\bar{G}) \rightarrow C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)$ are bounded;
(2) there exist a number $p>0$ such that ${ }^{15}$

$$
\left\|\hat{\mathbf{B}}_{\beta i}^{1}\right\|< \begin{cases}\frac{1}{c_{1}}, & \text { if } \alpha_{j}(y, \bar{G})=0 \text { for all } y \in \Gamma_{j}, j=1, \ldots, N, \\ \frac{1}{c_{1}\left(1+c_{1}\right)} & \text { otherwise },\end{cases}
$$

where $c_{1}$ is a constant from Corollary 23.3.

[^14]Remark 24.3. The operators $\hat{\mathbf{B}}_{\beta i}^{1}, \tilde{\mathbf{B}}_{\beta i}^{1}: C_{\mathcal{N}}(\bar{G}) \rightarrow C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)$ are bounded if and only if the operator $\hat{\mathbf{B}}_{\beta i}^{1}+\tilde{\mathbf{B}}_{\beta i}^{1}: C_{\mathcal{N}}(\bar{G}) \rightarrow C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)$ is bounded. This follows from the relations

$$
\hat{\mathbf{B}}_{\beta i}^{1} u=\left(\hat{\mathbf{B}}_{\beta i}^{1}+\tilde{\mathbf{B}}_{\beta i}^{1}\right)\left(\hat{\zeta}_{p} u\right), \quad \tilde{\mathbf{B}}_{\beta i}^{1} u=\left(\hat{\mathbf{B}}_{\beta i}^{1}+\tilde{\mathbf{B}}_{\beta i}^{1}\right)\left(\tilde{\zeta}_{p} u\right)
$$

and the continuity of the functions $\hat{\zeta}_{p}$ and $\tilde{\zeta}_{p}$.
Condition 24.8. The perators $\mathbf{B}_{\beta i}^{2}: C_{\mathcal{N}}(\bar{G}) \rightarrow C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right), i=1, \ldots, N$, are compact.
It follows from Eqs. (24.12) and (24.22) that the measures $\mu_{i}(y, \cdot)$ can be represented in the form

$$
\mu_{i}(y, \cdot)=\sum_{s=1}^{S_{i}} \delta_{i s}(y, \cdot)+\alpha_{i}(y, \cdot)+\beta_{i}^{1}(y, \cdot)+\beta_{i}^{2}(y, \cdot), \quad y \in \Gamma_{i} .
$$

The measures $\delta_{i s}(y, \cdot)$ correspond to nonlocal terms supported near the set $\mathcal{K}$ of conjugation points. The measures $\alpha_{i}(y, \cdot)$ correspond to nonlocal terms supported outside the set $\mathcal{K}$. The measures $\beta_{i}^{1}(y, \cdot)$ and $\beta_{i}^{2}(y, \cdot)$ correspond to nonlocal terms whose supports have arbitrary geometric structure (in particular, it can intersect with the set $\mathcal{K}$ ); however, the measure $\beta_{i}^{1}\left(y, \mathcal{M}_{p}\right)$ of the set $\mathcal{M}_{p}$ must be small for small $p$ (condition 24.7), and the measure $\beta_{i}^{2}(y, \cdot)$ must generate a compact operator (condition 24.8).

Lemma 24.6. Let conditions 23.1, 23.2, 24.1-24.5, and 24.6-24.8 hold. Then there exists a bounded operator $\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)^{-1}: \mathcal{C}_{\mathcal{N}}(\partial G) \rightarrow \mathcal{C}_{\mathcal{N}}(\partial G), q \geq q_{1}$, where $q_{1}>0$ is sufficiently small.
Proof. 1. Let us consider the bounded operators $\hat{\mathbf{B}}_{\beta}^{1}=\left\{\hat{\mathbf{B}}_{\beta i}^{1}\right\}, \tilde{\mathbf{B}}_{\beta}^{1}=\left\{\tilde{\mathbf{B}}_{\beta i}^{1}\right\}, \mathbf{B}_{\beta}^{2}=\left\{\mathbf{B}_{\beta i}^{2}\right\}$, and $\mathbf{B}_{\alpha}=\left\{\mathbf{B}_{\alpha i}\right\}$ acting from $C_{\mathcal{N}}(\bar{G})$ to $\mathcal{C}_{\mathcal{N}}(\partial G)$ (cf. (24.20)).

Prove that the operator $\mathbf{I}-\mathbf{B}_{\alpha} \mathbf{S}_{q}: \mathcal{C}_{\mathcal{N}}(\partial G) \rightarrow \mathcal{C}_{\mathcal{N}}(\partial G)$ has a bounded inverse operator. Introduce a function $\zeta \in C^{\infty}(\bar{G})$ such that $0 \leq \zeta(y) \leq 1, \zeta(y)=1$ for $y \in \overline{G_{\sigma}}$, and $\zeta(y)=0$ for $y \notin G_{\sigma / 2}$, where $\sigma>0$ is a number from condition 24.4.

We have

$$
\begin{equation*}
\mathbf{I}-\mathbf{B}_{\alpha} \mathbf{S}_{q}=\mathbf{I}-\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}-\mathbf{B}_{\alpha} \zeta \mathbf{S}_{q} . \tag{24.23}
\end{equation*}
$$

1a. First, we prove that the operator $\mathbf{I}-\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}$ has a bounded inverse operator. By Lemma 24.5 and Corollary 23.3, we have

$$
\begin{equation*}
\left\|\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right\| \leq c_{1} . \tag{24.24}
\end{equation*}
$$

Further, $(1-\zeta) \mathbf{S}_{q} \psi=0$ in $\overline{G_{\sigma}}$ for any $\psi \in \mathcal{C}_{\mathcal{N}}(\partial G)$. Hence, by condition 24.4, we see that

$$
\begin{equation*}
\operatorname{supp} \mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q} \psi \subset \partial G \cap \overline{\mathcal{O}_{\varkappa_{2}}(\mathcal{K})} \tag{24.25}
\end{equation*}
$$

Let us show that

$$
\begin{equation*}
\left\|\left[\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right]^{2}\right\| \leq \frac{c}{q}, \quad q \geq q_{1} \tag{24.26}
\end{equation*}
$$

where $q_{1}>0$ is sufficiently large and $c>0$ is independent of $q$. Applying sequentially Lemma 24.5, Lemma 24.2, relation (24.25), Lemma 24.5, and Corollary 23.3, we obtain the inequality

$$
\begin{aligned}
&\left\|\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q} \mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q} \psi\right\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \leq\left\|\mathbf{S}_{q} \mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q} \psi\right\|_{C_{\mathcal{N}}\left(\bar{G} \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K})\right)} \\
& \leq \frac{c_{3}}{q}\left\|\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q} \psi\right\|_{C_{\mathcal{N}}\left(\partial G \cap \overline{\left.\mathcal{O}_{\varkappa_{2}}(\mathcal{K})\right)}\right.} \leq \frac{c_{3} c_{1}}{q}\|\psi\|_{\mathcal{C}_{\mathcal{N}}(\partial G)}
\end{aligned}
$$

Equation (24.26) with $c=c_{3} c_{1}$ follows from here.
If $q \geq 2 c$, then the operator $\mathbf{I}-\left[\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right]^{2}$ has a bounded inverse operator. Then the operator $\mathbf{I}-\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}$ also has a bounded inverse operator and

$$
\begin{equation*}
\left[\mathbf{I}-\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right]^{-1}=\left[\mathbf{I}+\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right]\left[\mathbf{I}-\left(\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right)^{2}\right]^{-1} . \tag{24.27}
\end{equation*}
$$

It follows from (24.27), Lemma 24.5, Corollary 23.3, and relations (24.24) and (24.26) that

$$
\begin{equation*}
\left\|\left[\mathbf{I}-\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right]^{-1}\right\|=1+c_{1}+O\left(q^{-1}\right), \quad q \rightarrow+\infty \tag{24.28}
\end{equation*}
$$

1b. Now let us estimate the norm of the operator $\mathbf{B}_{\alpha} \zeta \mathbf{S}_{q}$. Lemmas 24.5 and 24.2 yield

$$
\begin{equation*}
\left\|\mathbf{B}_{\alpha} \zeta \mathbf{S}_{q} \psi\right\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \leq\left\|\mathbf{S}_{q} \psi\right\|_{C\left(\overline{G_{\sigma / 2}}\right)} \leq \frac{c_{2}}{q}\|\psi\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \tag{24.29}
\end{equation*}
$$

Hence, using representation (24.23), we see that the operator $\mathbf{I}-\mathbf{B}_{\alpha} \mathbf{S}_{q}$ has a bounded inverse for all sufficiently large $q$ and

$$
\begin{equation*}
\left(\mathbf{I}-\mathbf{B}_{\alpha} \mathbf{S}_{q}\right)^{-1}=\left[\mathbf{I}-\left(\mathbf{I}-\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right)^{-1} \mathbf{B}_{\alpha} \zeta \mathbf{S}_{q}\right]^{-1}\left[\mathbf{I}-\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right]^{-1} \tag{24.30}
\end{equation*}
$$

It follows from (24.28)-(24.30) that

$$
\begin{equation*}
\left\|\left(\mathbf{I}-\mathbf{B}_{\alpha} \mathbf{S}_{q}\right)^{-1}\right\|=1+c_{1}+O\left(q^{-1}\right), \quad q \rightarrow+\infty \tag{24.31}
\end{equation*}
$$

2. Let us prove that the operator $\mathbf{I}-\left(\mathbf{B}_{\alpha}+\hat{\mathbf{B}}_{\beta}^{1}+\tilde{\mathbf{B}}_{\beta}^{1}\right) \mathbf{S}_{q}: \mathcal{C}_{\mathcal{N}}(\partial G) \rightarrow \mathcal{C}_{\mathcal{N}}(\partial G)$ has a bounded inverse operator.
2a. It follows from the definition of the operator $\tilde{\mathbf{B}}_{\beta}^{1}$ and Lemma 24.1 (with $Q_{1}=\overline{\mathcal{M}}$ and $Q_{2}=$ $\bar{G} \backslash \mathcal{M}_{p / 2}$ ) that

$$
\begin{equation*}
\left\|\tilde{\mathbf{B}}_{\beta i}^{1} \mathbf{S}_{q} \psi\right\|_{C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)} \leq\left\|\mathbf{S}_{q} \psi\right\|_{C\left(\bar{G} \backslash \mathcal{M}_{p / 2}\right)} \leq \frac{c_{2}}{q}\|\psi\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \tag{24.32}
\end{equation*}
$$

because $\left(\bar{G} \backslash \mathcal{M}_{p / 2}\right) \cap \overline{\mathcal{M}}=\varnothing$ and $\left.\operatorname{supp}\left(\mathbf{S}_{q} \psi\right)\right|_{\partial G} \subset \overline{\mathcal{M}}$ for $\psi \in \mathcal{C}_{\mathcal{N}}(\partial G)$.
2b. Let $\alpha_{j}(y, \bar{G}) \neq 0$ for some $j$ and $y \in \Gamma_{j}$. By virtue of condition 24.7 (item 2) and Corollary 23.3 there is a number $d$ such that $0<2 d<1 /\left(1+c_{1}\right)$ and

$$
\begin{equation*}
\left\|\hat{\mathbf{B}}_{\beta i}^{1} \mathbf{S}_{q} \psi\right\|_{C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)} \leq\left(\frac{1}{c_{1}\left(1+c_{1}\right)}-\frac{2 d}{c_{1}}\right)\left\|\mathbf{S}_{q} \psi\right\|_{C_{\mathcal{N}}(\bar{G})} \leq\left(\frac{1}{1+c_{1}}-2 d\right)\|\psi\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \tag{24.33}
\end{equation*}
$$

Inequalities (24.32) and (24.33) yields the inequality

$$
\begin{equation*}
\left\|\left(\hat{\mathbf{B}}_{\beta}^{1}+\tilde{\mathbf{B}}_{\beta}^{1}\right) \mathbf{S}_{q}\right\| \leq \frac{1}{1+c_{1}}-d \tag{24.34}
\end{equation*}
$$

for all sufficiently large $q$. It follows from (24.31) and (24.34) that

$$
\left\|\left(\mathbf{I}-\mathbf{B}_{\alpha} \mathbf{S}_{q}\right)^{-1}\left(\hat{\mathbf{B}}_{\beta}^{1}+\tilde{\mathbf{B}}_{\beta}^{1}\right) \mathbf{S}_{q}\right\|<1
$$

for sufficiently large $q$. Hence, there exists a bounded inverse operator

$$
\begin{equation*}
\left[\mathbf{I}-\left(\mathbf{B}_{\alpha}+\hat{\mathbf{B}}_{\beta}^{1}+\tilde{\mathbf{B}}_{\beta}^{1}\right) \mathbf{S}_{q}\right]^{-1}=\left[\mathbf{I}-\left(\mathbf{I}-\mathbf{B}_{\alpha} \mathbf{S}_{q}\right)^{-1}\left(\hat{\mathbf{B}}_{\beta}^{1}+\tilde{\mathbf{B}}_{\beta}^{1}\right) \mathbf{S}_{q}\right]^{-1}\left[\mathbf{I}-\mathbf{B}_{\alpha} \mathbf{S}_{q}\right]^{-1} \tag{24.35}
\end{equation*}
$$

2c. If $\alpha_{j}(y, \bar{G})=0$ for $y \in \Gamma_{j}, j=1, \ldots, N$, then, by condition 24.7 (item 1 ), inequality (24.33) has the form

$$
\left\|\hat{\mathbf{B}}_{\beta i}^{1} \mathbf{S}_{q} \psi\right\|_{C_{\mathcal{N}}\left(\overline{\Gamma_{i}}\right)} \leq\left(\frac{1}{c_{1}}-\frac{2 d}{c_{1}}\right)\left\|\mathbf{S}_{q} \psi\right\|_{C_{\mathcal{N}}(\bar{G})} \leq(1-2 d)\|\psi\|_{\mathcal{C}_{\mathcal{N}}(\partial G)}
$$

Hence, inequality (24.34) has the form

$$
\begin{equation*}
\left\|\left(\hat{\mathbf{B}}_{\beta}^{1}+\tilde{\mathbf{B}}_{\beta}^{1}\right) \mathbf{S}_{q}\right\| \leq 1-d . \tag{24.36}
\end{equation*}
$$

Since $\mathbf{B}_{\alpha}=0$ in this case, we see that Eq. (24.36) implies that the operator

$$
\mathbf{I}-\left(\mathbf{B}_{\alpha}+\hat{\mathbf{B}}_{\beta}^{1}+\tilde{\mathbf{B}}_{\beta}^{1}\right) \mathbf{S}_{q}=\mathbf{I}-\left(\hat{\mathbf{B}}_{\beta}^{1}+\tilde{\mathbf{B}}_{\beta}^{1}\right) \mathbf{S}_{q}
$$

has a bounded inverse operator.
3. It remains to prove that the operator $\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}$ also has a bounded inverse operator. By condition 24.8, the operator $\mathbf{B}_{\beta}^{2}$ is compact. Hence, the operator $\mathbf{B}_{\beta}^{2} \mathbf{S}_{q}$ is also compact. Since the index of a Fredholm operator is steady with respect to compact perturbations, we see that the operator
$\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}$ is a Fredholm operator and $\operatorname{ind}\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)=0$. It suffices to show that $\operatorname{dim} \operatorname{ker}\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)=$ 0 to prove that $\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}$ has a bounded inverse operator.

Let $\psi \in \mathcal{C}_{\mathcal{N}}(\partial G)$ and $\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right) \psi=0$. The the function $u=\mathbf{S}_{q} \psi \in C^{\infty}(G) \cap C_{\mathcal{N}}(\bar{G})$ is a solution of the problem

$$
\begin{array}{ll}
\mathbf{P}(y, D) u-q u=0, & y \in G, \\
u(y)-\mathbf{B}_{i} u(y)-\mathbf{B}_{\alpha i} u(y)-\mathbf{B}_{\beta i} u(y)=0, & y \in \Gamma_{i}, \\
u(y)=0, & y \in \mathcal{K} .
\end{array}
$$

By Corollary 24.1, we have $u=0$. Hence, $\psi=\mathbf{B}_{\alpha \beta} \mathbf{S}_{q} \psi=\mathbf{B}_{\alpha \beta} u=0$.
24.4. Existence of Feller semigroups. Here we prove that bounded perturbations of elliptic operators with nonlocal terms which satisfy the conditions of Secs. 24.1-24.3 generate Feller semigroups.

Reducing nonlocal problems to the boundary and using Lemma 24.6, we prove that nonlocal problems are solvable in spaces of continuous functions.

Lemma 24.7. Let conditions 23.1, 23.2, 24.2-24.5, and 24.6-24.8 hold and let $q_{1}>0$ be sufficiently large. Then for any $q \geq q_{1}$ and $f_{0} \in C(\bar{G})$, the problem

$$
\begin{align*}
q u(y)-\mathbf{P}(y, D) u(y)=f_{0}(y), & y \in G  \tag{24.37}\\
u(y)-\mathbf{B}_{i} u(y)-\mathbf{B}_{\alpha i} u(y)-\mathbf{B}_{\beta i} u(y)=0, \quad y \in \Gamma_{i} ; & u(y)=0, \quad y \in \mathcal{K}, \tag{24.38}
\end{align*}
$$

has a unique solution $u \in C_{B}(\bar{G}) \cap W_{\mathrm{loc}}^{2}(G)$.
Proof. Consider the following auxiliary problem:

$$
\begin{equation*}
q v(y)-\mathbf{P}(y, D) v(y)=f_{0}(y), \quad y \in G ; \quad v(y)-\mathbf{B}_{i} v(y)=0, \quad y \in \Gamma_{i}, \quad i=1, \ldots, N . \tag{24.39}
\end{equation*}
$$

Since $f_{0} \in C(\bar{G})$, we see that by Corollary 23.3 there exists a unique solution $v \in C_{\mathcal{K}}(\bar{G})$ of problem (24.39). Hence, $v \in C_{\mathcal{N}}(\bar{G})$.
2. Let $w=u-v$. The unknown function $w$ belongs to $C_{\mathcal{N}}(\bar{G})$ and, by Eqs. (24.37)-(24.39), it satisfies the relations

$$
\begin{array}{ll}
q w(y)-\mathbf{P}(y, D) w(y)=0, & y \in G, \\
w(y)-\mathbf{B}_{i} w(y)-\mathbf{B}_{\alpha i} w(y)-\mathbf{B}_{\beta i} w(y)=\mathbf{B}_{\alpha i} v(y)+\mathbf{B}_{\beta i} v(y), & y \in \Gamma_{i}, \quad i=1, \ldots, N, \\
w(y)=0, & y \in \mathcal{K} . \tag{24.40}
\end{array}
$$

By condition 24.6, problem (24.40) is equivalent to the operator equation $\psi-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q} \psi=\mathbf{B}_{\alpha \beta} v$ with respect to the unknown function $\psi \in \mathcal{C}_{\mathcal{N}}(\partial G)$. By Lemma 24.6, this equation has a unique solution $\psi \in \mathcal{C}_{\mathcal{N}}(\partial G)$. Then problem (24.37), (24.38) also has a unique solution

$$
u=v+w=v+\mathbf{S}_{q} \psi=v+\mathbf{S}_{q}\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)^{-1} \mathbf{B}_{\alpha \beta} v \in C_{B}(\bar{G}) .
$$

Moreover, by the theorem on the inner smoothness of solutions of elliptic equations, we have $u \in$ $W_{\text {loc }}^{2}(G)$.

Using Lemma 24.7 and condition 24.1, we prove that problems with bounded perturbations are also solvable in spaces of continuous functions.

Lemma 24.8. Let conditions 23.1, 23.2, and 24.1-24.8 hold and let $q_{1}>0$ be sufficiently large. Then for any $q \geq q_{1}$ and $f_{0} \in C(\bar{G})$, the problem

$$
\begin{gather*}
q u-\left(\mathbf{P}(y, D)+\mathbf{P}_{1}\right) u=f_{0}(y), \quad y \in G,  \tag{24.41}\\
u(y)-\mathbf{B}_{i} u(y)-\mathbf{B}_{\alpha i} u(y)-\mathbf{B}_{\beta i} u(y)=0, \quad y \in \Gamma_{i} ; \quad u(y)=0, \quad y \in \mathcal{K}, \tag{24.42}
\end{gather*}
$$

has a unique solution $u \in C_{B}(\bar{G}) \cap W_{\mathrm{loc}}^{2}(G)$.

Proof. Denote the identity operator in the space $C(\bar{G})$ by $I$. Consider the operator $q I-\mathbf{P}(y, D)$ as an operator acting from $C(\bar{G})$ to $C(\bar{G})$ with the domain

$$
\mathrm{D}(q I-\mathbf{P}(y, D))=\left\{u \in C_{B}(\bar{G}) \cap W_{\mathrm{loc}}^{2}(G): \mathbf{P}(y, D) u \in C(\bar{G})\right\}
$$

By Lemma 24.7 and Corollary 24.1, there exists a bounded operator $(q I-\mathbf{P}(y, D))^{-1}: C(\bar{G}) \rightarrow C(\bar{G})$ and

$$
\left\|(q I-\mathbf{P}(y, D))^{-1}\right\| \leq \frac{1}{q}
$$

Introduce the operator $q I-\mathbf{P}(y, D)-\mathbf{P}_{1}: C(\bar{G}) \rightarrow C(\bar{G})$ with the domain

$$
\mathrm{D}\left(q I-\mathbf{P}(y, D)-\mathbf{P}_{1}\right)=\mathrm{D}(q I-\mathbf{P}(y, D)) .
$$

Since

$$
q I-\mathbf{P}(y, D)-\mathbf{P}_{1}=\left(I-\mathbf{P}_{1}(q I-\mathbf{P}(y, D))^{-1}\right)(q I-\mathbf{P}(y, D)),
$$

we see that the operator $q I-\mathbf{P}(y, D)-\mathbf{P}_{1}: C(\bar{G}) \rightarrow C(\bar{G})$ has a bounded inverse for $q \geq q_{1}$, where $q_{1}$ is so large that $\left\|\mathbf{P}_{1}\right\| \cdot\left\|(q I-\mathbf{P}(y, D))^{-1}\right\| \leq 1 / 2, q \geq q_{1}$.

Consider the unbounded operator $\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ defined by the formula

$$
\begin{align*}
\mathbf{P}_{B} u & =\mathbf{P}(y, D) u+\mathbf{P}_{1} u \\
u \in \mathrm{D}\left(\mathbf{P}_{B}\right)=\left\{u \in C_{B}(\bar{G})\right. & \left.\cap W_{\mathrm{loc}}^{2}(G): \mathbf{P}(y, D) u+\mathbf{P}_{1} u \in C_{B}(\bar{G})\right\} . \tag{24.43}
\end{align*}
$$

Lemma 24.9. Let conditions 23.1, 23.2, and 24.1-24.8 hold. Then $\mathrm{D}\left(\mathbf{P}_{B}\right)$ is dense in $C_{B}(\bar{G})$.
Proof. We prove the lemma using the scheme described in [20].

1. Let $u \in C_{B}(\bar{G})$. Since $C_{B}(\bar{G}) \subset C_{\mathcal{N}}(\bar{G})$ (by Eq. (24.14)), we see that there exists a function $u_{1} \in C^{\infty}(\bar{G}) \cap C_{\mathcal{N}}(\bar{G})$ such that for any $\varepsilon>0$ and $q \geq q_{1}$

$$
\begin{equation*}
\left\|u-u_{1}\right\|_{C(\bar{G})} \leq \min \left(\varepsilon, \varepsilon /\left(2 c_{1} k_{q}\right)\right) \tag{24.44}
\end{equation*}
$$

where $k_{q}=\left\|\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)^{-1}\right\|$.
Let

$$
\begin{array}{lll}
f_{0}(y) \equiv q u_{1}-\mathbf{P}(y, D) u_{1}, & y \in G, \\
\psi_{i}(y) \equiv u_{1}(y)-\mathbf{B}_{i} u_{1}(y)-\mathbf{B}_{\alpha i} u_{1}(y)-\mathbf{B}_{\beta i} u_{1}(y), & y \in \Gamma_{i}, \quad i=1, \ldots, N . \tag{24.45}
\end{array}
$$

Since $u_{1} \in C_{\mathcal{N}}(\bar{G})$, by condition 24.6 we have $\left\{\psi_{i}\right\} \in \mathcal{C}_{\mathcal{N}}(\partial G)$. Using the relation

$$
u(y)-\mathbf{B}_{i} u(y)-\mathbf{B}_{\alpha i} u(y)-\mathbf{B}_{\beta i} u(y)=0, \quad y \in \Gamma_{i}
$$

inequality (24.44), and Lemma 24.5, we obtain the inequality

$$
\begin{equation*}
\left\|\left\{\psi_{i}\right\}\right\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \leq\left\|u-u_{1}\right\|_{C(\bar{G})}+\left\|\left(\mathbf{B}+\mathbf{B}_{\alpha \beta}\right)\left(u-u_{1}\right)\right\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \leq \frac{\varepsilon}{c_{1} k_{q}} . \tag{24.46}
\end{equation*}
$$

Consider the following auxiliary nonlocal problem:

$$
\begin{gather*}
q u_{2}-\mathbf{P}(y, D) u_{2}=f_{0}(y), \quad y \in G \\
u_{2}(y)-\mathbf{B}_{i} u_{2}(y)-\mathbf{B}_{\alpha i} u_{2}(y)-\mathbf{B}_{\beta i} u_{2}(y)=0, \quad y \in \Gamma_{i} ; \quad u_{2}(y)=0, \quad y \in \mathcal{K} . \tag{24.47}
\end{gather*}
$$

Since $f_{0} \in C^{\infty}(\bar{G})$, by Lemma 24.7 we see that problem (24.47) has a unique solution $u_{2} \in C_{B}(\bar{G}) \subset C_{\mathcal{N}}(\bar{G})$.

Using Eqs. (24.45), (24.47), and the relations $u_{1}(y)=u_{2}(y)=0, y \in \mathcal{K}$, we see that the function $w_{1}=u_{1}-u_{2}$ satisfies the relations

$$
\begin{gather*}
q w_{1}-\mathbf{P}(y, D) w_{1}=0, \quad y \in G \\
w_{1}(y)-\mathbf{B}_{i} w_{1}(y)-\mathbf{B}_{\alpha i} w_{1}(y)-\mathbf{B}_{\beta i} w_{1}(y)=\psi_{i}(y), \quad y \in \Gamma_{i} ; \quad w_{1}(y)=0, \quad y \in \mathcal{K} . \tag{24.48}
\end{gather*}
$$

It follows from condition 24.6 that problem (24.48) is equivalent to the operator equation $\varphi-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q} \varphi=\psi$ in $\mathcal{C}_{\mathcal{N}}(\partial G)$, where $w_{1}=\mathbf{S}_{q} \varphi$. By Lemma 24.6, this equation has a unique solution $\varphi \in \mathcal{C}_{\mathcal{N}}(\partial G)$. Thus, using Corollary 23.3 and inequality (24.46), we obtain

$$
\begin{equation*}
\left\|w_{1}\right\|_{C(\bar{G})} \leq c_{1}\left\|\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)^{-1}\right\| \cdot\left\|\left\{\psi_{i}\right\}\right\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \leq c_{1} k_{q} \varepsilon /\left(c_{1} k_{q}\right)=\varepsilon \tag{24.49}
\end{equation*}
$$

2. Finally, we consider the problem

$$
\begin{gather*}
\lambda u_{3}-\mathbf{P}(y, D) u_{3}-\mathbf{P}_{1} u_{3}=\lambda u_{2}, \quad y \in G \\
u_{3}(y)-\mathbf{B}_{i} u_{3}(y)-\mathbf{B}_{\alpha i} u_{3}(y)-\mathbf{B}_{\beta i} u_{3}(y)=0, \quad y \in \Gamma_{i} ; \quad u_{3}(y)=0, \quad y \in \mathcal{K} . \tag{24.50}
\end{gather*}
$$

Since $u_{2} \in C_{B}(\bar{G})$, we see (by Lemma 24.8) that problem (24.50) has a unique solution $u_{3} \in \mathrm{D}\left(\mathbf{P}_{B}\right)$ for all sufficiently large $\lambda$.

Denote $w_{2}=u_{2}-u_{3}$. It follows from Eq. (24.50) that

$$
\lambda w_{2}-\mathbf{P}(y, D) w_{2}-\mathbf{P}_{1} w_{2}=-\mathbf{P}(y, D) u_{2}-\mathbf{P}_{1} u_{2}=f_{0}-q u_{2}-\mathbf{P}_{1} u_{2}
$$

Applying Corollary 24.1, we have

$$
\left\|w_{2}\right\|_{C(\bar{G})} \leq \frac{1}{\lambda}\left\|f_{0}-q u_{2}-\mathbf{P}_{1} u_{2}\right\|_{C(\bar{G})}
$$

Choosing a sufficiently large $\lambda$, we obtain

$$
\begin{equation*}
\left\|w_{2}\right\|_{C(\overline{\bar{G}})} \leq \varepsilon \tag{24.51}
\end{equation*}
$$

It follows from inequalities (24.44), (24.49), and (24.51) that

$$
\left\|u-u_{3}\right\|_{C(\bar{G})} \leq\left\|u-u_{1}\right\|_{C(\bar{G})}+\left\|u_{1}-u_{2}\right\|_{C(\bar{G})}+\left\|u_{2}-u_{3}\right\|_{C(\bar{G})} \leq 3 \varepsilon .
$$

Now we prove the main result of the paper.
Theorem 24.1. Let conditions 23.1, 23.2, and 24.1-24.8 hold. Then the operator

$$
\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})
$$

generates a Feller semigroup.
Proof. 1. By Lemma 24.8 and Corollary 24.1, there exists a bounded operator $\left(q I-\mathbf{P}_{B}\right)^{-1}: C_{B}(\bar{G}) \rightarrow$ $C_{B}(\bar{G})$ for all sufficiently large $q>0$ and

$$
\left\|\left(q I-\mathbf{P}_{B}\right)^{-1}\right\| \leq 1 / q
$$

2. Since the operator $\left(q I-\mathbf{P}_{B}\right)^{-1}$ is bounded and is defined in the whole space $C_{B}(\bar{G})$, it is closed. Hence, the operator $q I-\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ is closed. Therefore, the operator $\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ is also closed.
3. Prove that the operator $\left(q I-\mathbf{P}_{B}\right)^{-1}$ is nonnegative. Assume the converse; then there exists a function $f_{0} \geq 0$ such that the solution $u \in \mathrm{D}\left(\mathbf{P}_{B}\right)$ of the equation $q u-\mathbf{P}_{B} u=f_{0}$ has a negative minimum at some point $y^{0} \in \bar{G}$. In this case, the function $v=-u$ has a positive maximum at the point $y^{0}$. By Lemma 24.4, there exists a point $y^{1} \in G$ such that $v\left(y^{1}\right)=v\left(y^{0}\right)$ and $\mathbf{P}_{B} v\left(y^{1}\right) \leq 0$. Hence, $0<v\left(y^{0}\right)=v\left(y^{1}\right)=\left(\mathbf{P}_{B} v\left(y^{1}\right)-f_{0}\left(y^{1}\right)\right) / q \leq 0$. This contradiction proves that $u \geq 0$.

Thus, all conditions of the Hille-Yosida theorem are fulfilled (Theorem 23.1), and the operator $\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ generates a Feller semigroup.

In the next subsection, we give examples of nonlocal operators that satisfy the conditions of Theorem 24.1.
24.5. Example. Let $\partial G=\Gamma_{1} \cup \Gamma_{2} \cup \mathcal{K}$, where $\Gamma_{1}$ and $\Gamma_{2}$ are open and connected (in the topology of $\partial G$ ) curves of the class $C^{\infty}$; moreover, $\Gamma_{1} \cap \Gamma_{2}=\varnothing$ and $\overline{\Gamma_{1}} \cap \overline{\Gamma_{2}}=\mathcal{K}$; the set $\mathcal{K}$ consists of two points $g_{1}$ and $g_{2}$. Assume that the domain $G$ coincides with a plane angle in an $\varepsilon$-neighborhood of the points $g_{i}, i=1,2$. Let $\Omega_{j}, j=1, \ldots, 4$, be continuous transformations given at $\overline{\Gamma_{1}}$ and satisfying the following conditions (see Fig. 24.1):
(1) $\Omega_{1}(\mathcal{K}) \subset \mathcal{K}, \Omega_{1}\left(\Gamma_{1} \cap \mathcal{O}_{\varepsilon}(\mathcal{K})\right) \subset G, \Omega_{1}\left(\Gamma_{1} \backslash \mathcal{O}_{\varepsilon}(\mathcal{K})\right) \subset G \cup \Gamma_{2}$ and $\Omega_{1}(y)$ is the composition of operators of argument shift, rotation, and dilation when $y \in \overline{\Gamma_{1}} \cap \mathcal{O}_{\varepsilon}(\mathcal{K})$;
(2) there exist numbers $\varkappa_{1}>\varkappa_{2}>0$ and $\sigma>0$ such that $\Omega_{2}\left(\overline{\Gamma_{1}}\right) \subset \bar{G} \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K})$ and $\Omega_{2}\left(\overline{\Gamma_{1}} \backslash \mathcal{O}_{\varkappa_{2}}(\mathcal{K})\right) \subset \overline{G_{\sigma}} ;$ moreover, $\Omega_{2}\left(g_{1}\right) \in \Gamma_{1}$ and $\Omega_{2}\left(g_{2}\right) \in G$;
(3) $\Omega_{3}\left(\overline{\Gamma_{1}}\right) \subset G \cup \Gamma_{2}$ and $\Omega_{3}(\mathcal{K}) \subset \Gamma_{2}$;
(4) $\Omega_{4}\left(\overline{\Gamma_{1}}\right) \subset G \cup \overline{\Gamma_{2}}$ and $\Omega_{4}(\mathcal{K}) \subset \mathcal{K}$.


Fig. 24.1. Nontransversal nonlocal conditions
Let $b_{1} \in C\left(\overline{\Gamma_{1}}\right) \cap C^{\infty}\left(\overline{\Gamma_{1}} \cap \mathcal{O}_{\varepsilon}(\mathcal{K})\right), b_{2}, b_{3}, b_{4} \in C\left(\overline{\Gamma_{1}}\right)$, and $b_{j} \geq 0, j=1, \ldots, 4$.
Let $G_{1}$ be a bounded domain, $G_{1} \subset G$, and $\Gamma \subset \bar{G}$ be a curve of the class $C^{1}$. Introduce nonnegative functions $c(y, \eta), y \in \overline{\Gamma_{1}}, \eta \in \overline{G_{1}}$, and $d(y, \eta), y \in \overline{\Gamma_{1}}, \eta \in \bar{\Gamma}$.

Consider the following nonlocal conditions:

$$
\begin{array}{ll}
u(y)-\sum_{j=1}^{4} b_{j}(y) u\left(\Omega_{j}(y)\right)-\int_{G_{1}} c(y, \eta) u(\eta) d \eta-\int_{\Gamma} d(y, \eta) u(\eta) d \Gamma_{\eta}=0, & y \in \Gamma_{1},  \tag{24.52}\\
u(y)=0, & y \in \overline{\Gamma_{2}} .
\end{array}
$$

Let $Q \subset \bar{G}$ be an arbitrary Borel set. Introduce the following measures $\mu(y, \cdot), y \in \partial G$ :

$$
\begin{array}{lll}
\mu(y, Q)=\sum_{j=1}^{4} b_{j}(y) \chi_{Q}\left(\Omega_{j}(y)\right)+\int_{G_{1} \cap Q} c(y, \eta) d \eta+\int_{\Gamma \cap Q} d(y, \eta) u(\eta) d \Gamma_{\eta}, & y \in \Gamma_{1},  \tag{24.53}\\
\mu(y, Q)=0, & & y \in \overline{\Gamma_{2}} .
\end{array}
$$

Let $\mathcal{N}$ and $\mathcal{M}$ be defined as above. Assume that

$$
\begin{cases}\mu(y, \bar{G})=\sum_{j=1}^{4} b_{j}(y)+\int_{G_{1}} c(y, \eta) d \eta+\int_{\Gamma} d(y, \eta) d \Gamma_{\eta} \leq 1, & y \in \partial G  \tag{24.54}\\ \int_{\Gamma \cap \mathcal{M}} d(y, \eta) d \Gamma_{\eta}<1, & y \in \mathcal{M} \\ b_{2}\left(g_{1}\right)=0 \text { or } \mu\left(\Omega_{2}\left(g_{1}\right), \bar{G}\right)=0, \quad b_{2}\left(g_{2}\right)=0, & \\ b_{4}\left(g_{j}\right)=0, \quad c\left(g_{j}, \cdot\right)=0, \quad d\left(g_{j}, \cdot\right)=0\end{cases}
$$

Setting $b(y)=1-\mu(y, \bar{G})$, we can rewrite Eq. (24.52) in the following form (cf. (24.9)):

$$
b(y) u(y)+\int_{\bar{G}}[u(y)-u(\eta)] \mu(y, d \eta)=0, \quad y \in \partial G .
$$

Introduce a patch function $\zeta \in C^{\infty}\left(\mathbb{R}^{2}\right)$ with support in $\mathcal{O}_{\varepsilon}(\mathcal{K})$. This function is equal to 1 on $\mathcal{O}_{\varepsilon / 2}(\mathcal{K})$ and is such that $0 \leq \zeta(y) \leq 1$ for $y \in \mathbb{R}^{2}$. Let $y \in \overline{\Gamma_{1}}$, and $Q \subset \bar{G}$ be an arbitrary Borel set; we denote

$$
\begin{gather*}
\delta(y, Q)=\zeta(y) b_{1}(y) \chi_{Q}\left(\Omega_{1}(y)\right), \quad \alpha(y, Q)=b_{2}(y) \chi_{Q}\left(\Omega_{2}(y)\right), \\
\beta^{1}(y, Q)=(1-\zeta(y)) b_{1}(y) \chi_{Q}\left(\Omega_{1}(y)\right)+\sum_{j=3,4} b_{j}(y) \chi_{Q}\left(\Omega_{j}(y)\right),  \tag{24.55}\\
\beta^{2}(y, Q)=\int_{G_{1} \cap Q} c(y, \eta) d \eta+\int_{\Gamma \cap Q} d(y, \eta) u(\eta) d \Gamma_{\eta} .
\end{gather*}
$$

It can be directly verified that these measures satisfy conditions 6.3, 23.2, and 24.2-24.8.

## 25. Unbounded Perturbations of Diffusion Processes

In this section, we prove the existence of a Feller semigroup generated by an unbounded in $C(\bar{G})$ perturbation of an elliptic operator.
25.1. Assumptions about unbounded perturbations and nonlocal operators. Consider the same nonlocal conditions as in Sec. 24 (see (24.9), (24.10) or (24.13)). Let $\mathcal{N} \subset \partial G$ and $\mathcal{M}=\partial G \backslash \mathcal{N}$ be the same sets as in Sec. 24. We fix a natural number $l$ and a real number $a$ such that $l \geq 2$, $a=l+1-\delta$, where $\delta \in(0,1)$ is the same as in Sec. 23.4.
Remark 25.1. By Theorem 23.2 and Corollary 23.3, the operators

$$
\mathbf{S}_{q}: \mathcal{H}_{a}^{l+3 / 2}(\partial G) \rightarrow H_{a}^{l+2}(G), \quad \mathbf{S}_{q}: \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G) \rightarrow H_{\mathcal{N}, a}^{l+2}(G)
$$

are bounded in the corresponding norms $\|\cdot\|$ uniformly with respect to $q, q \geq q_{1}$, where $q_{1}>0$ is a sufficiently large number (the stated spaces are defined in Sec. 5.3).

Consider a linear bounded operator $\mathbf{P}_{1}: H_{a}^{l+2}(G) \rightarrow H_{a-1}^{l}(G)$ satisfying the following condition.
Condition 25.1. (1) If the function $u \in H_{a}^{l+2}(G)$ has a positive maximum at a point $y^{0} \in G$, then $\mathbf{P}_{1} u\left(y^{0}\right) \leq 0$.
(2) If $u \in C(\bar{G}) \cap H_{a}^{l+2}(G)$, then the function $\mathbf{P}_{1} u$ is bounded in the domain $G$.
(3) For all sufficiently small $\varrho>0$, the following representation holds:

$$
\mathbf{P}_{1}=\mathbf{P}_{1 \varrho}^{1}+\mathbf{P}_{1 \varrho}^{2}
$$

where the operators $\mathbf{P}_{1 \varrho}^{1}, \mathbf{P}_{1 \varrho}^{2}: H_{a}^{l+2}(G) \rightarrow H_{a-1}^{l}(G)$ are such that
(a) $\left\|\mathbf{P}_{1 \varrho}^{1} u\right\|_{H_{a-1}^{l}(G)} \leq c(\varrho)\|u\|_{H_{a}^{l+2}(G)}$, where $c(\varrho)>0$ is independent of $u$ and $c(\varrho) \rightarrow 0$ for $\varrho \rightarrow 0$;
(b) the operator $\mathbf{P}_{1 \varrho}^{2}$ is compact.

Note that $\mathrm{D}\left(\mathbf{P}_{1}\right) \subset C^{l}(\bar{G} \backslash \mathcal{K}) \subset C^{2}(G)$ and $\mathcal{R}\left(\mathbf{P}_{1}\right) \subset C^{l-2}(\bar{G} \backslash \mathcal{K}) \subset C(G)$ since $l \geq 2$ and by the Sobolev embedding theorem. Moreover, if $u \in C(\bar{G}) \cap H_{a}^{l+2}(G)$, then the function $\mathbf{P}_{1} u$ is bounded in the domain $G$, but it is not necessarily continuous on $\bar{G}$.

Consider an example of the operator $\mathbf{P}_{1}$.
Example 25.1. 1. Let $l \geq 2,0<\delta<1$, and $a=l+1-\delta$ be the same as above. Let $F$ be a space with a $\sigma$-algebra $\mathcal{F}$ and a Borel measure $\pi$. Let us consider a vector-valued function $z(y, \eta)$ with values in $\mathbb{R}^{2}$ and a scalar nonnegative function $m(y, \eta)$, where $y \in \bar{G}$ and $\eta \in F$.

In the description of a motion of a particle with "jumps" in a domain $G$, the function $z(y, \eta)$ characterizes the direction of the jump and its length, and the function $m(y, \eta)$ characterizes the density of the jump.

Assume that $z(\cdot, \eta), m(\cdot, \eta) \in C^{l}(\bar{G})$ for any fixed $\eta \in F$ and the functions $D_{y}^{\alpha} z(y, \eta)$ and $D_{y}^{\alpha} m(y, \eta)$ are bounded and $\pi$-measurable with respect to the variable $\eta$ for $|\alpha| \leq l, y \in \bar{G}$.

Let the function $z(y, \eta)$ satisfy the following conditions:

$$
\begin{gather*}
y+\theta z(y, \eta) \in \bar{G} \quad \forall y \in \bar{G}, \quad \eta \in F, \quad \theta \in[0,1],  \tag{25.1}\\
\left|D_{y}^{\alpha} z(y, \eta)\right| \leq Z(\eta) \quad \forall y \in \bar{G}, \quad \eta \in F, \quad|\alpha| \leq l,  \tag{25.2}\\
\int_{Z \leq \varrho} Z^{2}(\eta) \pi(d \eta) \leq c_{1}(\varrho), \quad \int_{Z>\varrho} \pi(d \eta) \leq c_{2}(\varrho), \tag{25.3}
\end{gather*}
$$

where $Z(\eta)$ is a nonnegative $\pi$-measurable function, $c_{1}(\varrho), c_{2}(\varrho)>0$, and $c_{1}(\varrho) \rightarrow 0$ as $\varrho \rightarrow 0$.
In particular, condition (25.1) means that jumps outward the $\bar{G}$ are impossible. Condition (25.3) characterizes the behavior of the measure $\pi$ with respect to small and large jumps.

To prove estimates in weight spaces, we assume that

$$
\begin{equation*}
c\left|y-y^{\prime}\right| \leq\left|\left(y-y^{\prime}\right)+\theta\left(z(y, \eta)-z\left(y^{\prime}, \eta\right)\right)\right| \leq C\left|y-y^{\prime}\right|, \quad y, y^{\prime} \in \bar{G}, \quad \eta \in F, \quad \theta \in[0,1] \tag{25.4}
\end{equation*}
$$

where $c, C>0$. Using inequalities (25.4), it is easy to show that the change of variables $Y: y \mapsto y+\theta z(y, \eta)$ is a diffeomorphism of class $C^{1}$ (even of class $C^{l}$, since $z(\cdot, \eta) \in C^{l}(\bar{G})$ for any $\eta \in F)$ mapping $\bar{G}$ to $Y(\bar{G}) \subset \bar{G}$ for any $\eta \in F$ and $\theta \in[0,1]$ and

$$
\begin{equation*}
c^{2} \leq J_{\eta, \theta}(y) \leq C^{2}, \quad y \in G, \quad \eta \in F, \quad \theta \in[0,1], \tag{25.5}
\end{equation*}
$$

where $J_{\eta, \theta}(y)$ is the absolute value of the Jacobian of this change of variables.
Now we impose some restrictions on the size, amplitude, and density of jumps near the set $\mathcal{K}$ and outside $\mathcal{K}$. For any $\varrho>0$, we assume that

$$
G_{\varrho}=G \cap \mathcal{O}_{\varrho}(\mathcal{K}), \quad G_{\varrho}^{\prime}=G \backslash \overline{G_{\varrho}} .
$$

Fix sufficiently small numbers $\varrho_{1}>\varrho_{2}>0$. For $y \in G_{\varrho_{1}}^{\prime}$, we assume that

$$
\begin{equation*}
y+\theta z(y, \eta) \in G_{\varrho_{2}}^{\prime} \quad \forall y \in G_{\varrho_{1}}^{\prime}, \quad \eta \in F, \quad \theta \in[0,1] . \tag{25.6}
\end{equation*}
$$

To describe the functions $z(y, \eta)$ and $m(y, \eta)$ for $y \in G_{\varrho_{1}}$, we fix an arbitrary point $g_{j}$ and assume that it coincides with the origin: $g_{j}=0$. Let

$$
\begin{align*}
y+\theta z(y, \eta) \in \overline{\mathcal{O}_{\tilde{\chi} r}(\mathcal{K})} \backslash \mathcal{O}_{\chi r}(\mathcal{K}) & \forall y \in G \cap \mathcal{O}_{\varrho_{1}}\left(g_{j}\right), \tag{25.7}
\end{align*} \quad \eta \in F,
$$

where $\omega$ and $r$ are the polar coordinates of the point $y, \mu_{j}(\omega)$ is a nonnegative, $l+2$ times continuously differentiable function, and $\tilde{\chi} \geq \chi>0, M_{j}(\eta)$ is a nonnegative, bounded, $\pi$-measurable function.

We assume that $\varrho_{1}$ is so small that the domain $G$ coincides with a plane angle in a neighborhood $\mathcal{O}_{\tilde{\chi} \varrho_{1}}\left(g_{j}\right), j=1, \ldots, N$, and $\mathcal{O}_{\tilde{\chi} \varrho_{1}}\left(g_{i}\right) \cap \mathcal{O}_{\tilde{\chi} \varrho_{1}}\left(g_{j}\right)=\varnothing, i \neq j$.

Condition (25.6) means that a particle that is "far" from the set $\mathcal{K}$ (i.e., is outside $G_{\varrho_{1}}$ ), cannot "jump" to a small neighborhood of the set $\mathcal{K}$ (i.e., to $G_{\varrho_{2}}$ ). Condition (25.7) means that a particle is "near" the set $\mathcal{K}$ (i.e., inside $G_{\varrho_{1}}$ ), that it cannot "jump" far from the set $\mathcal{K}$ (i.e., it stays inside $G_{\tilde{\chi}_{\varrho_{1}}}$ ); moreover, by condition (25.8), the density of such a "jump" tends to zero as $y$ tends to the set $\mathcal{K}$.

Let us define the operators $\mathbf{P}_{1}, \mathbf{P}_{1 \varrho}^{1}$, and $\mathbf{P}_{1 \varrho}^{2}$ on the set $C_{0}^{\infty}(\bar{G} \backslash \mathcal{K})$ by the formulas

$$
\begin{aligned}
& \mathbf{P}_{1} u(y)=\int_{F}[u(y+z(y, \eta))-u(y)-(\nabla u(y), z(y, \eta))] m(y, \eta) \pi(d \eta), \\
& \mathbf{P}_{1 \varrho}^{1} u(y)=\int_{Z \leq \varrho}[u(y+z(y, \eta))-u(y)-(\nabla u(y), z(y, \eta))] m(y, \eta) \pi(d \eta), \\
& \mathbf{P}_{1 \varrho}^{2} u(y)=\int_{Z>\varrho}[u(y+z(y, \eta))-u(y)-(\nabla u(y), z(y, \eta))] m(y, \eta) \pi(d \eta),
\end{aligned}
$$

where $(\cdot, \cdot)$ is the scalar product in $\mathbb{R}^{2}$ (cf. $\left.[6,20,21,102]\right)$. It will be shown below that the operators $\mathbf{P}_{1}, \mathbf{P}_{1 \varrho}^{1}, \mathbf{P}_{1 \varrho}^{2}: H_{l+1-\delta}^{l+2}(G) \rightarrow H_{l-\delta}^{l}(G)$ with the domain $C_{0}^{\infty}(\bar{G} \backslash \mathcal{K})$ are bounded. Hence, they can be extended by the continuity to the whole space $H_{l+1-\delta}^{l+2}(G)$.
2. Let us show that the operator $\mathbf{P}_{1}$ satisfies condition 25.1.
2.1. First, we prove that

$$
\begin{equation*}
\left\|\mathbf{P}_{1 \varrho}^{1} u\right\|_{W^{l}(G)} \leq c(\varrho)\|u\|_{H_{l+1-\delta}^{l+2}(G)} \tag{25.9}
\end{equation*}
$$

where $c(\varrho)>0$ is independent of $u$ and $c(\varrho) \rightarrow 0$ as $\varrho \rightarrow 0$.
Let us denote

$$
\begin{align*}
U(y, \eta)=u(y+z(y, \eta))-u(y)-(\nabla u(y), & z(y, \eta)) \\
& =\int_{0}^{1} d \theta \int_{0}^{\theta} \sum_{i, j=1}^{2} u_{y_{i} y_{j}}\left(y+\theta^{\prime} z(y, \eta)\right) z_{i}(y, \eta) z_{j}(y, \eta) d \theta^{\prime} \tag{25.10}
\end{align*}
$$

and write

$$
\begin{align*}
&\left\|\mathbf{P}_{1 \varrho}^{1} u\right\|_{W^{l}(G)}^{2}=\sum_{|\alpha| \leq l} \int_{G_{e_{1}}}\left|\int_{Z \leq \varrho} D_{y}^{\alpha}(U(y, \eta) m(y, \eta)) \pi(d \eta)\right|^{2} d y \\
&+\sum_{|\alpha| \leq l} \int_{G_{\rho_{1}}^{\prime}} \mid  \tag{25.11}\\
&\left.\int_{Z \leq \varrho} D_{y}^{\alpha}(U(y, \eta) m(y, \eta)) \pi(d \eta)\right|^{2} d y .
\end{align*}
$$

Using the Schwartz inequality, Eq. (??), the explicit form of the function $m(y, \eta)$ (see (25.8)), and Eq. (25.10), we have

$$
\begin{aligned}
& \sum_{|\alpha| \leq l} \int_{G_{Q_{1}}}\left|\int_{Z \leq \varrho} D_{y}^{\alpha}(U(y, \eta) m(y, \eta)) \pi(d \eta)\right|^{2} d y \\
& \quad \leq c_{1} \sum_{|\beta| \leq l+2} \int_{G_{Q_{1}}} \rho^{2(|\beta|-\delta-1)} \int_{0}^{1} d \theta \int_{0}^{\theta} d \theta^{\prime} \int_{Z \leq \varrho}\left|\left(D_{y}^{\beta} u\right)\left(y+\theta^{\prime} z(y, \eta)\right)\right|^{2} Z^{2}(\eta) \pi(d \eta) \int_{Z \leq \varrho} Z^{2}(\eta) \pi(d \eta),
\end{aligned}
$$

where $\rho(y)=\operatorname{dist}(y, \mathcal{K})$ and $c_{1}, c_{2}, \ldots>0$ are independent of $u$ and $\varrho$. By virtue of Eq. (25.1), (25.4), and (25.5), we can change the variables $Y=y+\theta^{\prime} z(y, \eta)$; then, using (25.7), from the last inequality,
we obtain

$$
\begin{equation*}
\sum_{|\alpha| \leq l} \int_{G_{\varrho_{1}}}\left|\int_{Z \leq \varrho} D_{y}^{\alpha}(U(y, \eta) m(y, \eta)) \pi(d \eta)\right|^{2} d y \leq c_{2}\|u\|_{H_{l+1-\delta}^{l+2}\left(G_{\tilde{\chi} \varrho_{1}}\right)}^{2}\left(\int_{Z \leq \varrho} Z^{2}(\eta) \pi(d \eta)\right)^{2} \tag{25.12}
\end{equation*}
$$

Similarly, using Eq. (25.6), we obtain the inequality

$$
\begin{align*}
\sum_{|\alpha| \leq l} \int_{G_{\varrho_{1}}^{\prime}} & \left|\int_{Z \leq \varrho} D_{y}^{\alpha}(U(y, \eta) m(y, \eta)) \pi(d \eta)\right|^{2} d y \\
& \leq c_{3}\|u\|_{W^{l+2}\left(G_{\varrho_{2}}^{\prime}\right)}^{2}\left(\int_{Z \leq \varrho} Z^{2}(\eta) \pi(d \eta)\right)^{2} \leq c_{4}\|u\|_{H_{l+1-\delta}^{l+2}\left(G_{\varrho_{2}}^{\prime}\right)}^{2}\left(\int_{Z \leq \varrho} Z^{2}(\eta) \pi(d \eta)\right)^{2} \tag{25.13}
\end{align*}
$$

Equations (25.11)-(25.13) and the first inequality in Eq. (25.3) yield (25.9).
2.2. Dividing the domain $G$ into two parts $G_{\varrho_{1}}$ and $G_{\varrho_{1}}^{\prime}$ and using the estimate

$$
\left|D^{\alpha} U(y, \eta)\right| \leq c_{5}\left(\sum_{|\beta| \leq|\alpha|}\left|\left(D_{y}^{\beta} u\right)(y+z(y, \eta))\right|+\sum_{|\beta| \leq|\alpha|+1}\left|D_{y}^{\beta} u(y)\right|\right)
$$

and the second estimate in Eq. (25.3), we see that

$$
\begin{equation*}
\left\|\mathbf{P}_{1 \varrho}^{2} u\right\|_{W^{l+1}(G)} \leq \hat{c}(\varrho)\|u\|_{H_{l+1-\delta}^{l+2}(G)} \tag{25.14}
\end{equation*}
$$

where $\hat{c}(\varrho)>0$ are independent of $u$.
From Eqs. (25.9) and (25.14) and the Sobolev embedding theorem (recall that $l \geq 2$ ), we obtain that the function $\mathbf{P}_{1} u=\mathbf{P}_{1 \varrho}^{1} u+\mathbf{P}_{1 \varrho}^{2} u$ is continuous on $\bar{G}$; hence, it is bounded on $\bar{G}$. Thus, item 2 in condition 25.1 is fulfilled.

Item 3 in Eq. 25.1 follows from estimates (25.9) and (25.14), the boundedness of the embedding operator $W^{l}(G) \subset H_{l-\delta}^{l}(G)$ (see item 1 of Lemma 5.2 ), and the compactness of the embedding operator $W^{l+1}(G) \subset W^{l}(G)$.

Item 1 in condition 25.1 follows from the nonnegativeness of the function $m(y, \eta)$ and the measure $\pi$.
Let $\mathbf{B}_{\alpha i}, \mathbf{B}_{\beta i}$, etc., be operators defined by the measures $\alpha_{i}(y, \cdot)$ and $\beta_{i}(y, \cdot)$ (see Sec. 24). In this section, in addition to conditions $23.1,23.2,24.2-24.6$, and 25.1 , we consider the following conditions.
Condition 25.2. For $u \in H_{\mathcal{N}, a}^{l+2}(G)$ and $q \geq q_{1}$, the following conditions hold:

$$
\begin{gather*}
\left\|\mathbf{B}_{\alpha i} u\right\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i}\right)} \leq c\|u\|_{H_{a}^{l+2}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)}  \tag{25.15}\\
\left\|\mathbf{B}_{\alpha i} u\right\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i} \backslash \overline{\mathcal{O}_{\varkappa_{2}}(\mathcal{K})}\right)} \leq c\|u\|_{H_{a}^{l+2}\left(G_{\sigma}\right)} \tag{25.16}
\end{gather*}
$$

where $q_{1}>0$ is sufficiently large, $i=1, \ldots, N$, the numbers $\varkappa_{1}, \varkappa_{2}$, and $\sigma$ are the same as in condition 24.4, and $c>0$ is independent of $u$ and $q$.

Note that the norms in the weight spaces $H_{a}^{l+2}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right), H_{a}^{l+3 / 2}\left(\Gamma_{i} \backslash \overline{\mathcal{O}_{\varkappa_{2}}(\mathcal{K})}\right)$, and $H_{a}^{l+2}\left(G_{\sigma}\right)$ are equivalent to the norms in the corresponding Sobolev spaces since the sets $G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}, \Gamma_{i} \backslash \overline{\mathcal{O}_{\varkappa_{2}}(\mathcal{K})}$, and $G_{\sigma}$ are separated from the set $\mathcal{K}$.

Condition 25.3. There exists $q_{1}>0$ such that for $i=1, \ldots, N$ and any sufficiently small $p>0$, the following statements are valid:
(1) the operator $\hat{\mathbf{B}}_{\beta i}^{1}: H_{\mathcal{N}, a}^{l+2}\left(G \cap \mathcal{M}_{p}\right) \rightarrow H_{\mathcal{N}, a}^{l+3 / 2}\left(\Gamma_{i}\right)$ is bounded in the norms $\|\cdot\| \|$ if $q \geq q_{1}$ and

$$
\left\|\hat{\mathbf{B}}_{\beta i}^{1}\right\|_{H_{\mathcal{N}, a}^{l+2}\left(G \cap \mathcal{M}_{p}\right) \rightarrow H_{\mathcal{N}, a}^{l+3 / 2}\left(\Gamma_{i}\right)} \rightarrow 0
$$

uniformly with respect to $q$ as $p \rightarrow 0$;
(2) the operator $\tilde{\mathbf{B}}_{\beta i}^{1}: H_{\mathcal{N}, a}^{l+2}\left(G \backslash \overline{\mathcal{M}_{p / 2}}\right) \rightarrow H_{\mathcal{N}, a}^{l+3 / 2}\left(\Gamma_{i}\right)$ is bounded in the norms $\|\cdot\|$ uniformly with respect to $q$ for $q \geq q_{1}$.
Condition 25.4. The operators $\mathbf{B}_{\beta i}^{2}: H_{\mathcal{N}, a}^{l+2}(G) \rightarrow H_{\mathcal{N}, a}^{l+3 / 2}\left(\Gamma_{i}\right), i=1, \ldots, N$, are compact.
Conditions 25.2, 25.3 and 25.4 in weight spaces are analogs of conditions 24.4, 24.7, and 24.8, respectively.
25.2. Reduction to the boundary. Introduce the operators

$$
\begin{aligned}
\mathbf{B} & =\left\{\mathbf{B}_{i}\right\}: H_{\mathcal{N}, a}^{l+2}(G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G), \\
\mathbf{B}_{\alpha \beta} & =\left\{\mathbf{B}_{\alpha i}+\mathbf{B}_{\beta i}\right\}: H_{\mathcal{N}, a}^{l+2}(G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)
\end{aligned}
$$

Using the operator $\mathbf{S}_{q}$ defined in Sec. 24.1, we introduce the bounded operator

$$
\begin{equation*}
\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}: \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G), \quad q \geq q_{1} \tag{25.17}
\end{equation*}
$$

By Remark 25.1, the operator (25.17) is well defined.
The following Lemma allows us to reduce nonlocal problems in bounded domains to operator equations on the boundary.

Lemma 25.1. For sufficiently large $q_{1}>0$, there exists the bounded operator

$$
\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)^{-1}: \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G), \quad q \geq q_{1}
$$

Proof. 1. Consider the bounded operators

$$
\hat{\mathbf{B}}_{\beta}^{1}=\left\{\hat{\mathbf{B}}_{\beta i}^{1}\right\}, \quad \tilde{\mathbf{B}}_{\beta}^{1}=\left\{\tilde{\mathbf{B}}_{\beta i}^{1}\right\}, \quad \mathbf{B}_{\beta}^{2}=\left\{\mathbf{B}_{\beta i}^{2}\right\}, \quad \mathbf{B}_{\alpha}=\left\{\mathbf{B}_{\alpha i}\right\}
$$

that act from $H_{\mathcal{N}, a}^{l+2}(G)$ into $\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)$.
Prove that the operator

$$
\mathbf{I}-\mathbf{B}_{\alpha} \mathbf{S}_{q}: \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)
$$

has a bounded inverse operator.
Introduce a function $\zeta \in C^{\infty}(\bar{G})$ such that $0 \leq \zeta(y) \leq 1, \zeta(y)=1$ for $y \in \overline{G_{\sigma}}$ and $\zeta(y)=0$ for $y \notin G_{\sigma / 2}$, where $\sigma>0$ is a number from conditions 24.4 and 25.2.

We have

$$
\begin{equation*}
\mathbf{I}-\mathbf{B}_{\alpha} \mathbf{S}_{q}=\mathbf{I}-\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}-\mathbf{B}_{\alpha} \zeta \mathbf{S}_{q} . \tag{25.18}
\end{equation*}
$$

1a. First, we prove that the operator $\mathbf{I}-\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}$ has a bounded inverse operator.
Assume $\Phi=\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q} \psi$. It follows from condition 25.2 and Theorem 23.2 that

$$
\begin{equation*}
\|\Phi\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)} \leq k_{1}\|\psi\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)} \tag{25.19}
\end{equation*}
$$

where $k_{1}, k_{2}, \ldots>0$ are independent of $q$ and $\psi$.
Further, $(1-\zeta) \mathbf{S}_{q} \psi=0$ in $\overline{G_{\sigma}}$ for any $\psi \in \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)$. Hence, by condition 24.4,

$$
\begin{equation*}
\operatorname{supp} \Phi=\operatorname{supp} \mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q} \psi \subset \partial G \cap \overline{\mathcal{O}_{\varkappa_{2}}(\mathcal{K})} \tag{25.20}
\end{equation*}
$$

We show that

$$
\begin{equation*}
\left\|\left[\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right]^{2} \psi\right\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)} \leq k_{2} q^{-1 / 2}\|\psi\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)}, \quad q \geq q_{1} \tag{25.21}
\end{equation*}
$$

where $q_{1}>0$ is sufficiently large. By condition 25.2,

$$
\begin{equation*}
\left\|\left[\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right]^{2} \psi\right\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)}=\left\|\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q} \Phi\right\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)} \leq\left\|\mathbf{S}_{q} \Phi\right\|_{\mathcal{H}_{a}^{l+2}\left(G \backslash \overline{\mathcal{O}_{\varkappa_{1}}(\mathcal{K})}\right)} \tag{25.22}
\end{equation*}
$$

Using the local a priori estimate (Corollary 23.1), the relation

$$
\mathbf{P}(y, D) \mathbf{S}_{q} \Phi-q \mathbf{S}_{q} \Phi=0
$$

and Remark 25.1, we have

$$
\begin{align*}
& \left\|\mathbf{S}_{q} \Phi\right\|_{\mathcal{H}_{a}^{l+2}\left(G \backslash \overline{\left.\mathcal{O}_{\varkappa_{1}}(\mathcal{K})\right)}\right.} \leq k_{4}\left(\sum_{i=1}^{N}\left\|\left.\mathbf{S}_{q} \Phi\right|_{\Gamma_{i} \backslash \overline{\mathcal{O}_{\left(\varkappa_{1}+\varkappa_{2}\right) / 2}(\mathcal{K})}}\right\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i} \backslash \overline{\left.\mathcal{O}_{\left(\varkappa_{1}+\varkappa_{2}\right) / 2}(\mathcal{K})\right)}\right.}\right. \\
& \left.\quad+q^{-1 / 2}\left\|\mathbf{S}_{q} \Phi\right\|_{H_{a}^{l+2}\left(G \backslash \overline{\left.\mathcal{O}_{\left(\varkappa_{1}+\varkappa_{2}\right) / 2}(\mathcal{K})\right)}\right.}\right) \\
& \leq k_{5}\left(\sum_{i=1}^{N}\left\|\left.\mathbf{S}_{q} \Phi\right|_{\Gamma_{i} \backslash \overline{\mathcal{O}_{\left(\varkappa_{1}+\varkappa_{2}\right) / 2}(\mathcal{K})}}\right\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i} \backslash \overline{\left.\mathcal{O}_{\left(\varkappa_{1}+\varkappa_{2}\right) / 2}(\mathcal{K})\right)}\right.}+q^{-1 / 2}\|\Phi\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)}\right) . \tag{25.23}
\end{align*}
$$

Since the functions $\left.\mathbf{S}_{q} \Phi\right|_{\Gamma_{i}}-\mathbf{B}_{i} \mathbf{S}_{q} \Phi=\Phi_{i}$ vanish on $\Gamma_{i} \backslash \overline{\mathcal{O}_{\left(\varkappa_{1}+\varkappa_{2}\right) / 2}(\mathcal{K})}$ by Eq. (25.20), we have

$$
\begin{aligned}
& \left\|\left.\mathbf{S}_{q} \Phi\right|_{\Gamma_{i} \backslash \overline{\mathcal{O}_{\left(\varkappa_{1}+\varkappa_{2}\right) / 2}(\mathcal{K})}}\right\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i} \backslash \overline{\left.\mathcal{O}_{\left(\varkappa_{1}+\varkappa_{2}\right) / 2}(\mathcal{K})\right)}\right.} \\
& \quad=\left\|\left.\mathbf{B}_{i} \mathbf{S}_{q} \Phi\right|_{\Gamma_{i} \backslash \overline{\mathcal{O}_{\left(\varkappa_{1}+\varkappa_{2}\right) / 2}(\mathcal{K})}}\right\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i} \backslash \overline{\left.\mathcal{O}_{\left(\varkappa_{1}+\varkappa_{2}\right) / 2}(\mathcal{K})\right)}\right.} \leq k_{6}\left\|\mathbf{S}_{q} \Phi\right\|_{H_{a}^{l+2}\left(G_{1}\right)},
\end{aligned}
$$

where $\overline{G_{1}} \subset G$. Using the local a priori estimate (Corollary 23.1), the relations $\mathbf{P}(y, D) \mathbf{S}_{q} \Phi-q \mathbf{S}_{q} \Phi=0$ and $\overline{G_{1}} \cap \partial G=\varnothing$, and Remark 25.1, we obtain the following inequality:

$$
\begin{align*}
\left\|\left.\mathbf{S}_{q} \Phi\right|_{\Gamma_{i} \backslash \overline{\mathcal{O}_{\left(\varkappa_{1}+\varkappa_{2}\right) / 2}(\mathcal{K})}}\right\|_{H_{a}^{l+3 / 2}\left(\Gamma_{i} \backslash \overline{\left.\mathcal{O}_{\left(\varkappa_{1}+\varkappa_{2}\right) / 2}(\mathcal{K})\right)}\right.} \leq k_{7} q^{-1 / 2}\left\|\mathbf{S}_{q} \Phi\right\|_{H_{a}^{l+2}\left(G_{2}\right)} & \\
& \leq k_{8} q^{-1 / 2}\|\Phi\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)} \tag{25.24}
\end{align*}
$$

where $\overline{G_{1}} \subset G_{2}$ and $\overline{G_{2}} \subset G$.
Inequalities (25.22)-(25.24) and (25.19) yield estimate (25.21). From here and estimate (24.26) in the proof of Lemma 24.6, we obtain the inequality

$$
\begin{equation*}
\left\|\left[\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right]^{2} \psi\right\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)} \leq k_{9} q^{-1 / 2}\|\psi\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)} . \tag{25.25}
\end{equation*}
$$

It follows from Eq. (25.25) that there exists a bounded operator

$$
\left(\mathbf{I}-\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right)^{-1}: \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)
$$

and

$$
\begin{equation*}
\left\|\left(\mathbf{I}-\mathbf{B}_{\alpha}(1-\zeta) \mathbf{S}_{q}\right)^{-1}\right\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)} \leq k_{10} \tag{25.26}
\end{equation*}
$$

(cf. Eqs. (24.27) and (24.28)).
1 b . Now let us estimate the norm of the operator $\mathbf{B}_{\alpha} \zeta \mathbf{S}_{q}$. Using condition 25.2, the local a priori estimate (Corollary 23.1), the relation $\mathbf{P}(y, D) \mathbf{S}_{q} \psi-q \mathbf{S}_{q} \psi=0$, and Remark 25.1, we obtain

$$
\left\|\mathbf{B}_{\alpha} \zeta \mathbf{S}_{q} \psi\right\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)} \leq k_{11}\left\|\mathbf{S}_{q} \psi\right\|_{H_{a}^{l+2}\left(G_{\sigma / 2}\right)} \leq k_{12} q^{-1 / 2}\left\|\mathbf{S}_{q} \psi\right\|_{H_{a}^{l+2}\left(G_{\sigma / 4}\right)} \leq k_{13} q^{-1 / 2}\|\psi\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)}
$$

From here and inequality (24.29) in the proof of Lemma 24.6, we have

$$
\left\|\mathbf{B}_{\alpha} \zeta \mathbf{S}_{q} \psi\right\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)} \leq k_{14} q^{-1 / 2}\|\psi\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)}
$$

Hence, taking into account inequality (25.26), we see that there exists a bounded operator

$$
\left(\mathbf{I}-\mathbf{B}_{\alpha} \mathbf{S}_{q}\right)^{-1}: \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)
$$

and

$$
\begin{equation*}
\left\|\left(\mathbf{I}-\mathbf{B}_{\alpha} \mathbf{S}_{q}\right)^{-1}\right\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)} \leq k_{15} \tag{25.27}
\end{equation*}
$$

(cf. (24.30) and (24.31)).
2. Let us prove that the operator

$$
\mathbf{I}-\left(\mathbf{B}_{\alpha}+\hat{\mathbf{B}}_{\beta}^{1}+\tilde{\mathbf{B}}_{\beta}^{1}\right) \mathbf{S}_{q}: \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)
$$

has a bounded inverse operator.
Let us fix arbitrary $\varepsilon>0$. Using condition 25.3 (item 1) and Remark 25.1, we have

$$
\begin{equation*}
\left\|\hat{\mathbf{B}}_{\beta}^{1} \mathbf{S}_{q} \psi\right\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)} \leq k_{16} \varepsilon\left\|\mathbf{S}_{q} \psi\right\|_{\mathcal{H}_{\mathcal{N}, a}^{l+2}(G)} \leq k_{17} \varepsilon\|\psi\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)} \tag{25.28}
\end{equation*}
$$

for sufficiently small $p>0$ (recall that $p$ presents in the definition of the operator $\hat{\mathbf{B}}_{\beta}^{1}$ ), where $k_{16}, k_{17}, \ldots>0$ are independent of $\psi, q$, and $\varepsilon$.

Now let us fix $p$. By condition 25.3 (item 2),

$$
\begin{equation*}
\left\|\tilde{\mathbf{B}}_{\beta i}^{1} \mathbf{S}_{q} \psi\right\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}\left(\Gamma_{i}\right)} \leq k_{18}\left\|\mathbf{S}_{q} \psi\right\|_{\mathcal{H}_{\mathcal{N}, a}^{l+2}\left(G \backslash \mathcal{M}_{p / 2}\right)} \tag{25.29}
\end{equation*}
$$

Using the local a priori estimate (Corollary 23.1), the relations

$$
\mathbf{P}(y, D) \mathbf{S}_{q} \psi-q \mathbf{S}_{q} \psi=0,\left.\quad \mathbf{S}_{q} \psi\right|_{\partial G \backslash \mathcal{O}_{p / 4}(\mathcal{M})}=0
$$

and Remark 25.1, we obtain the inequality

$$
\begin{equation*}
\left\|\mathbf{S}_{q} \psi\right\|_{\mathcal{H}_{a}^{l+2}\left(G \backslash \mathcal{M}_{p / 2}\right)} \leq k_{19} q^{-1 / 2}\left\|\mathbf{S}_{q} \psi\right\|_{\mathcal{H}_{a}^{l+2}\left(G \backslash \mathcal{M}_{p / 4}\right)} \leq k_{20} q^{-1 / 2}\|\psi\|_{\mathcal{H}_{a}^{l+3 / 2}(\partial G)} \tag{25.30}
\end{equation*}
$$

On the other hand, the following inequality follows from Lemma 24.1:

$$
\begin{equation*}
\left\|\mathbf{S}_{q} \psi\right\|_{C\left(\bar{G} \backslash \mathcal{M}_{p / 2}\right)} \leq c_{2} q^{-1}\|\psi\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \tag{25.31}
\end{equation*}
$$

(cf. (24.32)). Combining (25.29)-(25.31), we obtain

$$
\begin{equation*}
\left\|\tilde{\mathbf{B}}_{\beta i}^{1} \mathbf{S}_{q} \psi\right\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}\left(\Gamma_{i}\right)} \leq k_{21} q^{-1 / 2}\|\psi\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)} \tag{25.32}
\end{equation*}
$$

Inequalities (25.28) and (25.32) show us that the value

$$
\left\|\left(\hat{\mathbf{B}}_{\beta}^{1}+\tilde{\mathbf{B}}_{\beta}^{1}\right) \mathbf{S}_{q}\right\|_{\mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)}
$$

can be made arbitrarily small if we first choose sufficiently small $p>0$ and then sufficiently large $q>0$. Hence, taking into account (25.27), we see that there exists a bounded operator

$$
\left[\mathbf{I}-\left(\mathbf{B}_{\alpha}+\hat{\mathbf{B}}_{\beta}^{1}+\tilde{\mathbf{B}}_{\beta}^{1}\right) \mathbf{S}_{q}\right]^{-1}: \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)
$$

3. Now we prove that the operator

$$
\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}: \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)
$$

also has a bounded inverse operator. It follows from condition 25.4 and Remark 25.1 that the operator $\mathbf{B}_{\beta}^{2} \mathbf{S}_{q}$ is compact. Using [56, Theorem 16.4] (theorem on compact perturbations of Fredholm operators), we see that $\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}$ is also a Fredholm operator and $\operatorname{ind}\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)=0$. It follows from item 3 of the proof of Lemma 24.6 that $\operatorname{dim} \operatorname{ker}\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)=0$. This means that the operator $\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}$ has a bounded inverse operator.
25.3. Existence of Feller semigroups. In this subsection, we assume that conditions 23.1, 23.2, $24.2-24.6$, and $25.1-25.4$ hold. We prove that unbounded perturbations of an elliptic operator with nonlocal boundary condition described above are generators of Feller semigroups.

Lemma 25.2. If the function $u \in H_{a}^{l+2}(G)$ has a positive maximum at a point $y^{0} \in G$, then $\mathbf{P}(y, D) u\left(y^{0}\right)+\mathbf{P}_{1} u\left(y^{0}\right) \leq 0$.
Proof. Let $u \in H_{a}^{l+2}(G)$ have a positive maximum at a point $y^{0} \in G$. Since $\mathbf{P}(y, D) u \in H_{a}^{l}(G) \subset C(G)$, it follows from the maximum principle 24.1 that $\mathbf{P}(y, D) u\left(y^{0}\right) \leq 0$. From here and condition 25.1 (item 1), we obtain $\mathbf{P}(y, D) u\left(y^{0}\right)+\mathbf{P}_{1} u\left(y^{0}\right) \leq 0$.
Lemma 25.3. Let the function $u \in C_{B}(\bar{G}) \cap H_{a}^{l+2}(G)$ have a positive maximum at a point $y^{0} \in \bar{G}$. Then there exists a point $y^{1} \in G$ such that $u\left(y^{1}\right)=u\left(y^{0}\right)$ and $\mathbf{P}(y, D) u\left(y^{1}\right)+\mathbf{P}_{1} u\left(y^{1}\right) \leq 0$.

The proof is similar to the proof of Lemma 24.4, where the references to Lemma 24.3 must be replaced by the references to Lemma 25.2.
Corollary 25.1. Let $u \in C_{B}(\bar{G}) \cap H_{a}^{l+2}(G)$. Assume that

$$
f_{0}(y)=q u(y)-\mathbf{P}(y, D) u(y)-\mathbf{P}_{1} u(y), \quad y \in G,
$$

where $q>0$. Then there exists a point $y^{1} \in G$ such that

$$
\begin{equation*}
\|u\|_{C(\bar{G})} \leq \frac{1}{q}\left|f_{0}\left(y^{1}\right)\right| . \tag{25.33}
\end{equation*}
$$

The proof is similar to the proof of Corollary 24.1, where the references to Lemma 24.4 must be replaced by the references to Lemma 25.3.

Reducing the problems to the boundary and using Lemma 25.1, we prove that the nonlocal problems under consideration are solvable in spaces of continuous functions for a wide class of right-hand sides.
Lemma 25.4. Let $q \geq q_{1}$, where $q_{1}$ is the same as in Lemma 25.1, and let $f_{0} \in H_{a-1}^{l}(G)$. Then the problem

$$
\begin{align*}
q u(y)-\mathbf{P}(y, D) u(y) & =f_{0}(y), & & y \in G,  \tag{25.34}\\
u(y)-\mathbf{B}_{i} u(y)-\mathbf{B}_{\alpha i} u(y)-\mathbf{B}_{\beta i} u(y) & =0, & & y \in \Gamma_{i}, \quad i=1, \ldots, N,  \tag{25.35}\\
u(y) & =0, & & y \in \mathcal{K},
\end{align*}
$$

has a unique solution $u \in C_{B}(\bar{G}) \cap H_{a}^{l+2}(G)$.
Proof. Consider the auxiliary problem

$$
\begin{align*}
q v(y)-\mathbf{P}(y, D) v(y) & =f_{0}(y), & & y \in G,  \tag{25.36}\\
v(y)-\mathbf{B}_{i} v(y) & =0, & & y \in \Gamma_{i}, \quad i=1, \ldots, N . \tag{25.37}
\end{align*}
$$

Similarly to the proof of Lemma 24.7, using Theorem 23.2 and asymptotic formulas for solutions of nonlocal problems, we can show that problem (25.36), (25.37) has a unique solution $v \in H_{\mathcal{N}, a}^{l+2}(G)$.

Assume that $w=u-v$. Obviously, the unknown function $w$ belongs to the space $H_{\mathcal{N}, a}^{l+2}(G)$ if and only if $u \in H_{\mathcal{N}, a}^{l+2}(G)$ and $u$ is a solution of the problem

$$
\begin{aligned}
q w-\mathbf{P}(y, D) w=0, & y \in G, \\
w(y)-\mathbf{B}_{i} w(y)-\mathbf{B}_{\alpha i} w(y)-\mathbf{B}_{\beta i} w(y)=\mathbf{B}_{\alpha i} v(y)+\mathbf{B}_{\beta i} v(y), & y \in \Gamma_{i}, \quad i=1, \ldots, N, \\
w(y)=0, & y \in \mathcal{K} .
\end{aligned}
$$

Using Remark 25.1 and the fact that the operator $\mathbf{B}_{\alpha \beta}: H_{\mathcal{N}, a}^{l+2}(G) \rightarrow \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)$ is bounded, we see that this problem is equivalent to the operator equation $\psi-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q} \psi=\mathbf{B}_{\alpha \beta} v$ for the unknown function
$\psi \in \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)$. It follows from Lemma 25.1 that this equation has a unique solution $\psi \in \mathcal{H}_{\mathcal{N}, a}^{l+3 / 2}(\partial G)$. Hence problem (25.34), (25.35) also has a unique solution

$$
u=v+w=v+\mathbf{S}_{q} \psi=v+\mathbf{S}_{q}\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)^{-1} \mathbf{B}_{\alpha \beta} v \in H_{\mathcal{N}, a}^{l+2}(G) .
$$

Since $u$ satisfy nonlocal condition (25.35), we see that $u \in C_{B}(\bar{G}) \cap H_{a}^{l+2}(G)$.
Note that the solution $u$ of problem (25.34), (25.35) satisfies the inequality

$$
\begin{equation*}
\|u\|_{H_{a}^{l+2}(G)} \leq c\left\|f_{0}\right\|_{H_{a-1}^{l}(G)} \tag{25.38}
\end{equation*}
$$

where $c>0$ is independent of $u$ and $f$. This follows from Theorem 23.2 and the boundedness of the embedding operator $H_{a-1}^{l}(G) \subset H_{a}^{l}(G)$.

Using Lemma 25.5 and assumptions about unbounded perturbations (see condition 25.1), we prove that the perturbed problem is also solvable in the space of continuous functions for a wide class of right-hand sides.
Lemma 25.5. Let $q \geq q_{1}$, where $q_{1}$ is sufficiently large, and let $f_{0} \in H_{a-1}^{l}(G)$. Then the problem

$$
\begin{align*}
q u-\left(\mathbf{P}(y, D)+\mathbf{P}_{1}\right) u & =f_{0}(y), & & y \in G,  \tag{25.39}\\
u(y)-\mathbf{B}_{i} u(y)-\mathbf{B}_{\alpha i} u(y)-\mathbf{B}_{\beta i} u(y) & =0, & & y \in \Gamma_{i}, \quad i=1, \ldots, N,  \tag{25.40}\\
u(y) & =0, & & y \in \mathcal{K},
\end{align*}
$$

has a unique solution $u \in C_{B}(\bar{G}) \cap H_{a}^{l+2}(G)$.
Proof. Denote by $I$ the bounded operator acting from $H_{a}^{l+2}(G)$ to $H_{a-1}^{l}(G)$ by the formula $I u=u$. Let us consider the operator $q I-\mathbf{P}(y, D)$ as an operator acting from $H_{a}^{l+2}(G)$ to $H_{a-1}^{l}(G)$ with the domain

$$
\mathrm{D}(q I-\mathbf{P}(y, D))=\left\{u \in C_{B}(\bar{G}) \cap H_{a}^{l+2}(G): q u-\mathbf{P}(y, D) u \in H_{a-1}^{l}(G)\right\} .
$$

By Lemma 25.4 and inequality (25.38), there exists a bounded operator

$$
(q I-\mathbf{P}(y, D))^{-1}: H_{a-1}^{l}(G) \rightarrow H_{a}^{l+2}(G)
$$

In particular, this means that $q I-\mathbf{P}(y, D)$ is a Fredholm operator and $\operatorname{ind}(q I-\mathbf{P}(y, D))=0$.
Introduce the operator

$$
q I-\mathbf{P}(y, D)-\mathbf{P}_{1}: H_{a}^{l+2}(G) \rightarrow H_{a-1}^{l}(G)
$$

with domain $\mathrm{D}\left(q I-\mathbf{P}(y, D)-\mathbf{P}_{1}\right)=\mathrm{D}(q I-\mathbf{P}(y, D))$. Rewrite this operator in the form

$$
\begin{equation*}
q I-\mathbf{P}(y, D)-\mathbf{P}_{1}=\left[I-\left(\mathbf{P}_{1 \varrho}^{1}+\mathbf{P}_{1 \varrho}^{2}\right)(q I-\mathbf{P}(y, D))^{-1}\right](q I-\mathbf{P}(y, D)) \tag{25.41}
\end{equation*}
$$

It follows from condition 25.1 (item 3) that if $\varrho=\varrho(q)>0$ is sufficiently small, then the operator

$$
I-\mathbf{P}_{1 \varrho}^{1}(q I-\mathbf{P}(y, D))^{-1}: H_{a-1}^{l}(G) \rightarrow H_{a-1}^{l}(G)
$$

is an isomorphism and the operator

$$
\mathbf{P}_{1 \varrho}^{2}(q I-\mathbf{P}(y, D))^{-1}: H_{a-1}^{l}(G) \rightarrow H_{a-1}^{l}(G)
$$

is compact. Hence, by virtue of the results of [56, Sec. 16] and [56, Theorem 12.2], the operator $q I-\mathbf{P}(y, D)-\mathbf{P}_{1}: H_{a}^{l+2}(G) \rightarrow H_{a-1}^{l}(G)$ is a Fredholm operator and $\operatorname{ind}\left(q I-\mathbf{P}(y, D)-\mathbf{P}_{1}\right)=0$.

If $u \in \operatorname{ker}\left(q I-\mathbf{P}(y, D)-\mathbf{P}_{1}\right)$, then $u \in C_{B}(\bar{G}) \cap H_{a}^{l+2}(G)$ and

$$
\left(q I-\mathbf{P}(y, D)-\mathbf{P}_{1}\right) u=0 .
$$

Hence, by Corollary 25.1, we have $u=0$. Thus, $\operatorname{dim} \operatorname{ker}\left(q I-\mathbf{P}(y, D)-\mathbf{P}_{1}\right)=0$ and the operator $q I-\mathbf{P}(y, D)-\mathbf{P}_{1}$ has a bounded inverse operator. To complete the proof, we note that

$$
\mathcal{R}\left(\left(q I-\mathbf{P}(y, D)-\mathbf{P}_{1}\right)^{-1}\right)=\mathrm{D}(q I-\mathbf{P}(y, D)) \subset C_{B}(\bar{G}) \cap H_{a}^{l+2}(G) .
$$

Consider the unbounded operator $\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ defined by the formula

$$
\begin{aligned}
\mathbf{P}_{B} u & =\mathbf{P}(y, D) u+\mathbf{P}_{1} u \\
u \in \mathrm{D}\left(\mathbf{P}_{B}\right)=\left\{u \in C_{B}(\bar{G})\right. & \left.\cap H_{a}^{l+2}(G): \mathbf{P}(y, D) u+\mathbf{P}_{1} u \in C_{B}(\bar{G})\right\} .
\end{aligned}
$$

Note that, by the relation $l \geq 2$ and the Sobolev embedding theorem, $\mathrm{D}\left(\mathbf{P}_{B}\right) \subset C^{2}(G) \cap C_{B}(\bar{G})$.
Prove that the domain of the nonlocal operator is dense in $C_{B}(\bar{G})$ (one of assumptions of the Hille-Yosida theorem).

Lemma 25.6. The set $\mathrm{D}\left(\mathbf{P}_{B}\right)$ is dense in $C_{B}(\bar{G})$.
Proof. 1. Let $u \in C_{B}(\bar{G})$. By Eq. (24.14), we have $C_{B}(\bar{G}) \subset C_{\mathcal{N}}(\bar{G})$. Then for any $\varepsilon>0$ and $q \geq q_{1}$, there exists a function $u_{1} \in C^{\infty}(\bar{G}) \cap C_{\mathcal{N}}(\bar{G})$ such that

$$
\begin{equation*}
\left\|u-u_{1}\right\|_{C(\bar{G})} \leq \min \left(\varepsilon, \varepsilon /\left(2 c_{1} k_{q}\right)\right) \tag{25.42}
\end{equation*}
$$

where $k_{q}=\left\|\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)^{-1}\right\|_{\mathcal{C}_{\mathcal{N}}(\partial G) \rightarrow \mathcal{C}_{\mathcal{N}}(\partial G)}$.
Assume that

$$
\begin{align*}
f_{0}(y) \equiv q u_{1}-\mathbf{P}(y, D) u_{1}, & y \in G, \\
\psi_{i}(y) \equiv u_{1}(y)-\mathbf{B}_{i} u_{1}(y)-\mathbf{B}_{\alpha i} u_{1}(y)-\mathbf{B}_{\beta i} u_{1}(y), & y \in \Gamma_{i}, \quad i=1, \ldots, N . \tag{25.43}
\end{align*}
$$

Since $u_{1} \in C_{\mathcal{N}}(\bar{G})$, we see that $\left\{\psi_{i}\right\} \in \mathcal{C}_{\mathcal{N}}(\partial G)$. Using the relation

$$
u(y)-\mathbf{B}_{i} u(y)-\mathbf{B}_{\alpha i} u(y)-\mathbf{B}_{\beta i} u(y)=0, \quad y \in \Gamma_{i},
$$

inequality (25.42), and Lemma 24.5, we obtain the following inequality:

$$
\begin{equation*}
\left\|\left\{\psi_{i}\right\}\right\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \leq\left\|u-u_{1}\right\|_{C(\bar{G})}+\left\|\left(\mathbf{B}+\mathbf{B}_{\alpha \beta}\right)\left(u-u_{1}\right)\right\|_{C_{\mathcal{N}}(\partial G)} \leq \varepsilon /\left(c_{1} k_{q}\right) . \tag{25.44}
\end{equation*}
$$

Consider the following auxiliary nonlocal problem:

$$
\begin{align*}
q u_{2}-\mathbf{P}(y, D) u_{2}=f_{0}(y), & y \in G, \\
u_{2}(y)-\mathbf{B}_{i} u_{2}(y)-\mathbf{B}_{\alpha i} u_{2}(y)-\mathbf{B}_{\beta i} u_{2}(y)=0, & y \in \Gamma_{i}, \quad i=1, \ldots, N,  \tag{25.45}\\
u_{2}(y)=0, & y \in \mathcal{K} .
\end{align*}
$$

Since $f_{0} \in C^{\infty}(\bar{G}) \subset H_{a-1}^{l}(G)$, we obtain, by Lemma 25.4, that problem (25.45) has a unique solution $u_{2} \in C_{B}(\bar{G}) \cap H_{a}^{l+2}(G)$.

Using (25.43), (25.45), and the relations $u_{1}(y)=u_{2}(y)=0$ and $y \in \mathcal{K}$, we see that the function $w_{1}=u_{1}-u_{2}$ satisfies the relations

$$
\begin{align*}
q w_{1}-\mathbf{P}(y, D) w_{1}=0, & y \in G, \\
w_{1}(y)-\mathbf{B}_{i} w_{1}(y)-\mathbf{B}_{\alpha i} w_{1}(y)-\mathbf{B}_{\beta i} w_{1}(y)=\psi_{i}(y), & y \in \Gamma_{i}, \quad i=1, \ldots, N,  \tag{25.46}\\
w_{1}(y)=0, & y \in \mathcal{K} .
\end{align*}
$$

By Eq. (24.6), problem (25.46) is equivalent to the equation $\varphi-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q} \varphi=\psi$ in $\mathcal{C}_{\mathcal{N}}(\partial G)$, where $w_{1}=$ $\mathbf{S}_{q} \varphi$. By Lemma 24.6, this equation has a unique solution $\varphi \in \mathcal{C}_{\mathcal{N}}(\partial G)$. Hence, using Theorem 23.3 and inequality (25.44), we obtain the inequality

$$
\begin{equation*}
\left\|w_{1}\right\|_{C(\bar{G})} \leq c_{1}\left\|\left(\mathbf{I}-\mathbf{B}_{\alpha \beta} \mathbf{S}_{q}\right)^{-1}\right\| \cdot\left\|\left\{\psi_{i}\right\}\right\|_{\mathcal{C}_{\mathcal{N}}(\partial G)} \leq c_{1} k_{q} \varepsilon /\left(c_{1} k_{q}\right)=\varepsilon . \tag{25.47}
\end{equation*}
$$

2. Finally, we consider the problem

$$
\begin{align*}
\lambda u_{3}-\mathbf{P}(y, D) u_{3}-\mathbf{P}_{1} u_{3}=\lambda u_{2}, & y \in G, \\
u_{3}(y)-\mathbf{B}_{i} u_{3}(y)-\mathbf{B}_{\alpha i} u_{3}(y)-\mathbf{B}_{\beta i} u_{3}(y)=0, & y \in \Gamma_{i}, \quad i=1, \ldots, N,  \tag{25.48}\\
u_{3}(y)=0, & y \in \mathcal{K} .
\end{align*}
$$

Since $u_{2} \in\left(C_{B}(\bar{G}) \cap H_{a}^{l+2}(G)\right) \subset\left(C_{B}(\bar{G}) \cap H_{a-1}^{l}(G)\right)$, using Lemma 25.5, we obtain that problem (25.48) has a unique solution $u_{3} \in \mathrm{D}\left(\mathbf{P}_{B}\right)$ if $\lambda$ are sufficiently large.

Denote $w_{2}=u_{2}-u_{3} \in C_{B}(\bar{G}) \cap H_{a}^{l+2}(G)$. It follows from Eqs. (25.45) and (25.48) that

$$
\lambda w_{2}-\mathbf{P}(y, D) w_{2}-\mathbf{P}_{1} w_{2}=-\mathbf{P}(y, D) u_{2}-\mathbf{P}_{1} u_{2}=f_{0}-q u_{2}-\mathbf{P}_{1} u_{2} .
$$

Using Corollary 25.1, we have

$$
\left\|w_{2}\right\|_{C(\bar{G})} \leq \frac{1}{\lambda}\left(\left\|f_{0}\right\|_{C(\bar{G})}+q\left\|u_{2}\right\|_{C(\bar{G})}+\sup _{y \in G}\left|\mathbf{P}_{1} u_{2}(y)\right|\right) .
$$

By condition 25.1 (item 2), the value $\sup _{y \in G}\left|\mathbf{P}_{1} u_{2}(y)\right|$ is finite. Thus, choosing sufficiently large $\lambda$, we obtain the inequality

$$
\begin{equation*}
\left\|w_{2}\right\|_{C(\bar{G})} \leq \varepsilon \tag{25.49}
\end{equation*}
$$

It follows from Eqs. (25.42), (25.47), and (25.49) that

$$
\left\|u-u_{3}\right\|_{C(\bar{G})} \leq\left\|u-u_{1}\right\|_{C(\bar{G})}+\left\|u_{1}-u_{2}\right\|_{C(\overline{\bar{G}})}+\left\|u_{2}-u_{3}\right\|_{C(\bar{G})} \leq 3 \varepsilon .
$$

Let us verify that other assumptions of the Hille-Yosida theorem also hold.
Lemma 25.7. (1) The operator $\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ admits a closure $\overline{\mathbf{P}_{B}}$.
(2) Let $q_{1}$ be sufficiently large. Then for any $q \geq q_{1}$, the operator

$$
q I-\overline{\mathbf{P}_{B}}: \mathrm{D}\left(\overline{\mathbf{P}_{B}}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})
$$

has a bounded inverse operator $\left(q I-\overline{\mathbf{P}_{B}}\right)^{-1}: C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ and

$$
\left\|\left(q I-\overline{\mathbf{P}_{B}}\right)^{-1}\right\| \leq 1 / q .
$$

(3) The operator $\left(q I-\overline{\mathbf{P}_{B}}\right)^{-1}$ is nonnegative.

Proof. 1. First, we consider the auxiliary operator

$$
\mathbf{P}: \mathrm{D}(\mathbf{P}) \subset C_{B}(\bar{G}) \rightarrow C(\bar{G})
$$

defined by the formula

$$
\begin{gathered}
\mathbf{P} u=\mathbf{P}(y, D) u+\mathbf{P}_{1} u \\
u \in \mathrm{D}(\mathbf{P})=\left\{u \in C_{B}(\bar{G}) \cap H_{a}^{l+2}(G): \mathbf{P}(y, D) u+\mathbf{P}_{1} u \in C(\bar{G})\right\} .
\end{gathered}
$$

Show that $\mathbf{P}$ is closable. Consider an arbitrary sequence $\left\{u_{n}\right\}_{n=1}^{\infty} \subset \mathrm{D}(\mathbf{P})$ such that $u_{n} \rightarrow 0$ and $\mathbf{P} u_{n} \rightarrow v$ in $C(\bar{G})$, where $v \in C(\bar{G})$. Assume that $v \neq 0$. Then there is a point $y^{0} \in G$ and its neighborhood $U \subset G$ such that

$$
\begin{equation*}
\mathbf{P} u_{n}(y)>\varepsilon, \quad y \in U, \tag{25.50}
\end{equation*}
$$

for some $\varepsilon>0$. Consider a function $h \in C_{0}^{\infty}(G)$ such that $h\left(y^{0}\right)=1$ and $h(y)=0$ for $y \in \bar{G} \backslash U$. Introduce the functions

$$
\hat{u}_{n}(y)=u_{n}(y)+\frac{\varepsilon h(y)}{1+\sup _{y^{\prime} \in G}\left|\mathbf{P} h\left(y^{\prime}\right)\right|}, \quad n=1,2, \ldots
$$

We have

$$
\left\{\begin{array}{l}
\hat{u}_{n}\left(y^{0}\right)=u_{n}\left(y^{0}\right)+\frac{\varepsilon}{1+\sup _{y^{\prime} \in G}\left|\mathbf{P} h\left(y^{\prime}\right)\right|} \geq \frac{\varepsilon}{2+2 \sup _{y^{\prime} \in G}\left|\mathbf{P} h\left(y^{\prime}\right)\right|}, \\
\hat{u}_{n}(y)=u_{n}(y), \quad y \in \bar{G} \backslash U,
\end{array}\right.
$$

for all sufficiently large $n$. Hence, for sufficiently large $n$, every function $\hat{u}_{n}$ has a positive maximum at some point $\hat{y}^{n} \in U$, and by Lemma 25.2, $\mathbf{P} \hat{u}_{n}\left(\hat{y}^{n}\right) \leq 0$.

However, by Eq. (25.50) we have

$$
\mathbf{P} \hat{u}_{n}\left(\hat{y}^{n}\right)=\mathbf{P} u_{n}\left(\hat{y}^{n}\right)+\varepsilon \frac{\mathbf{P} h\left(\hat{y}^{n}\right)}{1+\sup _{y^{\prime} \in G}\left|\mathbf{P} h\left(y^{\prime}\right)\right|}>\mathbf{P} u_{n}\left(\hat{y}^{n}\right)-\varepsilon>0 .
$$

This contradiction proves that $v=0$ and that the operator $\mathbf{P}$ is closable.
2. It is known that $C^{l}(\bar{G}) \subset H_{a-1}^{l}(G)$. Then, by Lemma $25.5, C^{l}(\bar{G}) \subset \mathcal{R}(q I-\mathbf{P})$. Hence, the image $\mathcal{R}(q I-\mathbf{P})$ is dense in $C(\bar{G})$.

On the other hand, according to Corollary 25.1, we have

$$
\begin{equation*}
\|u\|_{C_{B}(\bar{G})} \leq \frac{1}{q}\|(q I-\overline{\mathbf{P}}) u\|_{C(\bar{G})} \tag{25.51}
\end{equation*}
$$

for any $u \in \mathrm{D}(\mathbf{P})$. It follows from here, the fact that the operator $q I-\overline{\mathbf{P}}$ is closed, and the denseness of $\mathcal{R}(q I-\mathbf{P})$ in $C(\bar{G})$ that there exists a bounded operator $(q I-\overline{\mathbf{P}})^{-1}: C(\bar{G}) \rightarrow C_{B}(\bar{G})$ and for any $u \in \mathrm{D}(\overline{\mathbf{P}})$ estimate (25.51) is valid.
3. Let us prove that the operator $(q I-\overline{\mathbf{P}})^{-1}$ is nonnegative. First, we take an arbitrary function $f \in C(\bar{G})$ such that $f(y)>0, y \in \bar{G}$. Since the image $\mathcal{R}(q I-\mathbf{P})$ is dense in $C(\bar{G})$, we can find a sequence $f_{n} \in \mathcal{R}(q I-\mathbf{P})$ such that $f_{n}(y)>0, y \in \bar{G}$, and $f_{n} \rightarrow f$ in $C(\bar{G})$. Hence, using Lemma 25.3, we obtain the relation

$$
(q I-\overline{\mathbf{P}})^{-1} f=\lim _{n \rightarrow \infty}(q I-\mathbf{P})^{-1} f_{n} .
$$

If $f \in C(\bar{G})$ and $f(y) \geq 0, y \in \bar{G}$, then there exists a sequence $F_{n} \in C(\bar{G})$ such that $F_{n}(y) \geq 0, y \in \bar{G}$, and $F_{n} \rightarrow f$ in $C(\bar{G})$. Hence, $(q I-\overline{\mathbf{P}})^{-1} f \geq 0$ for any $f \in C(\bar{G})$ such that $f(y) \geq 0, y \in \bar{G}$.
4. Now we consider the operator $\mathbf{P}_{B}$. Since $\mathbf{P}_{B} \subset \mathbf{P}$, we see that $\mathbf{P}_{B}$ is closable (i.e., item 1 is proved).

Since $\mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \cap H_{a}^{l+2}(G)$ and, by Lemma $25.6, \mathrm{D}\left(\mathbf{P}_{B}\right)$ is dense in $C_{B}(\bar{G})$, we see that the set $C_{B}(\bar{G}) \cap H_{a}^{l+2}(G)$ is also dense in $C_{B}(\bar{G})$. Therefore, by Lemma 25.5 , the image $\mathcal{R}\left(q I-\mathbf{P}_{B}\right)$ is dense in $C_{B}(\bar{G})$.

On the other hand, according to Corollary 25.1

$$
\|u\|_{C_{B}(\bar{G})} \leq \frac{1}{q}\left\|\left(q I-\overline{\mathbf{P}}_{B}\right) u\right\|_{C(\bar{G})} \quad \forall u \in \mathrm{D}\left(\mathbf{P}_{B}\right) .
$$

Item 2 follows from here, the fact that the operator $q I-\overline{\mathbf{P}_{B}}$ is closed, and the denseness of $\mathcal{R}\left(q I-\mathbf{P}_{B}\right)$ in $C_{B}(\bar{G})$.
3. Item 3 follows from the nonnegativeness of $(q I-\overline{\mathbf{P}})^{-1}$ and the relation

$$
\left(q I-\overline{\mathbf{P}_{B}}\right)^{-1} \subset(q I-\overline{\mathbf{P}})^{-1} .
$$

The lemma is proved.
Lemmas 25.6 and 25.7 and the Hille-Yosida theorem (Theorem 23.1) yield the main result of this subsection.

Theorem 25.1. The operator $\overline{\mathbf{P}_{B}}: \mathrm{D}\left(\overline{\mathbf{P}_{B}}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ generates a Feller semigroup.
Further, we give an example of nonlocal operators that satisfy the conditions of this subsection.
25.4. Example. Let $\partial G=\Gamma_{1} \cup \Gamma_{2} \cup \mathcal{K}$, where $\Gamma_{1}$ and $\Gamma_{2}$ are curves of class $C^{\infty}$, open in topology $\partial G$, $\Gamma_{1} \cap \Gamma_{2}=\varnothing$ and $\overline{\Gamma_{1}} \cap \overline{\Gamma_{2}}=\mathcal{K}$; the set $\mathcal{K}$ consists of two points $g_{1}$ and $g_{2}$. Denote by $\Omega_{j}, j=1, \ldots, 4$, nondegenerate transformations of class $C^{l+2}$ defined in some neighborhood $\overline{\Gamma_{1}}$ and satisfying the following conditions (see Fig. 25.1):
(1) $\Omega_{1}(\mathcal{K})=\mathcal{K}, \Omega_{1}\left(\Gamma_{1} \cap \mathcal{O}_{\varepsilon}(\mathcal{K})\right) \subset G, \Omega_{1}\left(\Gamma_{1} \backslash \mathcal{O}_{\varepsilon}(\mathcal{K})\right) \subset G \cup \Gamma_{2}$ and $\Omega_{1}(y)$ is the composition of an argument shift, rotation, and dilation for $y \in \overline{\Gamma_{1}} \cap \mathcal{O}_{\varepsilon}(\mathcal{K})$;
(2) there exist numbers $\varkappa_{1}>\varkappa_{2}>0$ and $\sigma>0$ such that

$$
\Omega_{2}\left(\overline{\Gamma_{1}}\right) \subset \bar{G} \backslash \mathcal{O}_{\varkappa_{1}}(\mathcal{K}), \quad \Omega_{2}\left(\overline{\Gamma_{1}} \backslash \mathcal{O}_{\varkappa_{2}}(\mathcal{K})\right) \subset \overline{G_{\sigma}}
$$

moreover, $\Omega_{2}\left(g_{1}\right) \in \Gamma_{1}$ and $\Omega_{2}\left(g_{2}\right) \in G$;
(3) $\Omega_{3}\left(\overline{\Gamma_{1}}\right) \subset G \cup \Gamma_{2}$ and $\Omega_{3}(\mathcal{K}) \subset \Gamma_{2}$;
(4) $\Omega_{4}\left(\overline{\Gamma_{1}}\right) \subset G \cup \overline{\Gamma_{2}}$ and $\Omega_{4}(\mathcal{K})=\mathcal{K}$; the angle between rays tangent to $\Gamma_{1}$ and $\Omega_{4}\left(\Gamma_{1}\right)$ at the point $g_{j}$ is not zero.


Fig. 25.1. Nontransversal nonlocal conditions
Introduce the functions $b_{j} \in C^{l+2}\left(\overline{\Gamma_{1}}\right), b_{j} \geq 0, j=1, \ldots, 4$. Let $G_{1}$ be a bounded domain, $G_{1} \subset G$, and $\Gamma \subset \bar{G}$ be a curve of class $C^{1}$, and let $c(y, \eta)$ and $d(y, \eta)$ be nonnegative functions,

$$
D_{y}^{\alpha} c(y, \eta) \in C\left(\bar{G} \times \overline{G_{1}}\right), \quad D_{y}^{\alpha} d(y, \eta) \in C(\bar{G} \times \bar{\Gamma}), \quad|\alpha| \leq l+2 .
$$

Consider the following nonlocal conditions:

$$
\begin{gathered}
u(y)-\sum_{j=1}^{4} b_{j}(y) u\left(\Omega_{j}(y)\right)-\int_{G_{1}} c(y, \eta) u(\eta) d \eta-\int_{\Gamma} d(y, \eta) u(\eta) d \Gamma_{\eta}=0, \quad y \in \Gamma_{1}, \\
u(y)=0, \quad y \in \overline{\Gamma_{2}} .
\end{gathered}
$$

Let $Q \subset \bar{G}$ be an arbitrary Borel set. Introduce the following measures $\mu_{i}(y, \cdot)$ :

$$
\begin{gathered}
\mu_{1}(y, Q)=\sum_{j=1}^{4} b_{j}(y) \chi_{Q}\left(\Omega_{j}(y)\right)+\int_{G_{1} \cap Q} c(y, \eta) d \eta+\int_{\Gamma \cap Q} d(y, \eta) u(\eta) d \Gamma_{\eta}, \quad y \in \Gamma_{1}, \\
\mu_{2}(y, Q)=0, \quad y \in \Gamma_{2} .
\end{gathered}
$$

Define the sets $\mathcal{N}$ and $\mathcal{M}$ as above. Assume that

$$
\begin{gathered}
\mu_{1}(y, \bar{G}) \leq 1, y \in \Gamma_{1} ; \quad \int_{\Gamma \cap \mathcal{M}} d(y, \eta) d \Gamma_{\eta}<1, \quad y \in \mathcal{M} ; \\
b_{2}\left(g_{1}\right)=0 \quad \text { or } \quad \mu\left(\Omega_{2}\left(g_{1}\right), \bar{G}\right)=0, \quad b_{2}\left(g_{2}\right)=0 ; \quad b_{4}\left(g_{j}\right)=0 ; \quad c\left(g_{j}, \cdot\right)=0 ; \quad d\left(g_{j}, \cdot\right)=0 .
\end{gathered}
$$

Introduce a patch function $\zeta \in C^{\infty}\left(\mathbb{R}^{2}\right)$ with the support in $\mathcal{O}_{\varepsilon}(\mathcal{K})$, which is equal to 1 on $\mathcal{O}_{\varepsilon / 2}(\mathcal{K})$ and such that $0 \leq \zeta(y) \leq 1, y \in \mathbb{R}^{2}$. Let $y \in \overline{\Gamma_{1}}$ and let $Q \subset \bar{G}$ be an arbitrary Borel set. Then the measures

$$
\begin{gathered}
\delta(y, Q)=\zeta(y) b_{1}(y) \chi_{Q}\left(\Omega_{1}(y)\right), \quad \alpha(y, Q)=b_{2}(y) \chi_{Q}\left(\Omega_{2}(y)\right), \\
\beta^{1}(y, Q)=(1-\zeta(y)) b_{1}(y) \chi_{Q}\left(\Omega_{1}(y)\right)+\sum_{j=3,4} b_{j}(y) \chi_{Q}\left(\Omega_{j}(y)\right), \\
\beta^{2}(y, Q)=\int_{G_{1} \cap Q} c(y, \eta) d \eta+\int_{\Gamma \cap Q} d(y, \eta) u(\eta) d \Gamma_{\eta} .
\end{gathered}
$$

satisfy conditions $6.3,23.2,24.2-24.6$, and 25.2-25.4.

## 26. Nonexistence of Feller Semigroup

26.1. Laplace operator in weight spaces. In this section, we construct examples in which the closures of the operators corresponding to nonlocal problems do not generate Feller semigroups.

Obtain the asymptotic of solutions of the equation

$$
\begin{equation*}
\Delta u=f(y), \quad y \in \mathbb{R}^{2} \backslash\{0\} \tag{26.1}
\end{equation*}
$$

in the space $H_{a}^{l}\left(\mathbb{R}^{2}\right)$.
Writing Eq. (26.1) in the polar coordinates and applying the Mellin transform with respect to $r$, we obtain the following auxiliary problem:

$$
\begin{gathered}
\frac{d^{2} \tilde{u}(\omega, \lambda)}{d \omega^{2}}-\lambda^{2} \tilde{u}(\omega, \lambda)=\tilde{F}(\omega, \lambda), \quad 0<\omega<2 \pi \\
\tilde{u}(0, \lambda)=\tilde{u}(2 \pi, \lambda), \quad \frac{d \tilde{u}(0, \lambda)}{d \omega}=\frac{d \tilde{u}(2 \pi, \lambda)}{d \omega}
\end{gathered}
$$

where $\tilde{F}(\omega, \lambda)$ is the Mellin transform of the function $r^{2} f(\omega, r)$, and $\lambda$ is a complex parameter.
Consider the corresponding eigenvalue problem:

$$
\begin{gather*}
\varphi^{\prime \prime}(\omega)-\lambda^{2} \varphi(\omega)=0, \quad 0<\omega<2 \pi \\
\varphi(0)=\varphi(2 \pi), \quad \varphi^{\prime}(0)=\varphi^{\prime}(2 \pi) \tag{26.2}
\end{gather*}
$$

The numbers $\lambda_{s}=s i, s=0, \pm 1, \pm 2, \ldots$, are eigenvalues. The eigenvector $\varphi_{0}(\omega) \equiv 1$ corresponds to the eigenvalue $\lambda_{0}=0$; moreover, there exists an adjoint vector $\hat{\varphi}_{0}(\omega) \equiv 0$. The eigenvectors $\varphi_{s}(\omega)=\cos s \omega$ correspond to the eigenvalues $\lambda_{s}, s= \pm 1, \pm 2, \ldots$; if $s \neq 0$, then there are no adjoint vectors.

From [26, Theorem 5.1], we obtain the following result.
Lemma 26.1. Let $f \in H_{a}^{0}\left(\mathbb{R}^{2}\right) \cap H_{a^{\prime}}^{0}\left(\mathbb{R}^{2}\right)$. Assume that the number $a^{\prime}$ is not an integer, $a^{\prime} \neq a$, and $\mathcal{S}$ denotes the set of integers concentrated in the interval $\left(\min \left(a, a^{\prime}\right), \max \left(a, a^{\prime}\right)\right)$. If $u$ is a solution of problem (26.1) from the space $H_{a}^{2}\left(\mathbb{R}^{2}\right)$, then

$$
u= \begin{cases}c_{0}+\hat{c}_{0} \ln r+\sum_{s \in \mathcal{S} \backslash\{0\}} r^{s}\left(c_{s} \cos s \omega+d_{s} \sin s \omega\right)+u^{\prime}, & \text { if } 0 \in \mathcal{S}, \\ \sum_{s \in \mathcal{S}} r^{s}\left(c_{s} \cos s \omega+d_{s} \sin s \omega\right)+u^{\prime}, & \text { if } 0 \notin \mathcal{S},\end{cases}
$$

where $c_{s}$ and $d_{s}, s \in \mathcal{S}$, and $\hat{c}_{0}$ are some constants and $u^{\prime}$ is a solution of problem (26.1) from the space $H_{a^{\prime}}^{2}\left(\mathbb{R}^{2}\right)$.
26.2. "Jumps" outside the neighborhood of termination points of the process with nonzero probability.
26.2.1. Statement of nonlocal problem. Here we show that condition 24.6 is substantial.

Let $G \subset \mathbb{R}^{2}$ be a bounded domain with smooth boundary $\partial G=\Gamma_{1} \cup \Gamma_{2} \cup \mathcal{K}$, where $\Gamma_{1}$ and $\Gamma_{2}$ are open and connected (in the topology of $\partial G$ ) curves of class $C^{\infty}$ such that $\Gamma_{1} \cap \Gamma_{2}=\varnothing$ and $\overline{\Gamma_{1}} \cap \overline{\Gamma_{2}}=\mathcal{K}$. Let the set $\mathcal{K}$ consist of two points $g_{1}$ and $g_{2}$. Assume that in some neighborhood of the points $g_{i}$, $i=1,2$, the domain $G$ coincides with a plane angle $\pi$.

Consider the problem (see Fig. 26.1)

$$
\begin{align*}
\Delta u(y)=f_{0}(y), & y \in G,  \tag{26.3}\\
u(y)-b_{1}(y) u\left(\Omega_{1}(y)\right)=0, & y \in \Gamma_{1},  \tag{26.4}\\
u(y)=0, & y \in \overline{\Gamma_{2}}, \tag{26.5}
\end{align*}
$$

where the function $b_{1} \in C^{\infty}\left(\overline{\Gamma_{1}}\right)$ is such that
(1) $0 \leq b_{1}(y) \leq 1$,
(2) $b_{1}(y)=b_{1}^{*}>0$ for $y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right)$,
(3) $b_{1}(y)=0$ as $y \notin \mathcal{O}_{2 \varepsilon}\left(g_{1}\right)$,
and $\Omega_{1}$ is a smooth nondegenerate manifold defined in some neighborhood of a curve $\overline{\Gamma_{1}}$; moreover,
(1) $\Omega_{1}\left(\Gamma_{1}\right) \subset G, \Omega_{1}\left(g_{1}\right) \in G, \Omega_{1}\left(g_{2}\right)=g_{2}$,
(2) $\Omega_{1}(y)$ for $y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right)$ is the composition of a rotation about the point $g_{1}$ and a shift by some vector.


Fig. 26.1. Problem (26.3)-(26.5).
Assume that $g=\Omega_{1}\left(g_{1}\right)$. Let $\varepsilon>0$ be so small that

$$
\mathcal{O}_{\varepsilon}\left(g_{1}\right) \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right)=\varnothing, \quad \mathcal{O}_{\varepsilon}(g) \cap \partial G=\varnothing, \quad \mathcal{O}_{\varepsilon}(g) \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right)=\varnothing, \quad j=1,2 .
$$

Introduce a measure $\alpha(y, \cdot), y \in \partial G$, such that for any Borel set $Q \subset \bar{G}$,

$$
\begin{array}{ll}
\alpha(y, Q)=b_{1}(y) \chi_{Q}\left(\Omega_{1}(y)\right), & y \in \Gamma_{1}, \\
\alpha(y, Q)=0, & y \in \overline{\Gamma_{2}} .
\end{array}
$$

Then boundary conditions (26.4), (26.5) can be rewritten in the form

$$
b(y) u(y)+\int_{\bar{G}}[u(y)-u(\eta)] \alpha(y, d \eta)=0, \quad y \in \partial G
$$

where $b(y)=1-\alpha(y, \bar{G})\left(\right.$ cf. (24.9)). Obviously, $\overline{\Gamma_{2}} \subset \mathcal{N}, \overline{\mathcal{O}_{\varepsilon}\left(g_{2}\right)} \cap \Gamma_{1} \subset \mathcal{N}$, and $\mathcal{M} \subset \Gamma_{1} \backslash \overline{\mathcal{O}_{\varepsilon}\left(g_{2}\right)}$.
Consider the operator

$$
\mathbf{B}_{\alpha 1} u(y)=\int_{\bar{G}} u(\eta) \alpha(y, d \eta)=b_{1}(y) u\left(\Omega_{1}(y)\right), \quad y \in \Gamma_{1} .
$$

It is easy to verify that conditions 23.1, 23.2, 24.1-24.4, 24.7, and 24.8 hold (with $\mathbf{P}(y, D)=\Delta$, $\mathbf{P}_{1}=0$, and $\beta_{i}(y, \bar{G}) \equiv 0$ ). Show that condition 24.6 is violated. Indeed, $\mathbf{B}_{\alpha 1} u \in C\left(\overline{\Gamma_{1}}\right)$ for any $u \in C_{\mathcal{N}}(\bar{G})$ since the function $b_{1}$ and the transformation $\Omega_{1}$ are continuous. However, if $u \in C_{\mathcal{N}}(\bar{G})$ and $u\left(\Omega_{1}\left(g_{1}\right)\right) \neq 0$, then $\lim _{y \rightarrow g_{1}} \mathbf{B}_{\alpha 1} u(y)=b_{1}^{*} u\left(\Omega_{1}\left(g_{1}\right)\right) \neq 0$, i.e., $\mathbf{B}_{\alpha 1} u \notin C_{\mathcal{N}}\left(\overline{\Gamma_{1}}\right)$.

Consider the unbounded operator $\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ defined by the formula

$$
\begin{equation*}
\mathbf{P}_{B} u=\Delta u, \quad u \in \mathrm{D}\left(\mathbf{P}_{B}\right)=\left\{u \in C_{B}(\bar{G}): \Delta u \in C_{B}(\bar{G})\right\}, \tag{26.6}
\end{equation*}
$$

where $C_{B}(\bar{G})$ is the set of functions from $C(\bar{G})$ satisfying nonlocal conditions (26.4) and (26.5).
Lemma 26.2. If $u \in \mathrm{D}\left(\mathbf{P}_{B}\right)$, then $u \in W^{2}\left(G^{\prime}\right)$ for any domain $G^{\prime}$ such that $\overline{G^{\prime}} \subset \bar{G} \backslash \mathcal{K}$.

Proof. Assume that

$$
\begin{equation*}
f=\Delta u . \tag{26.7}
\end{equation*}
$$

Since $f \in L_{2}(G)$, it follows from the theorem on the internal smoothness (see, e.g., [57, Chap. 2, Theorem 3.2]) that $u \in W_{\text {loc }}^{2}(G)$. Hence, it remains to prove that $u \in W^{2}\left(G \cap \mathcal{O}_{R}\left(y^{0}\right)\right)$, where $y^{0}$ is an arbitrary point on $\Gamma_{j}, j=1,2, R=R\left(y^{0}\right)$, and $\overline{\mathcal{O}_{R}\left(y^{0}\right)} \cap \mathcal{K}=\varnothing$.

Consider a domain $G_{R}$ with a smooth boundary $\partial G_{R}$ such that

$$
G \cap \mathcal{O}_{R}\left(y^{0}\right) \subset G_{R} \subset G
$$

If $y^{0} \in \Gamma_{1} \cap \mathcal{O}_{R}\left(y^{0}\right)$, then we consider a function $\psi \in W^{3 / 2}\left(\partial G_{R}\right)$ such that $\psi(y)=b_{1}(y) u\left(\Omega_{1}(y)\right)$ for $y \in \Gamma_{1} \cap \mathcal{O}_{R}\left(y^{0}\right)$. If $y^{0} \in \Gamma_{2} \cap \mathcal{O}_{R}\left(y^{0}\right)$, then we consider a function $\psi \in W^{3 / 2}\left(\partial G_{R}\right)$ such that $\psi(y)=0$ for $y \in \Gamma_{2} \cap \mathcal{O}_{R}\left(y^{0}\right)$.

Let $v \in W^{2}\left(G_{R}\right) \subset C\left(\overline{G_{R}}\right)$ be a solution of the problem

$$
\begin{align*}
\Delta v & =f(y), & & y \in G_{R}, \\
\left.v\right|_{\partial G_{R}} & =\psi(y), & & y \in \partial G_{R} . \tag{26.8}
\end{align*}
$$

It follows from Eqs. (26.7) and (26.8) that the function $w=u-v \in C\left(\overline{G_{R}}\right)$ satisfies the relations

$$
\begin{align*}
\Delta w=0, & y \in G \cap \mathcal{O}_{R}\left(y^{0}\right) \\
w(y)=0, & y \in \partial G \cap \mathcal{O}_{R}\left(y^{0}\right) . \tag{26.9}
\end{align*}
$$

Applying the theorem on the internal smoothness to the Laplace equation in Eq. (26.9), we see that $w \in C^{\infty}\left(G \cap \mathcal{O}_{R}\left(y^{0}\right)\right) \cap C\left(\overline{G \cap \mathcal{O}_{R}\left(y^{0}\right)}\right)$. Further, applying [22, Lemma 6.18] to problem (26.9), we obtain that $w \in C^{2}\left(\overline{G \cap B_{R / 2}\left(y^{0}\right)}\right)$. Hence,

$$
u=w+v \in W^{2}\left(G \cap B_{R / 2}\left(y^{0}\right)\right) .
$$

Since $y^{0} \in \Gamma_{j}$ is arbitrary, the lemma is proved.
Lemma 26.3. The operator $\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ admits a closure $\overline{\mathbf{P}_{B}}$.
Proof. The operator $\mathbf{P}_{B}$ is a restriction of the operator $\mathbf{P}: C(\bar{G}) \rightarrow C(\bar{G})$ defined by the formula

$$
\mathbf{P} u=\Delta u, \quad u \in \mathrm{D}(\mathbf{P})=\{u \in C(\bar{G}): \Delta u \in C(\bar{G})\}
$$

Hence, it suffices to prove that the operator $\mathbf{P}$ admits a closure. Obviously, the inclusion $C^{2}(\bar{G}) \subset \mathrm{D}(\mathbf{P})$ is valid; hence, the set $\mathrm{D}(\mathbf{P})$ is dense in $C(\bar{G})$. Moreover, if $u \in \mathrm{D}(\mathbf{P})$ has a positive maximum at the point $y^{0}$, then by the maximum principle 24.1 we see that $\mathbf{P} u\left(y^{0}\right) \leq 0$. It follows from here and [101, Theorem 9.3.3] that the operator $\mathbf{P}$ admits a closure.

Here we will prove the following result.
Theorem 26.1. Let $\overline{\mathbf{P}_{B}}: \mathrm{D}\left(\overline{\mathbf{P}_{B}}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ be a closure of the operator (26.6). Then $\overline{\mathbf{P}_{B}}$ is not a generator of a Feller semigroup.

By the Hille-Yosida theorem (Theorem 23.1), it suffices to show that the image $\mathcal{R}\left(\overline{\mathbf{P}_{B}}-q \mathbf{I}\right)$ does not coincide with $C_{B}(\bar{G})$ for some $q>0$.
26.2.2. Proof of Theorem 26.1. To prove Theorem 26.1, we obtain an asymptotic of the solution $u \in \mathrm{D}\left(\mathbf{P}_{B}\right)$ of problem (26.3)-(26.5).

Consider the following model problem with a complex parameter $\lambda$, corresponding to the point $g_{1}$ :

$$
\begin{align*}
\varphi^{\prime \prime}(\omega)-\lambda^{2} \varphi(\omega) & =0, \quad 0<\omega<\pi  \tag{26.10}\\
\varphi(0)=\varphi(\pi) & =0, \tag{26.11}
\end{align*}
$$

where $\omega$ and $r$ are the polar coordinates with pole at the point $g_{1}$ such that $(0, r) \in \Gamma_{1}$ and $(\pi, r) \in \Gamma_{2}$ as $0<r<\varepsilon$. Eigenvalues of this problem have the form $\lambda_{1, k}=k i, k= \pm 1, \pm 2, \ldots$, and the corresponding eigenvectors are $\varphi_{1, k}=\sin k \omega$. There are no adjoint vectors.

Since $b_{1}(y)=0$ near the point $g_{2}$, we see that the same model problem corresponds to the point $g_{2}$. In this example, we use the weight spaces

$$
\begin{gathered}
H_{a}^{l}(G)=H_{a}^{l}(G, \mathcal{K} \cup\{g\}), \\
H_{a}^{l}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right)\right)=H_{a}^{l}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right),\left\{g_{j}\right\}\right), \quad H_{a}^{l}\left(\mathcal{O}_{\varepsilon}(g)\right)=H_{a}^{l}\left(\mathcal{O}_{\varepsilon}(g),\{g\}\right),
\end{gathered}
$$

and the corresponding trace spaces. We emphasize that the space $H_{a}^{l}(G)$ (unlike spaces from previous sections) consists of functions that can have power singularities not only near the set $\mathcal{K}$, but also near the point $g$.

Let us fix a number $\delta$ such that

$$
\begin{equation*}
0<\delta<1 \tag{26.12}
\end{equation*}
$$

Lemma 26.4. Let $u \in \mathrm{D}\left(\mathbf{P}_{B}\right)$. Then $u \in H_{1-\delta}^{2}(G)$.
Proof. It follows from Lemma 26.2 that $u \in W^{2}\left(G^{\prime}\right)$ for any domain $G^{\prime}$ such that $\overline{G^{\prime}} \subset \bar{G} \backslash \mathcal{K}$. Since $u(g)=u\left(g_{1}\right) / b_{1}^{*}=0$, by Lemma 5.2 we obtain that $u \in H_{1-\delta}^{2}\left(\mathcal{O}_{\varepsilon}(g)\right)$. Hence, using (26.4) and (26.5) and the fact that the transform $\Omega_{1}$ is a smooth and nondegenerate, we obtain

$$
\begin{gather*}
\Delta u \in C(\bar{G}) \subset H_{1-\delta}^{0}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right),  \tag{26.13}\\
\left.u\right|_{\Gamma_{1} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)} \in H_{1-\delta}^{3 / 2}\left(\Gamma_{1} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right),\left.\quad u\right|_{\Gamma_{2} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}=0 . \tag{26.14}
\end{gather*}
$$

Using (26.13) and (26.14) and the relation $u \in C(\bar{G}) \subset H_{-1+\delta}^{0}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$, we obtain from Lemma 14.2 that

$$
\begin{equation*}
u \in H_{1+\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right) \tag{26.15}
\end{equation*}
$$

According to Eq. (26.12), the strip $-\delta \leq \operatorname{Im} \lambda \leq \delta$ does not contain eigenvalues $\lambda_{1, k}$ of problem (26.10), (26.11). It follows from relation (26.13)-(26.15) and [26, Theorem 2.2] that $u \in H_{1-\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$.

It can be proved similarly that $u \in H_{1-\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right)\right)$.
Introduce the bounded operator $\mathbf{L}_{a}(q): H_{a}^{2}(G) \rightarrow \mathcal{H}_{a}^{0}(G, \partial G)$ by the formula

$$
\mathbf{L}_{a}(q) u=\left\{\Delta u-q u,\left.u\right|_{\Gamma_{1}}-\left.b_{1}(y) u\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}},\left.u\right|_{\Gamma_{2}}\right\}, \quad q \geq 0 .
$$

We also denote $\mathbf{L}_{a}=\mathbf{L}_{a}(0)$.
Lemma 26.5. Let $q>0$ be sufficiently small; then $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1-\delta}(q)=0$.
Proof. First, we assume that $q=0$ and prove that $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1-\delta}=0$. Let $u \in \operatorname{ker} \mathbf{L}_{1-\delta}$. Without loss of generality, we assume that the function $u$ is real-valued. Lemma 26.1 yields the following asymptotic of the function $u$ near the point $g$ :

$$
\begin{equation*}
u(y)=r\left(c_{1} \cos \omega+d_{1} \sin \omega\right)+v(y), \quad y \in \mathcal{O}_{\varepsilon}(g), \tag{26.16}
\end{equation*}
$$

where $\omega$ and $r$ are the polar coordinates with pole at the point $g$ such that $(0, r) \in \Omega\left(\Gamma_{1}\right)$ for $0<r<\varepsilon$, $c_{1}$ and $d_{1}$ are some constants, and $v \in H_{-\delta}^{2}\left(\mathcal{O}_{\varepsilon}(g)\right)$. In particular, it follows from Eq. (26.16) and the Sobolev embedding theorem that $u \in C(\bar{G} \backslash \mathcal{K})$. Applying relations (26.3)-(26.5) and (26.16), we obtain

$$
\begin{gather*}
\Delta u=0, \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right), \\
\left.u\right|_{\Gamma_{1} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}=b_{1}^{*} c_{1} r+w(r),\left.\quad u\right|_{\Gamma_{2} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}=0, \tag{26.17}
\end{gather*}
$$

where $w \in H_{-\delta}^{3 / 2}\left(\Gamma_{1} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$.
Hence, applying [26, Theorem 2.2, Lemma 4.3], we see that

$$
\begin{equation*}
u(y)=r \psi(\omega)+v(y), \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right), \tag{26.18}
\end{equation*}
$$

where $\omega$ and $r$ are the polar coordinates with pole at the point $g_{1}, \psi \in C^{\infty}([0, \pi])$, and $v \in H_{-\delta}^{2}(G \cap$ $\left.\mathcal{O}_{\varepsilon}\left(g_{1}\right)\right) \subset W^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$. Thus, taking into account the Sobolev embedding theorem, we have $u \in C\left(\overline{G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}\right)$.
Similarly, we can show that $u \in C\left(\overline{G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right)}\right)$. Hence, $u \in C(\bar{G})$, and we can apply the maximum principle 24.1; this yields $u=0$, i.e., $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1-\delta}=0$.

According to [85, Theorem 3.4], the operator $\mathbf{L}_{1-\delta}(q)$ is a Fredholm operator for any $q$. On the other hand, the dimension of the kernel of a Fredholm operator does not increase for small perturbations (see [56, Sec. 16]). Therefore, $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1-\delta}(q)=0$ for all sufficiently small $q>0$.

Lemma 26.6. Let $q>0$ be sufficiently small. Then $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta}(q) \leq 1$.
Proof. 1. First, we assume that $q=0$ and prove that $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta} \leq 1$. Let $u \in \operatorname{ker} \mathbf{L}_{1+\delta}$. Without loss of generality, we assume that the function $u$ is real-valued. Lemma 26.1 yields the following asymptotic of the function $u$ near the point $g$ :

$$
\begin{equation*}
u(y)=c_{0}+\hat{c}_{0} \ln r+r\left(c_{1} \cos \omega+d_{1} \sin \omega\right)+v(y), \quad y \in \mathcal{O}_{\varepsilon}(g), \tag{26.19}
\end{equation*}
$$

where $\omega$ and $r$ are the polar coordinates with pole at the point $g$ such that $(0, r) \in \Omega_{1}\left(\Gamma_{1}\right)$ for $0<r<\varepsilon$, $c_{0}, \hat{c}_{0}, c_{1}$, and $d_{1}$ are some constants, and $v \in H_{-\delta}^{2}\left(\mathcal{O}_{\varepsilon}(g)\right)$. Using Eqs. (26.3)-(26.5) and (26.19), we obtain the relations

$$
\begin{gather*}
\Delta u=0, \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right), \\
\left.u\right|_{\Gamma_{1} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}=b_{1}^{*}\left(c_{0}+\hat{c}_{0} \ln r+c_{1} r\right)+w(r),\left.\quad u\right|_{\Gamma_{2} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}=0, \tag{26.20}
\end{gather*}
$$

where $w \in H_{-\delta}^{3 / 2}\left(\Gamma_{1} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$.
Hence, applying [26, Theorem 2.2, Lemma 4.3], we see that

$$
\begin{equation*}
u(y)=c_{0} b_{1}^{*} \varphi(\omega)+\hat{c}_{0} b_{1}^{*} \ln r \hat{\varphi}(\omega)+r \psi(\omega)+v(y), \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right), \tag{26.21}
\end{equation*}
$$

where $\omega$ and $r$ are the polar coordinates with pole at the point $g_{1}$ such that $(0, r) \in \Gamma_{1}$ for $0<r<\varepsilon$, $\varphi, \hat{\varphi}, \psi \in C^{\infty}([0, \pi])$, the functions $\varphi$ and $\hat{\varphi}$ are independent of $u, v, c_{0}$, and $\hat{c}_{0}$, and $v \in H_{-\delta}^{2}(G \cap$ $\left.\mathcal{O}_{\varepsilon}\left(g_{1}\right)\right) \subset W^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$. In particular, $\varphi(\omega)$ is a function depending only on the polar angle $\omega$ and is a solution of the problem

$$
\begin{gathered}
\Delta_{y} \varphi=0, \quad r>0, \quad 0<\omega<\pi \\
\varphi(0)=1, \quad \varphi(\pi)=0
\end{gathered}
$$

i.e., $\varphi(\omega)$ has the form

$$
\begin{equation*}
\varphi(\omega)=1-\frac{\omega}{\pi} . \tag{26.22}
\end{equation*}
$$

Consider the behavior of the function $u$ near the point $g_{2}$. We have

$$
\begin{gather*}
\Delta u=0, \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right), \\
\left.u\right|_{\Gamma_{2} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}=0,\left.\quad u\right|_{\Gamma_{2} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}=0 . \tag{26.23}
\end{gather*}
$$

Hence, using [26, Theorem 2.2], we obtain that

$$
\begin{equation*}
u(y)=c r \sin \omega+v(y), \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right) \tag{26.24}
\end{equation*}
$$

where $\omega$ and $r$ are the polar coordinates with pole at the point $g_{2}$ and $v \in H_{-\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right)\right) \subset$ $W^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right)\right)$. Thus, $u \in W^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right)\right)$.

Equations (26.19), (26.21), and (26.24) yield that any function $u \in \operatorname{ker} \mathbf{L}_{1+\delta}$ can be written in the form

$$
\begin{equation*}
u(y)=c_{0} u_{0}(y)+\hat{c}_{0} \hat{u}_{0}(y)+U(y), \quad y \in G \tag{26.25}
\end{equation*}
$$

where $u_{0}, \hat{u}_{0} \in C^{\infty}\left(\bar{G} \backslash\left\{g_{1}, g\right\}\right)$ are such that ${ }^{16}$

$$
u_{0}(y)=\left\{\begin{array}{ll}
1, & y \in \mathcal{O}_{\varepsilon}(g),  \tag{26.26}\\
b_{1}^{*} \varphi(\omega), & y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right), \\
0, & y \in \mathcal{O}_{\varepsilon}\left(g_{2}\right)
\end{array} \quad \hat{u}_{0}(y)= \begin{cases}\ln r, & y \in \mathcal{O}_{\varepsilon}(g) \\
b_{1}^{*} \ln r \hat{\varphi}(\omega), & y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right) \\
0, & y \in \mathcal{O}_{\varepsilon}\left(g_{2}\right)\end{cases}\right.
$$

$U \in W^{2}(G) \subset C(\bar{G})$ and $U\left(g_{1}\right)=U\left(g_{2}\right)=U(g)=0$. It follows from here and Lemma 5.2 (item 2) that $U \in H_{1-\delta}^{2}(G)$.

It follows from representation (26.25) and Lemma 26.5 that $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta} \leq 2$.
2. Let us prove that $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta} \leq 1$. Assume the contrary: let there exist two linearly independent functions $v_{1}, v_{2} \in \operatorname{ker} \mathbf{L}_{1+\delta}$. Since each of these functions can be presented in the form of Eq. (26.25), we see that some of their nontrivial linear combinations (let us denote it by $u$ ) have the form

$$
\begin{equation*}
u(y)=u_{0}(y)+U(y) . \tag{26.27}
\end{equation*}
$$

Equations (26.22), (26.26), and (26.27) yield $u \in C\left(\bar{G} \backslash\left\{g_{1}\right\}\right)$ and

$$
M=\sup _{y \in \bar{G} \backslash\left\{g_{1}\right\}}|u(y)|<\infty
$$

Let us show that $M=0$. For this, we consider a sequence $\left\{y^{n}\right\}_{n=1}^{\infty} \subset \bar{G} \backslash\left\{g_{1}\right\}$ such that

$$
u\left(y^{n}\right) \rightarrow M, \quad n \rightarrow \infty
$$

Since the sequence $\left\{y^{n}\right\}$ is bounded, we can extract a subsequence converging to a point $y^{0}$; we denote this subsequence also by $\left\{y^{n}\right\}$, i.e.,

$$
y^{n} \rightarrow y^{0}, \quad n \rightarrow \infty
$$

If $y^{0} \in G$, then, by the continuity of $u$ in the domain $G$, we have $\left|u\left(y^{0}\right)\right|=M$. Hence, by the maximum principle 24.1, we have $u \equiv$ const. This is impossible since $\varphi(\omega) \not \equiv$ const.

Let $y^{0} \in \Gamma_{2} \cup\left\{g_{2}\right\}$. Using the continuity of $u$ on $\Gamma_{2} \cup\left\{g_{2}\right\}$ and boundary condition (26.5), we obtain

$$
0=\left|u\left(y^{0}\right)\right|=\lim _{n \rightarrow \infty}\left|u\left(y^{n}\right)\right|=M
$$

If $y^{0} \in \Gamma_{1}$, then, by the continuity of $u$ on $\Gamma_{1}$ and boundary condition (26.4), we have

$$
M=\lim _{n \rightarrow \infty}\left|u\left(y^{n}\right)\right|=\left|u\left(y^{0}\right)\right|=b_{1}^{*}\left|u\left(\Omega_{1}\left(y^{0}\right)\right)\right| .
$$

Hence $\left|u\left(\Omega_{1}\left(y^{0}\right)\right)\right|=M / b_{1}^{*} \geq M$. This is possible only if $b_{1}^{*}=1$. But in this case, $\left|u\left(\Omega_{1}\left(y^{0}\right)\right)\right|=M$. This contradicts the relation $\Omega_{1}\left(y^{0}\right) \in G$.

Finally, we consider the case where $y^{0}=g_{1}$. Without loss of generality, we assume that $y^{n} \in \mathcal{O}_{\varepsilon}\left(g_{1}\right)$, $n=1,2, \ldots$. Denote the polar coordinates near the point $y^{n}$ by $\left(\omega^{n}, r^{n}\right)$. Let $x^{n} \in \Gamma_{1}$ be the point with coordinates $\left(0, r^{n}\right)$. It follows from Eqs. (26.26) and (26.22) that

$$
\left|u_{0}\left(x^{n}\right)\right|=\left|b_{1}^{*} \varphi(0)\right| \geq\left|b_{1}^{*} \varphi\left(\omega^{n}\right)\right|=\left|u_{0}\left(y^{n}\right)\right| .
$$

Hence, taking into account the fact that $U \in C(\bar{G}), U\left(g_{1}\right)=0$, and the functions $u$ and $u_{0}$ are continuous on $\overline{\Gamma_{1}}$, we have

$$
M \geq \lim _{n \rightarrow \infty}\left|u\left(x^{n}\right)\right|=\lim _{n \rightarrow \infty}\left|u_{0}\left(x^{n}\right)\right| \geq \lim _{n \rightarrow \infty}\left|u_{0}\left(y^{n}\right)\right|=\lim _{n \rightarrow \infty}\left|u\left(y^{n}\right)\right|=M .
$$

Thus,

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left|u\left(x^{n}\right)\right|=M \tag{26.28}
\end{equation*}
$$

Using the fact that $x^{n} \in \Gamma_{1}$, the continuity of $\Omega_{1}$, the continuity of the function $u$ in the domain $G$, and relations (26.4) and (26.28), we obtain

$$
|u(g)|=\lim _{n \rightarrow \infty}\left|u\left(\Omega_{1}\left(x^{n}\right)\right)\right|=\lim _{n \rightarrow \infty}\left|u\left(x^{n}\right)\right| / b_{1}^{*}=M / b_{1}^{*} \geq M .
$$

[^15]This is impossible since $g \in G$. Thus, we have proved that $u=0$, i.e., functions $v_{1}$ and $v_{2}$ are linearly independent and dim $\operatorname{ker} \mathbf{L}_{1+\delta} \leq 1$.
3. According to [85, Theorem 3.4], $\mathbf{L}_{1+\delta}(q)$ is a Fredholm operator for any $q$. On the other hand, the dimension of the kernel of a Fredholm operator does not increase for small perturbations (see [56, Section 16]). Hence, $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta}(q) \leq 1$ for all sufficiently small $q>0$.

Lemma 26.7. Let $q>0$ be sufficiently small. Then $\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right) \geq 1$.
Proof. The strip $-\delta \leq \operatorname{Im} \lambda \leq \delta$ does not contain eigenvalues of model problem (26.10), (26.11) but contains the unique eigenvalue $\lambda_{0}=0$ of model problem (26.2); moreover, the algebraic multiplicity of the eigenvalue $\lambda_{0}=0$ is equal to 2. Hence, according to [31, Theorem 4.1], ind $\mathbf{L}_{1+\delta}(q)=\operatorname{ind} \mathbf{L}_{1-\delta}(q)+$ 2, i.e.,

$$
\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta}(q)-\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1+\delta}(q)\right)=\operatorname{dim} \operatorname{ker} \mathbf{L}_{1-\delta}(q)-\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)+2 .
$$

From here and Lemmas 26.5 and 26.6, we obtain that

$$
\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)=\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1+\delta}(q)\right)-\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta}(q)+2 \geq 1
$$

Now we fix a number $q>0$ for which $\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right) \geq 1$. Consider the set

$$
R_{1-\delta}^{0}(G)=\left\{f_{0} \in H_{1-\delta}^{0}(G):\left(f_{0}, 0,0\right) \in \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)\right\} .
$$

Obviously, $R_{1-\delta}^{0}(G)$ is a closed subset in $H_{1-\delta}^{0}(G)$ since the image $\mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)$ is a closed subset in $\mathcal{H}_{1-\delta}^{0}(G, \partial G)$.
Lemma 26.8. $\operatorname{codim} R_{1-\delta}^{0}(G) \geq 1$.
Proof. Assume the contrary: let

$$
\begin{equation*}
R_{1-\delta}^{0}(G)=H_{1-\delta}^{0}(G) . \tag{26.29}
\end{equation*}
$$

We show that in this case

$$
\begin{equation*}
\mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)=\mathcal{H}_{1-\delta}^{0}(G, \partial G) . \tag{26.30}
\end{equation*}
$$

Consider an arbitrary function $f=\left(f_{0}, f_{1}, f_{2}\right) \in \mathcal{H}_{1-\delta}^{0}(G, \partial G)$. By Lemma 11.1, there exists a function $v \in H_{1-\delta}^{2}(G)$ such that $\left.v\right|_{\Gamma_{j}}=f_{j}, j=1,2$, and the support of the function $v$ is located in an arbitrarily small neighborhood $\partial G$. Let this neighborhood be so small that $b_{1}(y) v\left(\Omega_{1}(y)\right)=0, y \in \Gamma_{1}$. Then the function $v$ satisfies the following nonlocal conditions:

$$
\begin{equation*}
\left.v\right|_{\Gamma_{1}}-\left.b_{1}(y) u\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}}=f_{1},\left.\quad v\right|_{\Gamma_{2}}=f_{2} . \tag{26.31}
\end{equation*}
$$

On the other hand, it follows from Eq. (26.29) that there exists a function $w \in H_{1-\delta}^{2}(G)$ such that

$$
\begin{gather*}
\Delta w-q w=f_{0}-(\Delta v-q v)  \tag{26.32}\\
\left.w\right|_{\Gamma_{1}}-\left.b_{1}(y) w\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}}=0,\left.\quad w\right|_{\Gamma_{2}}=0 . \tag{26.33}
\end{gather*}
$$

Equations (26.31)-(26.33) yield $\mathbf{L}_{1-\delta}(q) u=f$, where $u=v+w \in H_{1-\delta}^{2}(G)$. Thus, Eq. (26.30) is valid; this contradicts Lemma 26.7.

Now, using the Hille-Yosida theorem (Theorem 23.1) and Lemmas 26.4 and 26.8, we prove Theorem 26.1.

Proof of Theorem 26.1. 1. Assume the contrary: let $\overline{\mathbf{P}_{B}}$ be a generator of a Feller semigroup. Then by the Hille-Yosida theorem (Theorem 23.1) we have $\mathcal{R}\left(\overline{\mathbf{P}_{B}-q \mathbf{I}}\right)=\mathcal{R}\left(\overline{\mathbf{P}_{B}}-q \mathbf{I}\right)=C_{B}(\bar{G})$. Hence $\overline{\mathcal{R}\left(\mathbf{P}_{B}-q \mathbf{I}\right)}=C_{B}(\bar{G})$. By Lemma 26.4, this means that every function from $C_{B}(\bar{G})$ can be approximated by functions from $R_{1-\delta}^{0}(G) \cap C_{B}(\bar{G})$.
2. Show that in this case

$$
\begin{equation*}
R_{1-\delta}^{0}(G)=H_{1-\delta}^{0}(G) \tag{26.34}
\end{equation*}
$$

(this contradicts Lemma 26.8). For this, we choose an arbitrary function $f_{0} \in H_{1-\delta}^{0}(G)$. Since the set $C_{0}^{\infty}(\bar{G} \backslash(\mathcal{K} \cup\{g\}))$ is dense in $H_{1-\delta}^{0}(G)$, we see that for any $\kappa>0$, we can find a function $f_{0}^{\prime} \in C_{0}^{\infty}(\bar{G} \backslash(\mathcal{K} \cup\{g\}))$ such that

$$
\begin{equation*}
\left\|f_{0}-f_{0}^{\prime}\right\|_{H_{1-\delta}^{0}(G)} \leq \kappa \tag{26.35}
\end{equation*}
$$

Since $f_{0}^{\prime} \in L_{2}(G)$, there exists a function $f_{0}^{\prime \prime} \in C^{\infty}(\bar{G})$ that vanishes near $\partial G \cup \overline{\Omega_{1}\left(\Gamma_{1}\right)}$ such that

$$
\begin{equation*}
\left\|f_{0}^{\prime}-f_{0}^{\prime \prime}\right\|_{H_{1-\delta}^{0}(G)} \leq k_{1}\left\|f_{0}^{\prime}-f_{0}^{\prime \prime}\right\|_{L_{2}(G)} \leq \kappa, \tag{26.36}
\end{equation*}
$$

where $k_{1}>0$ is independent of $f_{0}^{\prime}$ and $f_{0}^{\prime \prime}$.
Since $f_{0}^{\prime \prime} \in C_{B}(\bar{G})$, it follows from item 1 of the proof that there exists a function $f_{0}^{\prime \prime \prime} \in R_{1-\delta}^{0}(G) \cap$ $C_{B}(\bar{G})$ such that

$$
\begin{equation*}
\left\|f_{0}^{\prime \prime}-f_{0}^{\prime \prime \prime}\right\|_{H_{1-\delta}^{0}(G)} \leq k_{2}\left\|f_{0}^{\prime \prime}-f_{0}^{\prime \prime \prime}\right\|_{C(\bar{G})} \leq \kappa, \tag{26.37}
\end{equation*}
$$

where $k_{2}>0$ is independent of $f_{0}^{\prime \prime}$ and $f_{0}^{\prime \prime \prime}$.
The following inequality follows from Eqs. (26.35)-(26.37):

$$
\left\|f_{0}-f_{0}^{\prime \prime \prime}\right\|_{H_{1-\delta}^{0}(G)} \leq 3 \kappa .
$$

Hence, taking into account the fact that the set $R_{1-\delta}^{0}(G)$ is closed in $H_{1-\delta}^{0}(G)$, we obtain Eq. (26.34). Thus, we arrive at the contradiction with Lemma 26.8.

Remark 26.1. Let $\widehat{\mathbf{P}}_{B}: C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ be a reduction of the operator $\overline{\mathbf{P}_{B}}$. Obviously, the following embedding holds:

$$
\mathcal{R}\left(\widehat{\mathbf{P}}_{B}-q \mathbf{I}\right) \subset \mathcal{R}\left(\overline{\mathbf{P}_{B}}-q \mathbf{I}\right)
$$

Since $\mathcal{R}\left(\overline{\mathbf{P}_{B}}-q \mathbf{I}\right)$ does not coincide with $C_{B}(\bar{G})$, we see that $\mathcal{R}\left(\widehat{\mathbf{P}}_{B}-q \mathbf{I}\right)$ also does not coincide with $C_{B}(\bar{G})$. Hence, by the Hille-Yosida theorem (Theorem 23.1), the operator $\widehat{\mathbf{P}}_{B}$ is not a generator of a Feller semigroup.

## 26.3. "Jumps" from conjugation points that are not termination points of the process.

26.3.1. Statement of a nonlocal problem. In this example, we show that condition 24.2 is substantial.

Let $G, g_{1}, g_{2}$, and $\mathcal{K}$ be the same as in Sec. 26.2. Consider the following nonlocal problem (see Fig. 26.2):

$$
\begin{array}{ll}
\Delta u(y)=f_{0}(y), & y \in G, \\
u(y)-b_{1}(y) u\left(\Omega_{1}(y)\right)=0, & y \in \overline{\Gamma_{1}}, \\
u(y)-b_{2}(y) u\left(\Omega_{2}(y)\right)=0, & y \in \Gamma_{2}, \tag{26.40}
\end{array}
$$

where $b_{j} \in C^{\infty}\left(\overline{\Gamma_{j}}\right), j=1,2$, are such that
(1) $0 \leq b_{j}(y) \leq 1$,
(2) $b_{j}(y)=b^{*}>0$ for $y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right)$,
(3) $b_{j}(y)=0$ for $y \notin \mathcal{O}_{2 \varepsilon}\left(g_{1}\right)$,
and $\Omega_{j}, j=1,2$, is a smooth nondegenerate transformation defined in a neighborhood of a curve $\overline{\Gamma_{j}}$ such that
(1) $\Omega_{j}\left(\Gamma_{j}\right) \subset G, \Omega_{j}\left(g_{1}\right) \in G, \Omega_{j}\left(g_{2}\right)=g_{2}$, and $\Omega_{1}\left(g_{1}\right) \neq \Omega_{2}\left(g_{1}\right)$,
(2) for $y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right), \Omega_{j}(y)$ is the composition of a rotation about the point $g_{1}$ and a shift by some vector.


Fig. 26.2. Problem (26.38)-(26.40).
Assume that $g=\Omega_{1}\left(g_{1}\right)$ and $h=\Omega_{2}\left(g_{1}\right)$; by the assumption, $g, h \in G$ and $g \neq h$. Let $\varepsilon>0$ be so small that $2 \varepsilon$-neighborhoods of the points $g_{1}$ and $g_{2}$ and $g$ and $h$ do not intersect and

$$
\mathcal{O}_{2 \varepsilon}(g) \cap \partial G=\varnothing, \quad \mathcal{O}_{2 \varepsilon}(h) \cap \partial G=\varnothing .
$$

Introduce measures $\alpha(y, \cdot)$ and $y \in \partial G$ as follows: for any Borel set $Q \subset \bar{G}$, we assume

$$
\begin{array}{ll}
\alpha(y, Q)=b_{1}(y) \chi_{Q}\left(\Omega_{1}(y)\right), & y \in \overline{\Gamma_{1}}, \\
\alpha(y, Q)=b_{2}(y) \chi_{Q}\left(\Omega_{2}(y)\right), & y \in \Gamma_{2} .
\end{array}
$$

Then boundary conditions (26.39) and (26.40) can be rewritten in the form

$$
b(y) u(y)+\int_{\bar{G}}[u(y)-u(\eta)] \alpha(y, d \eta)=0, \quad y \in \partial G
$$

where $b(y)=1-\alpha(y, \bar{G})$.
Obviously, $g_{2} \in \mathcal{N}$, but $g_{1} \notin \mathcal{N}$. Hence, condition 24.2 is violated. It easy to verify that conditions 23.1, 23.2, 24.1, and 24.3-24.8 hold (with $\mathbf{P}(y, D)=\Delta, \mathbf{P}_{1}=0$, and $\beta_{i}(y, \bar{G}) \equiv 0$ ).

Consider the unbounded operator $\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ defined by the formula

$$
\begin{equation*}
\mathbf{P}_{B} u=\Delta u, \quad u \in \mathrm{D}\left(\mathbf{P}_{B}\right)=\left\{u \in C_{B}(\bar{G}): \Delta u \in C_{B}(\bar{G})\right\}, \tag{26.41}
\end{equation*}
$$

where $C_{B}(\bar{G})$ is the set of functions from $C(\bar{G})$ that satisfy nonlocal conditions (26.39) and (26.40).
The following two results are proved similarly to Lemmas 26.2 and 26.3.
Lemma 26.9. If $u \in \mathrm{D}\left(\mathbf{P}_{B}\right)$, then $u \in W^{2}\left(G^{\prime}\right)$ for any domain $G^{\prime}$ such that $\overline{G^{\prime}} \subset \bar{G} \backslash \mathcal{K}$.
Lemma 26.10. The operator $\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ admits a closure $\overline{\mathbf{P}_{B}}$.
We will prove the following result.
Theorem 26.2. Let $\overline{\mathbf{P}_{B}}: \mathrm{D}\left(\overline{\mathbf{P}_{B}}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ be the closure of operator (26.41). Then $\overline{\mathbf{P}_{B}}$ is not a generator of a Feller semigroup.

By the Hille-Yosida theorem (Theorem 23.1), it suffices to show that the image $\mathcal{R}\left(\overline{\mathbf{P}_{B}}-q \mathbf{I}\right)$ does not coincide with $C_{B}(\bar{G})$ for sufficiently small $q>0$.
26.3.2. Proof of Theorem 26.2. In this example, we will use the weight spaces

$$
\begin{gathered}
H_{a}^{l}(G)=H_{a}^{l}(G, \mathcal{K} \cup\{g, h\}), \quad H_{a}^{l}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right)\right)=H_{a}^{l}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right),\left\{g_{j}\right\}\right), \\
H_{a}^{l}\left(\mathcal{O}_{\varepsilon}(g)\right)=H_{a}^{l}\left(\mathcal{O}_{\varepsilon}(g),\{g\}\right), \quad H_{a}^{l}\left(\mathcal{O}_{\varepsilon}(h)\right)=H_{a}^{l}\left(\mathcal{O}_{\varepsilon}(h),\{h\}\right)
\end{gathered}
$$

and the corresponding trace spaces. We emphasize that the space $H_{a}^{l}(G)$ consists of functions that can have power singularities not only near the set $\mathcal{K}$, but also near points $g$ and $h$.

To prove Theorem 26.2, let us obtain asymptotics of solutions $u \in \mathrm{D}\left(\mathbf{P}_{B}\right)$ of problem (26.38)(26.40).

Let us fix a number $\delta$ such that

$$
\begin{equation*}
0<\delta<1 \tag{26.42}
\end{equation*}
$$

Lemma 26.11. Let $u \in \mathrm{D}\left(\mathbf{P}_{B}\right)$. Then $u=c w_{0}+w_{1}$, where $c$ is a constant, the function $w_{0}$ is independent of $u$,

$$
w_{0} \in C_{B}(\bar{G}) \cap C^{\infty}(\bar{G}), \quad \Delta w_{0}(y)=0, \quad y \in \mathcal{O}_{\varepsilon / 2}(\mathcal{K}) \cup \mathcal{O}_{\varepsilon / 2}(g) \cup \mathcal{O}_{\varepsilon / 2}(h)
$$

and $w_{1} \in C_{B}(\bar{G}) \cap H_{1-\delta}^{2}(G)$.
Proof. 1. It follows from Lemma 26.9 that $u \in W^{2}\left(G^{\prime}\right)$ for any domain $G^{\prime}$ such that $\overline{G^{\prime}} \subset \bar{G} \backslash \mathcal{K}$. Since $u \in W^{2}\left(\mathcal{O}_{\varepsilon}(g)\right)$, by Lemma 5.2 we see that $u \in H_{1+\delta}^{2}\left(\mathcal{O}_{\varepsilon}(g)\right)$. Since $f_{0} \in C(\bar{G}) \subset H_{-\delta}^{0}\left(\mathcal{O}_{\varepsilon}(g)\right)$, similarly to Eq. (26.19), we obtain the equality

$$
u(y)=c_{g}+\hat{c}_{g} \ln r+r\left(c_{g 1} \cos \omega+d_{g 1} \sin \omega\right)+v(y), \quad y \in \mathcal{O}_{\varepsilon}(g)
$$

where $\omega$ and $r$ are polar coordinates with pole at the point $g$ such that $(0, r) \in \Omega_{1}\left(\Gamma_{1}\right)$ for $0<r<\varepsilon$, $c_{g}, \hat{c}_{g}, c_{g 1}$, and $d_{g 1}$ are constants, and $v \in H_{-\delta}^{2}\left(\mathcal{O}_{\varepsilon}(g)\right)$. Applying the Sobolev embedding theorem, we have

$$
v \in H_{-\delta}^{2}\left(\mathcal{O}_{\varepsilon}(g)\right) \subset W^{2}\left(\mathcal{O}_{\varepsilon}(g)\right) \subset C\left(\overline{\mathcal{O}_{\varepsilon}(g)}\right), \quad v(g)=0
$$

Taking into account the fact that $u \in C\left(\overline{\mathcal{O}_{\varepsilon}(g)}\right)$, we see that $\hat{c}_{g}=0$ and

$$
\begin{gather*}
u(y)=c_{g}+r\left(c_{g 1} \cos \omega+d_{g 1} \sin \omega\right)+v(y), \quad y \in \mathcal{O}_{\varepsilon}(g),  \tag{26.43}\\
u(g)=c_{g} \tag{26.44}
\end{gather*}
$$

Replacing $g$ by $h$, we similarly obtain the equality

$$
\begin{gather*}
u(y)=c_{h}+r\left(c_{h 1} \cos \omega+d_{h 1} \sin \omega\right)+v(y), \quad y \in \mathcal{O}_{\varepsilon}(h),  \tag{26.45}\\
u(h)=c_{h}, \tag{26.46}
\end{gather*}
$$

where $\omega$ and $r$ are polar coordinates with pole at the point $h$ such that $(0, r) \in \Omega_{2}\left(\Gamma_{2}\right)$ for $0<r<\varepsilon$, $c_{h}, c_{h 1}$, and $d_{h 1}$ are constants, and $v \in H_{-\delta}^{2}\left(\mathcal{O}_{\varepsilon}(h)\right) \subset W^{2}\left(\mathcal{O}_{\varepsilon}(g)\right) \subset C\left(\overline{\mathcal{O}_{\varepsilon}(g)}\right)$.

Since the function $u \in C(\bar{G})$ satisfies nonlocal conditions (26.39), (26.40), we see that $u(g)=u\left(g_{1}\right) / b^{*}=u(h)$, i.e., by Eqs. (26.44) and (26.46), we have $c_{g}=c_{h}$. Assume that

$$
\begin{equation*}
c=c_{g}=c_{h} . \tag{26.47}
\end{equation*}
$$

2. Applying (26.38)-(26.40), (26.43), (26.45), (26.47), and the fact that the transformations $\Omega_{1}$ and $\Omega_{2}$ are smooth and nondegenerate, we obtain

$$
\begin{gather*}
\Delta\left(u-b^{*} c\right)=\Delta u \in C(\bar{G}) \subset H_{1-\delta}^{0}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right),  \tag{26.48}\\
\left.\left(u-b^{*} c\right)\right|_{\Gamma_{j} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)} \in H_{1-\delta}^{3 / 2}\left(\Gamma_{j} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right), \quad j=1,2 . \tag{26.49}
\end{gather*}
$$

Using Eqs. (26.48) and (26.49) and the relation $u-b^{*} c \in C(\bar{G}) \subset H_{-1+\delta}^{0}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$, from Lemma 14.2 we obtain

$$
\begin{equation*}
u-b^{*} c \in H_{1+\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right) . \tag{26.50}
\end{equation*}
$$

By Eq. (26.42), the strip $-\delta \leq \operatorname{Im} \lambda \leq \delta$ does not contain eigenvalues of problem (26.10), (26.11). Hence, from Eqs. (26.48)-(26.50) and [26, Theorem 2.2] we obtain the following embedding:

$$
u-b^{*} c \in H_{1-\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right) .
$$

Since $b_{1}(y)=b_{2}(y)=0$ near the point $g_{2}$, we see that $u \in H_{1-\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right)\right)$ (cf. the proof of Lemma 26.4).
3. Introduce a patch function $\xi \in C^{\infty}([0, \infty))$ such that $\xi(r)=1$ for $r<\varepsilon / 2$ and $\operatorname{supp} \xi \subset[0, \varepsilon)$. Consider the function $w_{0} \in C^{\infty}(\bar{G})$ defined by the formula ${ }^{17}$

$$
w_{0}(y)= \begin{cases}b^{*} \psi(r), & y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right),  \tag{26.51}\\ \psi(r), & y \in \mathcal{O}_{\varepsilon}(g), \\ \psi(r), & y \in \mathcal{O}_{\varepsilon}(h), \\ 0, & y \notin \mathcal{O}_{\varepsilon}\left(g_{1}\right) \cup \mathcal{O}_{\varepsilon}(g) \cup \mathcal{O}_{\varepsilon}(h),\end{cases}
$$

and the function $w_{1}=u-c w_{0}$. It is easy to see that $w_{0}$ and $w_{1}$ are the required functions.
Introduce the bounded operator $\mathbf{L}_{a}(q): H_{a}^{2}(G) \rightarrow \mathcal{H}_{a}^{0}(G, \partial G)$ by the formula

$$
\mathbf{L}_{a}(q) u=\left\{\Delta u-q u,\left.u\right|_{\Gamma_{1}}-\left.b_{1}(y) u\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}},\left.u\right|_{\Gamma_{2}}\right\}, \quad q \geq 0 .
$$

We also denote $\mathbf{L}_{a}=\mathbf{L}_{a}(0)$.
Lemma 26.12. Let $q>0$ be sufficiently small. Then $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1-\delta}(q)=0$.
The proof is similar to the proof of Lemma 26.5.
Lemma 26.13. Let $q>0$ be sufficiently small. Then $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta}(q) \leq 2$.
Proof. 1. First, we assume that $q=0$ and prove that $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta} \leq 2$. Let $u \in \operatorname{ker} \mathbf{L}_{1+\delta}$. Without loss of generality, we assume that the function $u$ is real-valued.

As in the proof of Lemma 26.11, we have

$$
\begin{equation*}
u(y)=c_{g}+\hat{c}_{g} \ln r+r\left(c_{g 1} \cos \omega+d_{g 1} \sin \omega\right)+v(y), \quad y \in \mathcal{O}_{\varepsilon}(g), \tag{26.52}
\end{equation*}
$$

where $\omega$ and $r$ are the polar coordinates with pole at the point $g$ such that $(0, r) \in \Omega_{1}\left(\Gamma_{1}\right)$ for $0<r<\varepsilon$, $c_{g}, \hat{c}_{g}, c_{g 1}$ and $d_{g 1}$ are constants, and $v \in H_{-\delta}^{2}\left(\mathcal{O}_{\varepsilon}(g)\right)$.

Similarly, replacing the point $g$ by the point $h$, we obtain the equality

$$
\begin{equation*}
u(y)=c_{h}+\hat{c}_{h} \ln r+r\left(c_{h 1} \cos \omega+d_{h 1} \sin \omega\right)+v(y), \quad y \in \mathcal{O}_{\varepsilon}(h), \tag{26.53}
\end{equation*}
$$

where $\omega$ and $r$ are the polar coordinates with pole at the point $h$ such that $(0, r) \in \Omega_{2}\left(\Gamma_{2}\right)$ for $0<r<\varepsilon$, $c_{h}, \hat{c}_{h}, c_{h 1}$ and $d_{h 1}$ are constants, and $v \in H_{-\delta}^{2}\left(\mathcal{O}_{\varepsilon}(h)\right)$.

Using Eqs. (26.38)-(26.40), (26.52), and (26.53), we obtain

$$
\begin{gather*}
\Delta u=0, \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right), \\
\left.u\right|_{\Gamma_{1} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}=b^{*}\left(c_{g}+\hat{c}_{g} \ln r+c_{g 1} r\right)+w_{1}(r),  \tag{26.54}\\
\left.u\right|_{\Gamma_{2} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}=b^{*}\left(c_{h}+\hat{c}_{h} \ln r+c_{h 1} r\right)+w_{2}(r),
\end{gather*}
$$

where $w_{j} \in H_{-\delta}^{3 / 2}\left(\Gamma_{j} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$.
Hence, applying [26, Theorem 2.2, Lemma 4.3], we see that

$$
\begin{equation*}
u(y)=c_{g} \varphi_{g}(\omega)+c_{h} \varphi_{h}(\omega)+\hat{c}_{g} \ln r \hat{\varphi}_{g}(\omega)+\hat{c}_{h} \ln r \hat{\varphi}_{h}(\omega)+r \psi(\omega)+v(y), \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right) \tag{26.55}
\end{equation*}
$$

where $\omega$ and $r$ are the polar coordinates with pole at the point $g_{1}$ such that $(0, r) \in \Gamma_{1}$ for $0<r<\varepsilon$, $\varphi_{g}, \varphi_{h}, \hat{\varphi}_{g}, \hat{\varphi}_{h}, \psi \in C^{\infty}([0, \pi])$, the functions $\varphi_{g}, \varphi_{h}, \hat{\varphi}_{g}$, and $\hat{\varphi}_{h}$ are independent of $u, v, c_{g}, c_{h}, \hat{c}_{j}$, $\hat{c}_{h}$, and $v \in H_{-\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right) \subset W^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$. In particular, $\varphi_{g}(\omega)$ is a function that depends only on the polar angle $\omega$ and is a solution of the problem

$$
\begin{gathered}
\Delta_{y} \varphi_{g}=0, \quad r>0, \quad 0<\omega<\pi \\
\varphi_{g}(0)=1, \quad \varphi_{g}(\pi)=0
\end{gathered}
$$

i.e., $\varphi_{g}(\omega)$ has the form

$$
\begin{equation*}
\varphi_{g}(\omega)=1-\omega / \pi . \tag{26.56}
\end{equation*}
$$

[^16]The function $\varphi_{h}(\omega)$ depends only on the polar angle $\omega$ and is a solution of the problem

$$
\begin{gathered}
\Delta_{y} \varphi_{h}=0, \quad r>0, \quad 0<\omega<\pi \\
\varphi_{h}(0)=0, \quad \varphi_{h}(\pi)=1
\end{gathered}
$$

i.e., $\varphi_{h}(\omega)$ has the form

$$
\begin{equation*}
\varphi_{h}(\omega)=\frac{\omega}{\pi} \tag{26.57}
\end{equation*}
$$

Similarly to the proof of Lemma 26.6, we have

$$
u \in W^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right)\right) \subset C\left(\overline{G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right)}\right)
$$

It follows from Eqs. (26.52), (26.53), and (26.55) that any function $u \in \operatorname{ker} \mathbf{L}_{1+\delta}$ can be rewritten in the form

$$
\begin{equation*}
u(y)=c_{g} u_{g}(y)+c_{h} u_{h}(y)+\hat{c}_{g} \hat{u}_{g}(y)+\hat{c}_{h} \hat{u}_{h}(y)+U(y), \quad y \in G, \tag{26.58}
\end{equation*}
$$

where $u_{g}, u_{h}, \hat{u}_{g}, \hat{u}_{h} \in C^{\infty}\left(\bar{G} \backslash\left\{g_{1}, g, h\right\}\right)$,
$u_{g}(y)=\left\{\begin{array}{ll}1, & y \in \mathcal{O}_{\varepsilon}(g), \\ b^{*} \varphi_{g}(\omega), & y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right), \\ 0, & y \in \mathcal{O}_{\varepsilon}\left(g_{2}\right) \cup \mathcal{O}_{\varepsilon}(h),\end{array} \quad u_{h}(y)= \begin{cases}1, & y \in \mathcal{O}_{\varepsilon}(h), \\ b^{*} \varphi_{h}(\omega), & y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right), \\ 0, & y \in \mathcal{O}_{\varepsilon}\left(g_{2}\right) \cup \mathcal{O}_{\varepsilon}(g),\end{cases}\right.$
$\hat{u}_{g}(y)=\left\{\begin{array}{ll}\ln r, & y \in \mathcal{O}_{\varepsilon}(g), \\ b^{*} \ln r \hat{\varphi}_{g}(\omega), & y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right), \\ 0, & y \in \mathcal{O}_{\varepsilon}\left(g_{2}\right) \cup \mathcal{O}_{\varepsilon}(h),\end{array} \quad \hat{u}_{h}(y)= \begin{cases}\ln r, & y \in \mathcal{O}_{\varepsilon}(h), \\ b^{*} \ln r \hat{\varphi}_{h}(\omega), & y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right), \\ 0, & y \in \mathcal{O}_{\varepsilon}\left(g_{2}\right) \cup \mathcal{O}_{\varepsilon}(g),\end{cases}\right.$
$U \in W^{2}(G) \subset C(\bar{G})$ and $U\left(g_{1}\right)=U\left(g_{2}\right)=U(g)=U(h)=0$. The embedding $U \in H_{1-\delta}^{2}(G)$ follows from here and Lemma 5.2 (item 2).

It follows from Eq. (26.58) and Lemma 26.12 that $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta} \leq 4$.
2. Prove that $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta} \leq 2$. Assume the contrary: let there exist three linearly independent functions $v_{1}, v_{2}, v_{3} \in \operatorname{ker} \mathbf{L}_{1+\delta}$. Since each of these functions can be represented in the form of Eq. (26.58), we see that some of their nontrivial linear combinations (we denote it by $u$ ) have the form

$$
\begin{equation*}
u(y)=c_{g} u_{g}(y)+c_{h} u_{h}(y)+U(y) . \tag{26.59}
\end{equation*}
$$

Using (26.59), the explicit form (26.56) and (26.57) of the functions $\varphi_{g}$ and $\varphi_{h}$ (which describe the behavior of functions $u_{g}$ and $u_{h}$ near the point $g_{1}$ ), and reasoning similarly to the proof of Lemma 26.6, we obtain $u=0$. Hence, the functions $v_{1}, v_{2}$, and $v_{3}$ are linearly independent and $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta} \leq 2$.
3. Similarly to the proof of Lemma 26.6 we can show that

$$
\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta}(q) \leq 2
$$

for sufficiently small $q>0$.
Lemma 26.14. Let $q>0$ be sufficiently small. Then $\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right) \geq 2$.
Proof. The strip $-\delta \leq \operatorname{Im} \lambda \leq \delta$ does not contain eigenvalues of model problem (26.10), (26.11), but contains the unique eigenvalue $\lambda_{0}=0$ of model problem (26.2) that corresponds to the point $g$, and contains the unique eigenvalue $\lambda_{0}=0$ of the same model problem (26.2) that corresponds to the point $h$ (recall that $g \neq h$ ). The algebraic multiplicity of the eigenvalue $\lambda_{0}=0$ is 2 for both cases. Hence, according to [31, Theorem 4.1], we have ind $\mathbf{L}_{1+\delta}(q)=\operatorname{ind} \mathbf{L}_{1-\delta}(q)+4$, i.e.,

$$
\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta}(q)-\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1+\delta}(q)\right)=\operatorname{dim} \operatorname{ker} \mathbf{L}_{1-\delta}(q)-\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)+4
$$

The relation

$$
\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)=\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1+\delta}(q)\right)-\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta}(q)+4 \geq 2
$$

follows from here and Lemmas 26.12 and 26.13

Let us fix a number $q>0$ for which $\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right) \geq 2$. Consider the set

$$
R_{1-\delta}^{0}(G)=\left\{f_{0} \in H_{1-\delta}^{0}(G):\left(f_{0}, 0,0\right) \in \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)\right\} .
$$

Obviously, $R_{1-\delta}^{0}(G)$ is a closed subspace in $H_{1-\delta}^{0}(G)$ since the image $\mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)$ is a closed subspace in $\mathcal{H}_{1-\delta}^{0}(G, \partial G)$.
Lemma 26.15. $\operatorname{codim} R_{1-\delta}^{0}(G) \geq 2$.
Proof. 1. According to Lemma 26.14, it suffices to prove that

$$
\operatorname{codim} R_{1-\delta}^{0}(G)=\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right) .
$$

Let $f_{0} \in R_{1-\delta}^{0}(G)$, i.e., $f=\left(f_{0}, 0,0\right) \in \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)$. This is equivalent to the relations

$$
\begin{equation*}
\left(f, F_{l}\right)_{\mathcal{H}_{1-\delta}^{0}(G, \partial G)}=0, \quad l=1, \ldots, \operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right), \tag{26.60}
\end{equation*}
$$

where $F_{l} \in \mathcal{H}_{1-\delta}^{0}(G, \partial G)$ are linearly independent functions from the orthogonal complement to the subspace $\mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)$ in the space $\mathcal{H}_{1-\delta}^{0}(G, \partial G)$. It follows from Eq. (26.60) and the Riesz theorem on the general form of linear continuous functionals in Hilbert spaces that

$$
\operatorname{codim} R_{1-\delta}^{0}(G) \leq \operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)
$$

2. Now we prove the inverse inequality. Let $f=\left(f_{0}, f_{1}, f_{2}\right) \in \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)$, i.e., $\mathbf{L}_{1-\delta}(q) u=f$ for some function $u \in H_{1-\delta}^{2}(G)$. By Lemma 11.1, there exists a function $v \in H_{1-\delta}^{2}(G)$ such that $\left.v\right|_{\Gamma_{j}}=f_{j}$, $j=1,2$, the support $v$ is located in an arbitrarily small neighborhood $\mathcal{O}(\partial G)$ of the boundary $\partial G$, and

$$
\begin{equation*}
\|v\| \leq k\left(\left\|f_{1}\right\|_{H_{1-\delta}^{3 / 2}\left(\Gamma_{1}\right)}+\left\|f_{2}\right\|_{H_{1-\delta}^{3 / 2}\left(\Gamma_{2}\right)}\right), \tag{26.61}
\end{equation*}
$$

where $k>0$ depends on a neighborhood $\mathcal{O}(\partial G)$, but is independent of $f_{1}$ and $f_{2}$.
Assume that a neighborhood $\mathcal{O}(\partial G)$ is such that $b_{j}(y) v\left(\Omega_{j}(y)\right)=0, y \in \Gamma_{j}$. Then the function $v$ satisfies the nonlocal conditions

$$
\left.v\right|_{\Gamma_{j}}-b_{j}(y) u\left(\Omega_{j}(y)\right)_{\Gamma_{j}}=f_{j}, \quad j=1,2 .
$$

Hence, the function $w=u-v \in H_{1-\delta}^{2}(G)$ satisfies the relation

$$
\begin{equation*}
\mathbf{L}_{1-\delta}(q) w=\left(f_{0}-(\Delta v-q v), 0,0\right) . \tag{26.62}
\end{equation*}
$$

It follows from Eq. (26.62) that $f_{0}-(\Delta v-q v) \in R_{1-\delta}^{0}(G)$. This is equivalent to relations

$$
\begin{equation*}
\left(f_{0}-(\Delta v-q v), \Phi_{l}\right)_{H_{1-\delta}^{0}(G)}=0, \quad l=1, \ldots, \operatorname{codim} R_{1-\delta}^{0}(G), \tag{26.63}
\end{equation*}
$$

where $\Phi_{l} \in H_{1-\delta}^{0}(G)$ are linearly independent functions from the orthogonal complement to the subspace $R_{1-\delta}^{0}(G)$ in the space $H_{1-\delta}^{0}(G)$. Using (26.61) and (26.63) and the Riesz theorem on the general form of linear continuous functionals in Hilbert spaces we obtain $\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right) \leq$ $\operatorname{codim} R_{1-\delta}^{0}(G)$.

Let us prove Theorem 26.2 applying the Hille-Yosida theorem (see Theorem 23.1) and Lemmas 26.11 and 26.15.

Proof of Theorem 26.2. 1. Assume the contrary: let $\overline{\mathbf{P}_{B}}$ be a generator of a Feller semigroup. Then, by the Hille-Yosida theorem (Theorem 23.1),

$$
\mathcal{R}\left(\overline{\mathbf{P}_{B}-q \mathbf{I}}\right)=\mathcal{R}\left(\overline{\mathbf{P}_{B}}-q \mathbf{I}\right)=C_{B}(\bar{G}) .
$$

Hence, $\overline{\mathcal{R}\left(\mathbf{P}_{B}-q \mathbf{I}\right)}=C_{B}(\bar{G})$. According to Lemma 26.11, this means that any function from $C_{B}(\bar{G})$ can be approximated by functions from the set

$$
\operatorname{Span}\left(R_{1-\delta}^{0}(G), \Delta w_{0}\right) \cap C_{B}(\bar{G}),
$$

where $w_{0}$ is a function from Lemma 26.12.
2. Similarly to the prove of Theorem 26.2, we obtain from item 1 of the proof of this theorem that

$$
\begin{equation*}
\operatorname{Span}\left(R_{1-\delta}^{0}(G), \Delta w_{0}\right)=H_{1-\delta}^{0}(G) . \tag{26.64}
\end{equation*}
$$

It follows from here that $\operatorname{codim} R_{1-\delta}^{0}(G) \leq 1$. This contradicts Lemma 26.15.
Remark 26.2. Similarly to Sec. 26.2 , we see that no reduction $\widehat{\mathbf{P}}_{B}: C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ of the operator $\overline{\mathbf{P}_{B}}$ is a generator of a Feller semigroup.

## 26.4. "Jumps" with probability 1 inside a neighborhood of a termination point of the process.

26.4.1. Statement of nonlocal problem. Let us show that Eq. (23.3) in condition 23.2 is substantial.

Let $G, g_{1}, g_{2}$, and $\mathcal{K}$ be the same as in Sec. 26.2. Consider the nonlocal problem (see Fig. 26.3)

$$
\begin{align*}
\Delta u(y)=f_{0}(y), & y \in G,  \tag{26.65}\\
u(y)-b_{j}(y) u\left(\Omega_{j}(y)\right)=0, & y \in \Gamma_{j}, \quad j=1,2,  \tag{26.66}\\
u(y)=0, & y \in \mathcal{K}, \tag{26.67}
\end{align*}
$$

where the functions $b_{j} \in C^{\infty}\left(\overline{\Gamma_{j}}\right)$ are such that
(1) $0 \leq b_{j}(y) \leq 1$,
(2) $b_{j}(y)=1$ for $y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right)$,
(3) $b_{j}(y)=0$ for $y \notin \mathcal{O}_{2 \varepsilon}\left(g_{1}\right)$,
and $\Omega_{j}, j=1,2$, is a smooth nondegenerate transformation defined in a neighborhood of the curve $\overline{\Gamma_{j}}$ such that
(1) $\Omega_{j}\left(\Gamma_{j}\right) \subset G, \Omega_{j}\left(g_{1}\right)=g_{1}, \Omega_{j}\left(g_{2}\right)=g_{2}$,
(2) $\Omega_{j}(y)$ for $y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right)$ is the rotation by angle $\pi / 2$ degrees inward the domain $G$.


Fig. 26.3. Problem (26.65)-(26.67).
Further, we consider $\varepsilon>0$ so small that

$$
\mathcal{O}_{2 \varepsilon}\left(g_{1}\right) \cap \mathcal{O}_{2 \varepsilon}\left(g_{2}\right)=\varnothing
$$

Introduce the measure $\delta(y, \cdot), y \in \partial G$, as follows: for any Borel set $Q \subset \bar{G}$, we assume that

$$
\begin{array}{cl}
\delta(y, Q)=b_{j}(y) \chi_{Q}\left(\Omega_{j}(y)\right), \quad y \in \Gamma_{j}, \quad j=1,2, \\
\delta(y, Q)=0, \quad y \in \mathcal{K} . &
\end{array}
$$

Then boundary conditions (26.66), (26.67) can be rewritten in the form

$$
b(y) u(y)+\int_{\bar{G}}[u(y)-u(\eta)] \delta(y, d \eta)=0, \quad y \in \partial G,
$$

where $b(y)=1-\delta(y, \bar{G})$.
It is easy to verify that conditions 23.1 and $24.1-24.8$ hold (with $\mathbf{P}(y, D)=\Delta, \mathbf{P}_{1}=0$ and $\alpha_{i}(y, \bar{G}) \equiv \beta_{i}(y, \bar{G}) \equiv 0$ ). Obviously, relation (23.3) in condition 23.2 is not fulfilled since $b_{1}\left(g_{1}\right)+b_{2}\left(g_{1}\right)=2$.

Consider the unbounded operator $\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ defined by the formula

$$
\begin{equation*}
\mathbf{P}_{B} u=\Delta u, \quad u \in \mathrm{D}\left(\mathbf{P}_{B}\right)=\left\{u \in C_{B}(\bar{G}): \Delta u \in C_{B}(\bar{G})\right\}, \tag{26.68}
\end{equation*}
$$

where $C_{B}(\bar{G})$ is the set of functions from $C(\bar{G})$ satisfying nonlocal conditions (26.66) and (26.67).
The following two results are proved similarly to Lemmas 26.2 and 26.3.
Lemma 26.16. If $u \in \mathrm{D}\left(\mathbf{P}_{B}\right)$, then $u \in W^{2}\left(G^{\prime}\right)$ for any domain $G^{\prime}$ such that $\overline{G^{\prime}} \subset \bar{G} \backslash \mathcal{K}$.
Lemma 26.17. The operator $\mathbf{P}_{B}: \mathrm{D}\left(\mathbf{P}_{B}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ admits a closure $\overline{\mathbf{P}_{B}}$.
We prove the following result.
Theorem 26.3. Let $\overline{\mathbf{P}_{B}}: \mathrm{D}\left(\overline{\mathbf{P}_{B}}\right) \subset C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ be the closure of operator (26.68). Then $\overline{\mathbf{P}_{B}}$ is not a generator of a Feller semigroup.

By the Hille-Yosida theorem (Theorem 23.1), it suffices to show that the image $\mathcal{R}\left(\overline{\mathbf{P}_{B}}-q \mathbf{I}\right)$ does not coincide with $C_{B}(\bar{G})$ for some $q>0$.
26.4.2. Proof of Theorem 26.3. To prove Theorem 26.3, we obtain the asymptotic of a solution $u \in \mathrm{D}\left(\mathbf{P}_{B}\right)$ of problem (26.65)-(26.67).

Consider the following model problem corresponding to the point $g_{1}$ with a complex parameter $\lambda$ :

$$
\begin{gather*}
\varphi^{\prime \prime}(\omega)-\lambda^{2} \varphi(\omega)=0, \quad 0<\omega<\pi  \tag{26.69}\\
\varphi(-\pi / 2)-\varphi(0)=0, \quad \varphi(\pi / 2)-\varphi(0)=0 \tag{26.70}
\end{gather*}
$$

where $\omega$ and $r$ are the polar coordinates with pole at the point $g_{1}$ such that $(0, r) \in \Gamma_{1}$ and $(\pi, r) \in \Gamma_{2}$ for $0<r<\varepsilon$. The eigenvalues of this problem have the form $\lambda_{k}=2 k i, k=0, \pm 1, \pm 2, \ldots$. Further, we will be interested in the eigenvalue $\lambda_{0}=0$. The eigenvector corresponding to it has the form $\varphi_{0}(\omega) \equiv 1$. There exists an adjoint vector $\hat{\varphi}_{0}(\omega) \equiv 0$. There is no second adjoint vector.

Since $b_{1}(y)=0$ near the point $g_{2}$, problem (26.10), (26.11) corresponds to the point $g_{2}$ (see Sec. 26.2).
Here, similarly to Secs. 23-25), we use the weight spaces

$$
H_{a}^{l}(G)=H_{a}^{l}(G, \mathcal{K}), \quad H_{a}^{l}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right)\right)=H_{a}^{l}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{j}\right),\left\{g_{j}\right\}\right)
$$

and the corresponding trace spaces.
Fix a number $\delta$ such that

$$
\begin{equation*}
0<\delta<1 \tag{26.71}
\end{equation*}
$$

Lemma 26.18. Let $u \in \mathrm{D}\left(\mathbf{P}_{B}\right)$. Then $u \in H_{1-\delta}^{2}(G)$.
Proof. It follows from Lemma 26.16 that $u \in W^{2}\left(G^{\prime}\right)$ for any domain $G^{\prime}$ such that $\overline{G^{\prime}} \subset \bar{G} \backslash \mathcal{K}$. Further, using Eqs. (26.66) and (26.67), we obtain

$$
\begin{gather*}
\Delta u \in C(\bar{G}) \subset H_{-\delta}^{0}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right),  \tag{26.72}\\
\left.u\right|_{\Gamma_{j} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}-\left.u\left(\Omega_{j}(y)\right)\right|_{\Gamma_{j} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}=0, \quad j=1,2 . \tag{26.73}
\end{gather*}
$$

Using Eqs. (26.72) and (26.73), the relation $u \in C(\bar{G}) \subset H_{-1+\delta}^{0}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$, and Lemma 14.2, we obtain

$$
\begin{equation*}
u \in H_{1+\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right) \tag{26.74}
\end{equation*}
$$

According to (26.71), the strip $-1-\delta \leq \operatorname{Im} \lambda \leq \delta$ contains the unique eigenvalue $\lambda_{0}=0$ of problem (26.69), (26.70). Hence, applying Eqs. (26.72) and (26.73) and [26, Theorem 2.2], we obtain

$$
u(y)=c_{0} \varphi_{0}(\omega)+\hat{c}_{0}\left(\hat{\varphi}_{0}(\omega)+\varphi_{0}(\omega) \ln r\right)+v(y)=c_{0}+\hat{c} \ln r+v(y), \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right),
$$

where $c_{1}$ and $c_{2}$ are constants and $v \in H_{-\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$. Since $H_{-\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right) \subset W^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$, using the Sobolev embedding theorem, we obtain $v \in C\left(\overline{G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}\right)$ and $v\left(g_{1}\right)=0$. It follows from here and the relation $u \in C\left(\overline{G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}\right)$ that $\hat{c}_{0}=0$. Taking into account Eq. (26.67) and the relation $v\left(g_{1}\right)=0$, we obtain

$$
c_{0}=u\left(g_{1}\right)-v\left(g_{1}\right)=0 .
$$

Thus, $u \in H_{-\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right) \subset H_{1-\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right)$.
Similarly to Lemma 26.5, we prove that $u \in H_{1-\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right)\right)$.
Introduce the bounded operator $\mathbf{L}_{a}(q): H_{a}^{2}(G) \rightarrow \mathcal{H}_{a}^{0}(G, \partial G)$ by the formula

$$
\mathbf{L}_{a}(q) u=\left\{\Delta u-q u,\left.u\right|_{\Gamma_{1}}-\left.b_{1}(y) u\left(\Omega_{1}(y)\right)\right|_{\Gamma_{1}},\left.u\right|_{\Gamma_{2}}\right\}, \quad q \geq 0 .
$$

We also denote $\mathbf{L}_{a}=\mathbf{L}_{a}(0)$.
Lemma 26.19. Let $q>0$ be sufficiently small. Then $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1-\delta}(q)=0$.
Proof. Suppose $q=0$ and prove that $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1-\delta}=0$. Let $u \in \operatorname{ker} \mathbf{L}_{1-\delta}$. Without loss of generality, we assume that the function $u$ is real-valued. Using Eqs. (26.65)-(26.67), we obtain

$$
\begin{gathered}
\Delta u=0, \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right), \\
\left.u\right|_{\Gamma_{j} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}-\left.u\left(\Omega_{j}(y)\right)\right|_{\Gamma_{j} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}=0, \quad j=1,2 .
\end{gathered}
$$

Hence, reasoning similarly to the proof of Lemma 26.4, we obtain the embedding $u \in C\left(\overline{G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}\right)$. It is easy to show (similarly to Lemma 26.5) that $u \in C\left(\overline{G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right)}\right)$. Then $u \in C(\bar{G})$, and, according to maximum principle 24.1, we have $u=0$, i.e., $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1-\delta}=0$.

Similarly to Lemma 26.5, we prove that $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1-\delta}(q)=0$ for sufficiently small $q>0$.
Lemma 26.20. Let $q>0$ be sufficiently small. Then $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta}(q) \leq 1$.
Proof. 1. First, we prove that $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta} \leq 2$. Let $u \in \operatorname{ker} \mathbf{L}_{1+\delta}$. Without loss of generality, we assume that the function $u$ is real-valued. Using Eqs. (26.65)-(26.67), we obtain

$$
\begin{gather*}
\Delta u=0, \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right),  \tag{26.75}\\
\left.u\right|_{\Gamma_{j} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}-\left.u\left(\Omega_{j}(y)\right)\right|_{\Gamma_{j} \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}=0, \quad j=1,2 . \tag{26.76}
\end{gather*}
$$

By (26.71), the strip $-1-\delta \leq \operatorname{Im} \lambda \leq \delta$ contains only one eigenvalue $\lambda_{0}=0$ of problem (26.69), (26.70). Hence, using Eqs. (26.75) and (26.76), and [26, Theorem 2.2], we obtain

$$
\begin{equation*}
u(y)=c_{0} \varphi_{0}(\omega)+\hat{c}_{0}\left(\hat{\varphi}_{0}(\omega)+\varphi_{0}(\omega) \ln r\right)+v(y)=c_{0}+\hat{c} \ln r+v(y), \quad y \in G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right), \tag{26.77}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are constants,

$$
v \in H_{-\delta}^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right) \subset W^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)\right) \subset C\left(\overline{G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}\right)
$$

and $v\left(g_{1}\right)=0$.
As in the proof of Lemma 26.20, we have

$$
u \in W^{2}\left(G \cap \mathcal{O}_{\varepsilon}\left(g_{2}\right)\right) \subset C\left(\overline{G \cap \mathcal{O}_{\varepsilon}\left(g_{1}\right)}\right) .
$$

Hence, taking into account (26.77), we see that

$$
\begin{equation*}
u(y)=c_{0} u_{0}(y)+\hat{c}_{0} \hat{u}_{0}(y)+U(y), \quad y \in G \tag{26.78}
\end{equation*}
$$

where $u_{0}, \hat{u}_{0} \in C^{\infty}\left(\bar{G} \backslash\left\{g_{1}\right\}\right)$,

$$
u_{0}(y)=\left\{\begin{array}{ll}
1, & y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right), \\
0, & y \in \mathcal{O}_{\varepsilon}\left(g_{2}\right),
\end{array} \quad \hat{u}_{0}(y)= \begin{cases}\ln r, & y \in \mathcal{O}_{\varepsilon}\left(g_{1}\right) \\
0, & y \in \mathcal{O}_{\varepsilon}\left(g_{2}\right)\end{cases}\right.
$$

$U \in W^{2}(G) \subset C(\bar{G})$ and $U\left(g_{1}\right)=U\left(g_{2}\right)=U(g)=0$. It follows from here and Lemma 5.2 (item 2) that $U \in H_{1-\delta}^{2}(G)$.

From Eq. (26.78) and Lemma 26.19 we obtain that $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta} \leq 2$.
2. Prove that $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta} \leq 1$. Assume the contrary: let there exist two linearly independent functions $v_{1}, v_{2} \in \operatorname{ker} \mathbf{L}_{1+\delta}$. Since each of these functions can be represented in the form of Eq. (26.78), we see that some of their nontrivial linear combinations (that we denote by $u$ ) have the form

$$
\begin{equation*}
u(y)=u_{0}(y)+U(y), \tag{26.79}
\end{equation*}
$$

i.e., $u \in C(\bar{G})$. Using maximum principle 24.1, we have $u(y) \equiv 0$. Hence, the functions $v_{1}$ and $v_{2}$ are linearly dependent and $\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta} \leq 1$.
3. Similarly to the proof of Lemma 26.6, we can show that

$$
\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta}(q) \leq 1
$$

for sufficiently small $q>0$.
Lemma 26.21. Let $q>0$ be sufficiently small. Then $\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right) \geq 1$.
Proof. The strip $-\delta \leq \operatorname{Im} \lambda \leq \delta$ does not contain eigenvalues of the model problem (26.10), (26.11) (see Sec. 26.2) corresponding to the point $g_{2}$ and contains the unique eigenvalue $\lambda_{0}=0$ of the model problem (26.69), (26.70). Moreover, the algebraic multiplicity of the eigenvalue $\lambda_{0}=0$ is 2 . Hence, according to [31, Theorem 4.1],

$$
\operatorname{ind} \mathbf{L}_{1+\delta}(q)=\operatorname{ind} \mathbf{L}_{1-\delta}(q)+2,
$$

that is,

$$
\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta}(q)-\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1+\delta}(q)\right)=\operatorname{dim} \operatorname{ker} \mathbf{L}_{1-\delta}(q)-\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)+2
$$

The relation

$$
\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)=\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1+\delta}(q)\right)-\operatorname{dim} \operatorname{ker} \mathbf{L}_{1+\delta}(q)+2 \geq 1
$$

follows from here and Lemmas 26.19 and 26.20
Fix a number $q>0$ for which

$$
\operatorname{codim} \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right) \geq 1
$$

Consider the set

$$
R_{1-\delta}^{0}(G)=\left\{f_{0} \in H_{1-\delta}^{0}(G):\left(f_{0}, 0,0\right) \in \mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)\right\}
$$

Obviously, $R_{1-\delta}^{0}(G)$ is a closed subspace in $H_{1-\delta}^{0}(G)$ since the image $\mathcal{R}\left(\mathbf{L}_{1-\delta}(q)\right)$ is a closed subspace in $\mathcal{H}_{1-\delta}^{0}(G, \partial G)$.

The following result can be proved similarly to Lemma 26.8 (here, Lemma 26.7 must be replaced by Lemma 26.21).

Lemma 26.22. codim $R_{1-\delta}^{0}(G) \geq 1$.
Now, repeating the proof of Theorem 26.1 and applying Lemmas 26.18 and 26.22 instead of Lemmas 26.4 and 26.8, respectively, we obtain the conclusion of Theorem 26.3.
Remark 26.3. As in Sec. 26.2, we see that no reduction $\widehat{\mathbf{P}}_{B}: C_{B}(\bar{G}) \rightarrow C_{B}(\bar{G})$ of the operator $\overline{\mathbf{P}_{B}}$ is a generator of a Feller semigroup.

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[^0]:    ${ }^{1}$ Do not confuse it with the space dual to $\dot{W}^{l}(G)$, where $\dot{W}^{l}(G)$ is the closure of the set $C_{0}^{\infty}(G)$ with respect to the norm of the space $W^{l}(G)$.

[^1]:    ${ }^{2}$ The space $H_{-a}^{1 / 2}\left(\Gamma_{i}\right)$ is a Hilbert space; therefore, it is reflexive.

[^2]:    ${ }^{3}$ Strictly speaking, the angles $K_{j}$ introduced here can be obtained from the angles $K_{j}$ described in Sec. 5.1 by the shift by the vector $-\overrightarrow{O g_{j}}$ and a rotation. In what follows, we will identify them.

[^3]:    ${ }^{4}$ Theorem 3.2 in [89] is formulated for the case where the operators $\mathbf{B}_{i \mu}^{2}$ have the same form as in Sec. 6.2 (see Chap. 2). However, the proof of [89, Theorem 3.2] is based on inequalities (6.5) and (6.6) and is independent of the explicit form of the operators $\mathbf{B}_{i \mu}^{2}$ (see also [41]).

[^4]:    ${ }^{5}$ In assumptions of the lemma, according to Lemma 5.2, we have $U \in \mathcal{H}_{a}^{l+2 m}\left(K^{d}\right)$ for any $a>0$. Hence, $U \in \mathcal{H}_{0}^{l+2 m-1}\left(K^{d}\right)$, and the right-hand side of inequality (11.6) is finite.

[^5]:    ${ }^{6}$ If $\ell=2 m$, then the norms in the spaces $\mathbf{W}^{\ell}(G)$ and $W^{\ell}(G)$ are equivalent and these spaces can be identified, but we consider the case $\ell \leq 2 m-1$.

[^6]:    ${ }^{7}$ Since the function $\varphi_{\delta}$ vanishes near the set $\mathcal{K}$, we can replace the space $W^{\ell-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ in the second inequality by $H_{a}^{\ell-m_{i \mu}-1 / 2}\left(\Gamma_{i}\right)$ for any $a \in \mathbb{R}$.

[^7]:    ${ }^{8}$ Note that the closedness of an operator $\mathbf{P}$, generally speaking, is not implied by the closedness of its image in a Hilbert space and the finite dimension of its kernel and cokernel; this can be shown by using reasoning similar to that given, e.g., in [2, Chap. 2, Sec. 18]. However, if, in addition, we assume that $\mathbf{P}$ is an extension of a Fredholm operator, then we will see that $\mathbf{P}$ will be closed.

[^8]:    ${ }^{9}$ If $\lambda_{0} \in \mathbb{C}$ is an eigenvalue of problem (17.3), (17.4) and $\varphi_{0}(\omega)$ is a corresponding eigenvector, then an adjoint vector $\varphi_{1}(\omega)$ can be found as a solution (perhaps, zero) of the equation $\varphi_{1}^{\prime \prime}-\lambda_{0}^{2} \varphi_{1}+\left.\frac{d}{d \lambda}\left(\varphi_{0}^{\prime \prime}-\lambda^{2} \varphi_{0}\right)\right|_{\lambda=\lambda_{0}}=0$ with nonlocal conditions (17.4). Thus, if $\lambda_{0}=0$, then the adjoint vector $\varphi_{1}(\omega)$ is a solution of the equation $\varphi_{1}^{\prime \prime}=0$ with nonlocal conditions (17.4).

[^9]:    ${ }^{10}$ This solution exists and is unique since $-i s$ is not an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$.

[^10]:    ${ }^{11}$ The function $f_{i \mu}^{1}$ written in the coordinate system with origin at the point $g \in \overline{\Gamma_{i}} \cap \mathcal{K}$ either vanishes or is a monomial of degree $s-m_{j \sigma \mu}$.

[^11]:    ${ }^{12}$ In this example, for simplicity of notation, we will use the operators $\frac{\partial}{\partial \tau_{\sigma}}$ instead of $D_{\tau_{\sigma}}=-i \frac{\partial}{\partial \tau_{\sigma}}$.

[^12]:    ${ }^{13}$ In what follows, the operator $\mathbf{P}(y, D)$ acts in the sense of generalized function.

[^13]:    ${ }^{14}$ In [19, Lemma 1.3], it was assumed that the boundary of the domain is infinitely smooth. This assumption was used in the proof of the existence of classic solutions of elliptic equations with inhomogeneous boundary conditions. However, if we know that a classic solution exists, we can omit the assumption on the smoothness of the boundary in the proof of the first inequality in Eq. (24.1).

[^14]:    ${ }^{15}$ Item 2 of condition 24.7 can be replaced by a stronger assumption " $\left\|\hat{\mathbf{B}}_{\beta i}^{1}\right\| \rightarrow 0$ as $p \rightarrow 0$," which is easier for verification in concrete applications.

[^15]:    ${ }^{16}$ For (26.26), we assume that in any $\varepsilon$-neighborhood, a different coordinate system $\omega, r$ is given. This coordinate system is such that the point $r=0$ corresponds to the center of the neighborhood.

[^16]:    ${ }^{17}$ For Eq. (26.51), we assume that a specific polar coordinate system $\omega, r$ is given for any $\varepsilon$-neighborhood. The coordinate system is such that the point $r=0$ corresponds to the center of the neighborhood.

